Correlation Made Simple: Applications to Salience and Regret Theory*

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Abstract

We offer an axiomatization of risk models where the choices of the decision maker are correlation sensitive. By extending the techniques of conjoint measurement to the nondeterministic case, we show that transitivity is the vN-M axiom that has to be relaxed to allow for these richer patterns of behavior. To illustrate the advantages of our modeling choice, we provide a simple axiomatization for the salience theory model within our general framework. This approach leads to clear comparison to popular preexisting models such as regret and reference dependence, and lets us single out the ordering property as the feature that brings salience theory outside the prospect theory realm.

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1 Introduction

Correlation between risky alternatives can play a significant role in decisions. First, it may be relevant because the Decision Maker (henceforth DM) cares about what she would have received had she chosen differently, a channel emphasized by regret theory. For example, an agent who has decided not to invest part of her resources in stocks the day before a press release by the Fed may be better off if the market's effect is negative since she does not suffer for the foregone opportunity.

Second, the correlation structure can determine the attention and weight that the DM allocates to the various contingencies, as emphasized by salience theory. For example, when deciding whether to purchase comprehensive car insurance, the (unlikely) event in which the car is destroyed in a crash may disproportionately attract the DM's attention due to the vast difference between the two alternatives' consequences. In this paper, we study these possibilities from an axiomatic perspective.

We provide a simple axiomatization for a general class of correlation-sensitive preferences. The motivation is two-fold. First, we show that our general framework nests the recent models that highlight the role of correlation (see, e.g., Bordalo, Gennaioli, and Shleifer, 2012, henceforth BGS, and Koszegi and Szeidl, 2013) as particular cases so that we can characterize them in terms of additional testable axioms. The study of these axioms lets us understand better where they depart from the preexisting theories. Second, we use this axiomatization to provide new insights into the difference between classical models for which correlation is relevant (see Bell, 1982, Loomes and Sugden, 1982, Fishburn, 1989) and the benchmark model for choice under risk, expected utility (henceforth EU).

We accomplish these goals by taking a different route than those followed in the usual axiomatizations of correlation-sensitive preferences.[1] We represent the preferences of the DM in the space of lotteries. In doing so, we face a complication: when the correlation between alternatives matters, binary relations over lotteries are not sufficiently rich as modeling tools. To see why, suppose that we have the two

[1] See, e.g., Fishburn (1989), Sugden (1993), and Diecidue and Somaundaram (2017). All these papers represent the preferences as a binary relation over acts à la Savage.

lotteries $p = \left(10, \frac{1}{3}; 5, \frac{1}{3}; 0, \frac{1}{3}\right)$ and $q = \left(10, \frac{1}{3}; 4, \frac{1}{3}; 1, \frac{1}{3}\right)$, and consider the following two possible correlation structures:

π	1	4	10
0	0	1/3	0
5	0	0	1/3
10	1/3	0	0

π'	1	4	10
0	0	0	1/3
5	1/3	0	0
10	0	1/3	0

Both the joint distributions π and π' have marginal distributions p and q. However, we will see that a salience-sensitive DM may strictly prefer p under the first correlation structure (driven by the salient realization $(10, 1)$) and q under the second correlation structure (driven by the salient realization $(0, 10)$). Therefore, the classical approach of describing the DM's tastes using a binary relation over lotteries is not viable since the DM cannot rank p and q without additional information about their joint distribution. Indeed, applied researchers (e.g., Smith 1996, Braun and Muermann, 2004, Filiz-Ozbay and Ozbay, 2007) have shown that in various economically significant situations as auctions, insurance decisions, and health interventions, the correlation between lotteries impacts choices.

Instead of using a binary relation, we use the preference set concept introduced by Fishburn (1990a):[2] Given a fixed set of possible outcomes X, tastes are represented by a preference set $\Pi \subseteq \Delta(X \times X)$, with the following interpretation. The DM contemplates a joint distribution π over $X \times X$. Facing this joint lottery, the DM decides if, *given the marginals and the correlation structure,* she prefers to be paid according to the realized row or column outcome.[3] Then, we say that π belongs to the preference set Π if and only if the DM prefers to be paid according to the row outcome. In our previous example, we have $\pi \in \Pi$, and $\pi' \notin \Pi$.

There are several motivations for this modeling choice. On a theoretical side,

[2]Fishburn (1990a) introduces the concept of preference sets for intransitive preferences over multi-attribute products and applies it to choices between acts in Fishburn (1990b). To the best of our knowledge, this is the first work in which preference sets are used to axiomatize preferences under risk.

[3]As we can always represent a joint distribution in the tabular form used above, we will refer to the first and second marginal respectively as the row and column marginals.

it avoids introducing an ancillary state-space and provides a clear comparison with expected utility. If we want to test the theory, having an axiomatization for the case of choice under risk, instead of one for acts defined over a state space in which probabilities are not specified, allows us to disentangle violation of the axioms at the cornerstone of our correlation sensitive theory from the ubiquitous failures in formulating a unique, coherent probability measure over the states of the world.

The second motivation comes from our salience theory application. Indeed, BGS define their preferences on the joint distributions of two alternative random variables, and the correlation is part of the data exactly as under our proposed approach. Moreover, the subsequent experimental papers consider choices between lotteries, where the only state space is the one *defined* as the space of all the possible joint realizations of the two lotteries under scrutiny.[4] Therefore, axioms stated in terms of joint lotteries are more natural to map into the BGS model, and they can be directly challenged by the existing experimental evidence on the model. Finally, under the alternative state-space formulation, the characterization of the salience properties postulated by BGS is much more demanding in terms of the underlying state space's structural properties.

We first identify three axioms on the preference set Π necessary and sufficient to obtain a representation for correlation-sensitive preferences. This representation includes regret and salience theory as particular cases. These axioms are Completeness, Strong Independence, and Archimedean Continuity, and they are equivalent to the correlation-sensitive representation

$$\pi \in \Pi \Leftrightarrow \sum_{x,y} \phi(x,y)\, \pi(x,y) \geq 0$$

where ϕ is skew-symmetric. Here, $\phi(x,y)$ corresponds to how much the joint realization (x,y) contributes in favor of the row marginal. That is, we have $\phi(x,y) \geq 0$ if and only x is preferred to y, and larger values imply a comparison more favorable

[4] For the experimental tests of salience theory, see Dertwinkel-Kalt and Köster (2020), Frydman and Mormann, (2017), Konigsheim, Lukas, and Noth (2019), Dertwinkel-Kalt, Frey, and Koster (2021) Nielsen, Sebald, and Sorensen (2021).

to x. Under expected utility, $\phi(x, y)$ reduces to the separable form $u(x) - u(y)$, but more generally (e.g., in the salience model), the two components are entangled. Indeed, how much attention an outcome x attracts may depend on how much it contrasts with the counterfactual realization y. The skew symmetry of ϕ means that the "row" and "column" labeling are irrelevant: $\phi(x, y) = -\phi(y, x)$, so that the contribution of the joint realization (x, y) in favor of the row component is equal to the contribution of (y, x) to the column component. We show that these axioms are mild relaxations of their more familiar counterparts for binary relations and that if Transitivity is added, the representation reduces to EU.

After weakening the EU axioms to allow for this more general correlation-sensitive representation, we look for the additional axioms needed to characterize the particular case of salience theory. BGS's salience model provides a theory of choice under risk based on few psychological properties of salience detection: Ordering, Diminishing Sensitivity, and Weak Reflexivity. Most importantly, Ordering prescribes that joint realizations in which the two components are farther apart are overweighted. In addition, there is Diminishing Sensitivity to the differences between the components as their absolute values increase. At the same time, Weak Reflexivity can be loosely paraphrased as the requirement that the salience ranking between two joint realizations that only involve gains remains the same if all the gains are transformed into losses of the same size.

A payoff of our preference sets setup is that it allows us to state and characterize the testable versions of these properties in a straightforward manner. We also find that Ordering is the property that brings salience theory outside the prospect theory realm—instead, Diminishing Sensitivity and Weak Reflexivity combined amount to the usual risk-aversion in gains, risk-loving in loss property featured by prospect theory. We then characterize the salience model as the result of the Ordering, Diminishing Sensitivity, and Weak Reflexivity axioms combined with continuity and monotonicity requirements.

We also provide a partial solution to the problem of choice between multiple alternatives. A DM with correlation-sensitive preferences may not have an alternative that is weakly preferred to all the others when facing a set of at least three options.

However, we prove that an optimal stochastic choice rule always exists.

Related Literature This paper belongs to the literature studying the axiomatization of correlation-sensitive models of choice. This literature starts with the classical works of Fishburn (1989), Sugden (1993), and Quiggin (1994). Recently, Diecidue and Somasundaram (2017) significantly improve the regret model's previous representation, providing an axiomatization that delivers a continuous regret function on an arbitrary finite state space. Their main conceptual contribution is to single out the axioms for the more restrictive version of regret theory initially formulated by Loomes and Sugden (1982) and separate the edonic utility from the regret function. In this sense, their work is complementary to ours. In the first part of the paper, we want to axiomatize the more general form of correlation-sensitive preferences to characterize later regret theory and salience theory as particular cases of this model.

Fishburn (1990b) uses preference sets to provide an axiomatization of the Skew-Symmetric Additive (SSA) model. On a technical side, the object on which the preferences are defined is different: Fishburn defines the preference sets as subsets of the space of acts with two outcomes, whereas we focus on joint distribution over outcomes. Notice that by letting the preference sets being a subset of the multivariate acts, Fishburn (1990b) faces the general disadvantages discussed above: potential confusion with ambiguity aversion, axioms that are sufficient for the representation but not necessary, more difficult comparison with EU, and a more relevant departure from the version of the model that has been experimentally tested.[5] These disadvantages become even more relevant in our salience theory application: first, the additional properties characterizing salience theory as a particular case become much more involved under the act formulation. Second, generalizations that build on our axiomatization to combine salience theory for consumption and risk (see Köster,

[5] Among other things, the state space has to be atomless, a property at odds with the small finite set of joint realizations used as the state space in BGS. Moreover, the use of atomless state space in the classical axiomatizations of correlation-sensitive preferences is particularly unsatisfactory since it is a direct consequence of what Fishburn (1990b) calls axiom P6*. This axiom is made for technical convenience but is not necessary for the representation. Therefore, such a richness of the space is not an intrinsic feature of the model but more the result of a technically convenient assumption.

2021) cannot be conciliated with the "structure axiom" needed in Fishburn (1990b), therefore limiting the scope of his axiomatization.

Fishburn (1982) axiomatizes the class of SSB preferences over the space of lotteries. With these preferences, each alternative's realization has a value that depends on *all* the possible realizations of the other alternative. When restricted to the comparison between *independent* lotteries, the two models coincide.[6] In this sense, Theorem 1 provides an alternative set of axioms for the SSB model. More importantly, the two models are highly different in their predictions about correlation. Fishburn (1982) explicitly rules out any correlation effect, and so it excludes the salience model, where significantly different joint realizations attract the attention of the DM and imposes an awkward structure on the regret model.[7] Farther afield, the quadratic preferences of Chew, Epstein, and Segal (1991) and the reference-dependent model of Koszegi and Rabin (2007) also use of a joint evaluation of outcomes, although they do not allow for correlation sensitivity and satisfy Transitivity.

This paper is the first to axiomatize the salience theory of choice under risk. Ellis and Masatioglu (2021) provide an axiomatization of the salience theory of consumption (Bordalo, Gennaioli, and Shleifer, 2013). They focus on the rank-dependent version of the salience model, while we focus on the continuous version. Herweg and Muller (2021) provide a comparison between salience and regret theory, arguing that the former can be interpreted as a particular case of the latter, but they do not identify the axioms underlying the representation.

Outline The rest of the paper is structured as follows. Section 2 introduces preference sets. Then, in Section 3 we describe the *weakening* of EU that is necessary

[6]However, when Transitivity is imposed in the two models, the conclusion differ. Since by definition of the domain of preferences in the SSB model Transitivity can only be imposed on independent distributions, SSB reduces to the weighted utility model of Chew (1983), see Theorem 3 in Fishburn (1983). Instead when I impose Transitivity of the marginal regardless of the correlation structure, I obtain the Expected Utility model. I thank Chew Soo Hong for pushing me to explore this additional difference.

[7]For example, the form of regret compatible with the SSB model requires that when choosing not to bet on a horse in a race, the DM must feel regret for the foregone possibility of a significant payoff, regardless of whether the horse wins the race.

to capture sensitivity to correlation, while in Section 4 we provide the *additional* axioms characterizing salience theory. All the proofs of the results in the main text are in Online Appendix A. Online Appendix B establishes the formal connection between axioms stated for preference sets and their counterparts in terms of binary relations. Online Appendix C extends the model to choice from nonbinary subsets. Finally, Online Appendix D studies the rank-based version of salience theory.

2 Preference Sets

Let X be an arbitrary nonempty set of outcomes (or prizes), and denote as $\Delta(X \times X)$ the set of (joint) probability measures over $X \times X$ with finite support. We model the DM preferences by a subset Π (called preference set) of $\Delta(X \times X)$. The interpretation is that the DM faces a $\pi \in \Delta(X \times X)$, and she has to decide whether to be paid according to the row or column outcome. Then, we say that $\pi \in \Pi$ if and only if she (weakly) prefers to be paid according to the row outcome. The fact that the knowledge of the marginal π_1 and π_2 may be insufficient to determine whether $\pi \in \Pi$ is the deviation from the standard paradigm of rational choice.

2.1 Eliciting Preference Sets

Here is a roadmap of how to test axioms imposed on the preference set. The DM faces a finite-support joint distribution π over prizes that a table can summarize:

π	y_1	$\dots$	y_m
x_1	π_{11}	$\dots$	π_{1m}
$\dots$	$\dots$	$\dots$	$\dots$
x_n	π_{n1}	$\dots$	π_{nm}

That is, the DM knows that every pair of outcomes (x_i, y_j) realizes with probability π_{ij}. Then, given the correlation structure between the two alternatives, the subject chooses between being paid according to the row prizes (the x's) or the column prizes (the y's). If she chooses to be paid according to the rows (resp. the columns), if outcome (x_i, y_j) realizes she gets x_i (resp. y_j) regardless of the value of y_j (resp. x_i).[8]

A joint distribution belongs to the preference set if, when faced, the DM chooses to be paid according to the row prizes. The typical axioms we impose on preference sets have the form "if $\pi \in \Pi$ then π' belongs to Π," where π' has some particular relation with π.

2.2 Preference Sets and Binary Relations

For every joint distribution $\pi \in \Delta(X \times X)$, we denote as $\pi_1 \in \Delta(X)$ and $\pi_2 \in \Delta(X)$, respectively, the row and column marginals of π. Formally:

$$\pi_1(x) = \sum_{y \in X} \pi(x, y) \text{ and } \pi_2(y) = \sum_{x \in X} \pi(x, y).$$

Notice that a binary relation $\succsim$ over marginal distributions induces a *unique* preference set $\Pi_{\succsim}$ that contains a joint distribution if and only if the row marginal is preferred to the column according to $\succsim$.

Definition 1 *The preference set* $\Pi_{\succsim}$ *induced by a binary relation* $\succsim$ *is defined as*

$$\pi \in \Pi_{\succsim} \Leftrightarrow \pi_1 \succsim \pi_2.$$

It is easy to see that two different binary relations induce different preference sets, so no information is lost by describing the DM's tastes using preference sets rather than binary relations. Also, every preference set induces a (possibly incomplete) binary relation over marginal distributions.

Definition 2 *The binary relation* $\succsim^{\Pi}$ *induced by a preference set* Π *is defined as*

$$p \succsim^{\Pi} q \Leftrightarrow (\forall \pi \in \Delta(X \times X) : (\pi_1, \pi_2) = (p, q),\ \pi \in \Pi).$$

[8]Our theory is silent about the information revealed to the subject after a joint outcome (x, y) is drawn. One may expect that the behavior may differ, whether only the component paid out to the DM or the joint realization is revealed.

Requiring $p \succsim^{\Pi} q$ ensures that all the joint distributions with those marginals are in the preference set (i.e., p has to be preferred to q regardless of their correlation structure). Of course, when $\succsim^{\Pi}$ is complete, it describes the DM's tastes fully. However, $\succsim^{\Pi}$ may not be complete for a correlation-sensitive DM. In this case, the patterns of behavior that can be described using preference sets are much richer than those for binary relations.[9]

3 General Representation Theorem

We first define the general form of risk preferences we are interested in. Recall that a function $\phi : X \times X \to \mathbb{R}$ is skew symmetric if $\phi(x,y) = -\phi(y,x)$ for all $x, y \in X$.

Definition 3 *A preference set* Π *admits a correlation-sensitive representation if there exists a skew-symmetric* $\phi : X \times X \to \mathbb{R}$ *such that for all* $\pi \in \Delta(X \times X)$

$$\pi \in \Pi \Leftrightarrow \sum_{x,y} \phi(x,y)\,\pi(x,y) \geq 0. \tag{1}$$

To better understand this representation, it is helpful to compare it with expected utility. Let $\pi \in \Delta(X \times X)$. Under EU, there exists a utility function u such that

$$\begin{aligned} \pi_1 \succsim \pi_2 \quad &\Leftrightarrow \quad \sum_{x} u(x)\,\pi_1(x) \geq \sum_{y} u(y)\,\pi_2(y) &(2)\\ &\Leftrightarrow \quad \sum_{x,y} (u(x) - u(y))\,\pi(x,y) \geq 0. &(3)\end{aligned}$$

Given these equivalences, the difference between EU and the correlation-sensitive representation can be described in the following way. In principle, when contemplating a joint lottery π, two algorithmic procedures can determine according to which component to be paid. The first algorithm is the following: (i) Take marginal π_1. Consider the utility obtained under each realization. Aggregate these utilities

[9] Lemma 1 in the Online Appendix shows that for every binary relation $\succeq$, the binary relation $\succsim^{\Pi_{\succeq}}$ coincides with $\succeq$. A weaker notion of $\succsim^{\Pi}$ would have replaced "for all π" with "for some π" in its definition. Proposition 1 shows that our definition is more fruitful.

according to the probability measure π_1 to get a "score" $U(\pi_1) = \sum_x u(x)\pi_1(x)$. Note that this score is *independent of* π_2. (ii) Follow the same procedure for marginal π_2. (iii) Compare these scores obtained for the two alternatives, and choose to be paid according to the row outcome if and only if $U(\pi_1) \geq U(\pi_2)$. There is no role for correlation between the two marginal distributions under this procedure. This procedure consists of a *comparison of aggregations*, and in the case of EU is given by (2).

Alternatively, one may consider the following procedure: (i) Take a possible joint realization (x, y). Compare the two prizes and give a score $\phi(x, y)$, representing a combination of how much x is preferred to y and the attention diverted to that realization, with 0 meaning indifference or zero attention. (ii) Do the same for every joint realization. (iii) Aggregate all these comparisons according to the probability measure π obtaining $\Phi(\pi) = \sum_{x,y} \pi(x, y)\phi(x, y)$. (iv) Choose to be paid according to the row outcome if and only if $\Phi(\pi) \geq 0$.

This *aggregation of comparisons* allows for correlation to matter. It is the kind of reasoning that characterizes both regret and salience-sensitive DMs, and for EU it corresponds to line (3). The pioneering works by Bell (1982) and Loomes and Sugden (1982) already recognize the descriptive and normative value of this procedure. However, under expected utility, aggregation of comparisons reduces to $\phi(x, y) = u(x) - u(y)$, which makes correlation irrelevant because, for an EU agent, the value of receiving x is $u(x)$ independent of the realization of the counterfactual. Therefore, in this case, the two algorithms reach the same conclusion. We formalize this reasoning in the following definition.

Definition 4 *A preference set* Π *admits an expected utility representation if there exists* $u : X \to \mathbb{R}$ *such that*

$$\pi \in \Pi \iff \sum_{(x,y)\in X\times X} (u(x) - u(y))\,\pi(x, y) \geq 0. \tag{4}$$

Instead, our first step is to provide a set of axioms that characterize the general correlation-sensitive representation for a (possibly) nonseparable ϕ. We will call these axioms Completeness, Strong Independence, and Archimedean Continuity after the names of the standard axioms for binary relations they resemble. In Online Appendix B, we show formally how each of the axioms for preference sets is a weakening of the original one that only applies to joint distributions and that they coincide when Transitivity is satisfied.

Before going further, a piece of notation is needed. Given $\pi \in \Delta(X \times X)$, we define its conjugate distribution $\bar{\pi}$ as

$$\forall (x, y) \in X \times X \quad \bar{\pi}(x, y) = \pi(y, x).$$

Therefore, the conjugate distribution is just a relabeling of the row and column outcomes into each other.

Axiom 1 (Completeness) *For all* $\pi \in \Delta(X \times X)$

$$\pi \notin \Pi \Rightarrow \bar{\pi} \in \Pi.$$

Completeness is a minimal requirement about the rationality of the DM. If she prefers to be paid according to the column marginal when the joint distribution is π, she (weakly) prefers to be paid according to the row marginal after relabeling row outcomes into column ones and vice-versa.

Given a preference set $\Pi \subseteq \Delta(X \times X)$, the strict preference set is defined as

$$\hat{\Pi} = \{\pi \in \Pi : \bar{\pi} \notin \Pi\}.$$

In words, a joint distribution π is in the strict preference set if the DM weakly prefers to be paid according to the row outcome (i.e., $\pi \in \Pi$), and she does not prefer to be paid according to the column outcome (i.e., $\bar{\pi} \notin \Pi$). It is the counterpart of the asymmetric part of a binary relation in the language of preference sets. We use the strict preference set in our second axiom. This axiom is a generalization to intransitive preferences of the standard principle of reduction for compound lotteries. If there are two joint distributions π and π' such that under each of them the DM prefers to be paid according to the row outcome, it then seems reasonable she prefers to be paid according to the row outcome even if the joint distribution that is going to be used is π with probability α and π' with probability $(1-\alpha)$. The preference is strict whenever one of the initial preferences is.

Axiom 2 (Strong Independence) *For all $\pi, \pi' \in \Pi$, and all $\alpha \in (0,1)$*

$$\alpha\pi + (1-\alpha)\,\pi' \in \Pi.$$

Moreover, if $\pi' \in \hat{\Pi}$, then

$$\alpha\pi + (1-\alpha)\,\pi' \in \hat{\Pi}.$$

The difference between the previous axiom and the standard Strong Independence for binary relations can be understood in the setting of the Allais Paradox.

Example 1 *Recall that in the Allais paradox, the marginal distributions faced by the DM are*

$$\begin{aligned} p &= (2500, 0.33; 0, 0.01; z, 0.66) \\ q &= (2400, 0.34; z, 0.66) \end{aligned}$$

for $z \in \{0, 2400\}$. It is immediate to see that the Strong Independence axiom for binary relations implies that the choice of the DM does not depend on the particular value of z. The conclusion is more nuanced for our Strong Independence axiom. Indeed, the version of the Allais paradox in which the alternatives are independent corresponds to the joint distribution

$\pi_{ind,z}$	2400	z
2500	0.1122	0.2178
0	0.0034	0.0066
z	0.2244	0.4356

Here, Strong Independence formulated as above does not impose cross-restrictions for the behavior with different values of z. Therefore it accommodates the widely *documented pattern that for most of the DMs, $\pi_{ind,0} \in \Pi$ and $\pi_{ind,2400} \notin \Pi$. Instead, the correlated version of the Allais paradox corresponds to the joint distribution*

$\pi_{cor,z}$	2400	z
2500	0.33	0
0	0.01	0
z	0	0.66

Here, Strong Independence formulated as above has bite: it requires that $\pi_{cor,0} \in \Pi$ if and only if $\pi_{cor,2400} \in \Pi$. This is consistent with the empirical evidence in BGS, which shows how almost all the subjects do not change behavior when z changes in the correlated version of the problem. ▲

The example above highlights how preference sets allow us to disentangle two components of Strong Independence for binary relations: the sure-thing principle and probabilistic sophistication. The sure-thing principle is the part that is maintained by Strong Independence for preference sets, as realizations where the two alternatives pay the same are irrelevant for the evaluation. Instead, probabilistic sophistication requires that the marginal distributions are sufficient for the comparison, and therefore identical realizations can be canceled out even if they do not realize jointly. This probabilistic sophistication is not imposed by Strong Independence for preference sets.

Finally, we impose a weak continuity axiom guaranteeing the nonexistence of a joint distribution such that one marginal is "infinitely preferred" to the other.

Axiom 3 (Archimedean Continuity) *For all $\pi \in \hat{\Pi}$, $\pi' \notin \Pi$, there exist $\alpha, \beta \in (0,1)$ such that*

$$\alpha\pi + (1-\alpha)\,\pi' \in \hat{\Pi} \text{ and } \beta\pi + (1-\beta)\,\pi' \notin \Pi.$$

The following theorem provides a representation of the preference sets satisfying these three axioms.

Theorem 1 *A preference set* Π *satisfies Completeness, Strong Independence, and Archimedean Continuity if and only if* Π *admits a correlation-sensitive representation. Moreover, the representing* ϕ *is unique up to a positive linear transformation.*

The theorem's proof combines the standard techniques used to prove the vN-M theorem with those used to deal with preference sets (see Fishburn 1990a) and intransitive preferences over acts (see Fishburn 1989). The theorem's importance stems from the fact that it connects a subset of the EU axioms to a general representation sensitive to the alternatives' correlation. Moreover, the value $\phi(x, y)$ has a cardinal interpretation as the contribution of the joint outcome (x, y) in favor of the row distribution. This cardinal role is the reason why the representing ϕ is unique up to a positive linear transformation.[10]

The representation still meaningfully restricts the pattern of behavior of the DM. To begin, if the joint distribution π is such that the row distribution dominates *realization by realization* the column distribution, then the joint distribution must be in the preference set, that is, if for all $(x, y) \in \operatorname{supp} \pi$, $\delta_{(x,y)} \in \Pi$, then $\pi \in \Pi$.[11] Moreover, Section 3.1 shows that the conclusion can be strengthened from realization by realization dominance to first-order stochastic dominance if the two lotteries under consideration are independent.

As the names of the previous axioms suggest, when the Transitivity axiom is added, the correlation-sensitive representation reduces to EU. Proposition 1 shows that this interpretation is correct. To do so, we need to translate Transitivity into the language of preference sets.

Axiom 4 (Transitivity) *For all* $\pi, \chi, \rho \in \Delta(X \times X)$*, if* $\pi_2 = \chi_1$*,* $\rho_1 = \pi_1$*, and* $\rho_2 = \chi_2$*, then*

$$(\pi \in \Pi, \chi \in \Pi) \Rightarrow \rho \in \Pi.$$

[10] It may be interesting to explore a decision criterion that treats the two distributions asymmetrically because the row one is the status quo. We illustrate this possibility in Section 4.5 where we compare the correlation sensitive representation to the reference-dependent model of Koszegi and Rabin (2007). Note that if the preference set admits a correlation-sensitive representation, by Theorem 1 of Fishburn (1982) the function ϕ is determined by the independent joint distributions.

[11] We denote as $\delta_{(x,y)}$ the joint lottery such that with probability one, the row outcome is x, and the column outcome is y.

The axiom has the following interpretation: Since $\pi \in \Pi$, $\pi_1 = \rho_1$ is preferred to $\pi_2 = \chi_1$ (given the correlation structure described by π). Since $\chi \in \Pi$, $\chi_1 = \pi_2$ is preferred to $\chi_2 = \rho_2$ (given the correlation structure described by χ). For Transitivity to hold, we then need that ρ_1 is preferred to ρ_2, i.e., $\rho \in \Pi$.

Example 2 *The following three joint distributions illustrate a typical failure of Transitivity due to salience sensitivity. By changing the correlation structure between alternatives, the realization with the most striking difference between outcomes changes, reversing the comparison between a fixed marginal and two similar alternatives. Let*

π	7	2
10	0	$\frac{1}{4}$
5	$\frac{1}{2}$	0
0	0	$\frac{1}{4}$

χ	8	1
7	1/2	0
2	0	1/2

ρ	8	1
10	1/4	0
5	0	1/2
0	1/4	0

we will see that for a salience sensitive DM, it is reasonable to have $\pi \in \Pi$, $\chi \in \Pi$, *and* $\rho \notin \Pi$. *Indeed, in* π *the large difference in the realization* $(10, 2)$ *tilts the evaluation in favor of the row marginal, and in* ρ *the large difference in the realization* $(0, 8)$ *tilts the evaluation in favor of the column marginal. Moreover, the property of Diminishing Sensitivity implies that* $\chi \in \Pi$.

The following result proves that when Transitivity is added to the previous axioms, the decision criterion reduces to expected utility maximization, confirming a similar conclusion obtained by Bikhchandani and Segal (2011) in a slightly different setting.

Proposition 1 *If* Π *admits a correlation-sensitive representation then the following are equivalent:*

1. Π *satisfies Transitivity;*

2. $\succsim^{\Pi}$ *admits an expected utility representation.*

The intuition behind the additional strengthening imposed by Transitivity on the correlation-sensitive representation is the following. Since the preference set is complete, the representing ϕ must be skew symmetric, and imposing Archimedean Continuity and Strong Independence ensures that probabilities are correctly taken into account. However, only when Transitivity is added the alternatives are valued independently.

3.1 Monotonicity and Continuity

Since salience theory is defined for lotteries with monetary outcomes, from now on, we will focus on the case where $X = \mathbb{R}$ endowed with the usual topology. In this setting, we discuss using preference sets to axiomatically describe standard regularity conditions for the representing function, such as monotonicity and continuity.

Axiom 5 (Monotonicity) *For all $x, y, z \in X$ and $\pi \in \Delta(X \times X)$, if $x > y$ and $\alpha \in (0,1)$, then*

$$\alpha \delta_{(y,z)} + (1-\alpha)\pi \in \Pi \Rightarrow \alpha \delta_{(x,z)} + (1-\alpha)\pi \in \hat{\Pi}.$$

Since we do not, in general, impose Transitivity, our monotonicity axiom slightly departs from the usual one: it requires that whenever x is strictly larger than y, x is more favorably compared than y to every alternative z. Given a correlation-sensitive representation, Monotonicity is easily characterized in terms of ϕ.

Remark 1 *If Π admits a correlation-sensitive representation, Π satisfies Monotonicity if and only if ϕ is strictly increasing in the first argument and strictly decreasing in the second argument.*

Before proceeding with salience theory, a few observations about the connection between first-order stochastic dominance (FOSD) and Monotonicity in the general correlation sensitive representation are in order. It is worth noting that Monotonicity is not enough to guarantee that the preference set Π satisfies first-order stochastic dominance, where the latter is defined as the requirement that

$$\pi_1 \geq_{FOSD} \pi_2 \Rightarrow \pi \in \Pi \tag{5}$$

with $\pi \in \hat{\Pi}$ if $\pi_1 \neq \pi_2$. However, the decision criterion axiomatized in Theorem 1 has a few stochastic monotonicity implications. Indeed, the preference set Π satisfies (5) when π is an *independent* joint distribution, i.e., $\pi = \pi_1 \times \pi_2$.[12]

Finally, this setup also allows for a simple characterization of the continuity properties of ϕ.

Axiom 6 (Continuity in Outcomes) *Let* $(x_n)_{n\in\mathbb{N}} \rightarrow x$. *Then, for every* $\alpha \in [0,1]$, $y \in X$, $\pi \in \Delta(X \times X)$

$$\alpha\delta_{(x_n,y)} + (1-\alpha)\pi \in \Pi \quad \forall n \in \mathbb{N} \implies \alpha\delta_{(x,y)} + (1-\alpha)\pi \in \Pi$$

and

$$\alpha\delta_{(y,x_n)} + (1-\alpha)\pi \in \Pi \quad \forall n \in \mathbb{N} \implies \alpha\delta_{(y,x)} + (1-\alpha)\pi \in \Pi.$$

Given Completeness, Strong Independence, and Archimedean Continuity, Continuity in Outcomes is one to one with a continuous ϕ.

Remark 2 *If* Π *admits a correlation-sensitive representation,* Π *satisfies Continuity in Outcomes if and only if* ϕ *is continuous in both arguments.*

4 Salience Characterization

This section describes salience theory as introduced by BGS and shows why it is a particular case of our correlation sensitive representation in which $\phi(x,y) = (x-y)\sigma(x,y)$ and σ is a function that captures the salience of the joint realization (x,y), and satisfies some psychologically motivated conditions. We then propose

[12] Indeed, Remark 1 guarantees that a preference set that satisfies Completeness, Strong Independence, Archimedean Continuity, and Monotonicity, admits a representing ϕ that satisfies the OPT and I properties of Loomes and Sugden (1987).

an equivalent but testable formulation of salience theory's critical properties of Ordering, Diminishing Sensitivity, and Weak Reflexivity. Finally, we characterize the salience model entirely as the result of these Ordering, Diminishing Sensitivity and Weak Reflexivity axioms combined with continuity and monotonicity requirements.

As formulated in BGS, salience theory explains the behavior of a DM that is facing a joint lottery $\pi \in \Delta(X \times X)$. Salience's main departure from EU theory is that expectations are calculated with a distorted probability measure that overweights salient pairs of outcomes. To formalize this idea, BGS introduced the concept of *salience function*.

Definition 5 *A function $\sigma : \mathbb{R}^2 \to \mathbb{R}$ satisfies:*

1. Symmetry *if $\sigma(x, y) = \sigma(y, x)$;*
2. BGS-Ordering *if $x' < y'$, $x < y$ and $[x', y'] \subset [x, y]$ imply $\sigma(x', y') < \sigma(x, y)$;*
3. BGS-Diminishing Sensitivity *if $x, y, k \in \mathbb{R}_{++}$ and $x > y$ imply $\sigma(x + k, y + k) < \sigma(x, y)$;*
4. BGS-Weak Reflexivity *if for all $x, y, x', y' \in \mathbb{R}_+$ with $|x - y| = |x' - y'|$,*

$$\sigma(x, y) \geq \sigma(x', y') \Longleftrightarrow \sigma(-x, -y) \geq \sigma(-x', -y').$$

A salience function *is a function $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ satisfying Symmetry, BGS-Ordering, BGS-Diminishing Sensitivity, BGS-Weak Reflexivity and such that $\sigma(x, x) = 0$ for all $x \in \mathbb{R}$.*

We will interpret the properties momentarily when we introduce their testable counterparts. A fundamental feature is that a joint realization's salience depends only on its value, not its probability, a key difference with prospect theory. Indeed, relative to the original vN-M set of axioms, prospect theory relaxes even the weaker version of Strong Independence for joint distributions introduced by this paper, while salience theory relaxes Transitivity.

Definition 6 *A preference set* Π *admits a* σ*-distorted representation if there exists a continuous function* $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ *that satisfies symmetry such that*

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} (x-y)\,\sigma(x,y)\,\pi(x,y) \geq 0. \tag{6}$$

It admits a (smooth) salience representation if σ *is also a salience function.*

It is easy to see that a σ-distorted representation is a particular case of our correlation-sensitive model. The latter is much more general, and allows for behaviors that are at odds with salience theory's key idea that states where the alternatives differ more are overweighted. Therefore, we next characterize BGS-Ordering, BGS-Diminishing Sensitivity, and BGS-Weak Reflexivity in terms of testable axioms.

Notice that BGS mainly used the rank-based version of their model, but they recognized that its discontinuity causes some problems, and they suggest using the smooth version of Definition 6.[13] In what follows, we stick with the smooth version, which has been the most used in empirical studies of the salience model.[14] Online Appendix D analyzes the weaknesses of the rank-based version.

4.1 The Ordering Axiom

The idea behind the BGS-Ordering property is straightforward. Fix the outcomes $x > y$. Then, we can take some $\alpha, \beta \in (0,1)$, $\beta > \alpha$ and consider the two outcomes obtained by mixing x and y

$$x > \beta x + (1-\beta)\,y > \alpha x + (1-\alpha)\,y > y.$$

If we consider the two realizations (x, y) and $(\alpha x + (1-\alpha)\,y, \beta x + (1-\beta)\,y)$ the first pair of outcomes has more widespread values, and therefore BGS-Ordering implies

[13]In their words: *"A smooth specification would also address a concern with the current model that states with similar salience may obtain very different weights. This implies that (1) splitting states and slightly altering payoffs could have a large impact on choice, and (2) in choice problems with many states the (slightly) less salient states are effectively ignored."*

[14]See Dertwinkel-Kalt and Koster (2020), Dertwinkel-Kalt, Frey, and Koster (2021), Nielsen, Sebald, and Sorensen (2021) and the references therein.

that its contribution in favor of the row outcome will be relatively overweighted. However, distortions of probabilities are not observable, and therefore, we cannot directly test BGS-Ordering. Nevertheless, we can propose a testable version of the property.

Now, if we look at the joint distribution

$$\left((x,y)\,, \frac{\beta-\alpha}{1+\beta-\alpha}; (\alpha x+(1-\alpha)\,y, \beta x+(1-\beta)\,y)\,, \frac{1}{1+\beta-\alpha}\right)$$

the row and column marginals have the same expected value, and they should be indifferent to an expected value maximizer. However, a salience-sensitive DM's attention is disproportionately drawn to the outcome with the most significant difference between payoff (in the inclusion sense). Since this outcome is (x,y), and favors the row component, a salience-sensitive DM prefers (at least weakly) to be paid according to the row component. This reasoning is formalized in the Ordering axiom.

Axiom 7 (Ordering) *For every $x,y\in\mathbb{R}$, $\alpha,\beta\in[0,1]$ if $x>y$, $\beta>\alpha$, and at least one between β and α is in $(0,1)$, we have that*

$$\left((x,y)\,, \frac{\beta-\alpha}{1+\beta-\alpha}; (\alpha x+(1-\alpha)\,y, \beta x+(1-\beta)\,y)\,, \frac{1}{1+\beta-\alpha}\right)\in\hat{\Pi}.$$

The following proposition shows that the axiom corresponds to the original property of BGS.

Proposition 2 *Let Π admit a σ-distorted representation. Then Π satisfies Ordering if and only if σ satisfies BGS-Ordering. In that case, the representing ϕ satisfies*

$$\phi\,(x,y) > \frac{\phi\,(\beta x+(1-\beta)\,y, \alpha x+(1-\alpha)\,y)}{(\beta-\alpha)} \tag{7}$$

for all $x,y\in\mathbb{R}$ and $\alpha,\beta\in[0,1]$ with $x>y$, $\beta>\alpha$ and at least one between β and α in $(0,1)$.

Equation (7) confirms the intuition behind Ordering: under this axiom, the positive contribution of the realization $\phi\,(x,y)$ decreases sufficiently fast as the two components are mixed, because of the combined effect of a smaller difference and a decreased salience.

4.2 The Diminishing Sensitivity Axiom

The BGS-Diminishing Sensitivity property requires that when two pairs of outcomes have the same absolute difference, the one with the highest relative difference is overweighted. The interpretation is easier for two-outcome lotteries. Suppose that the DM is envisioning the joint probability distribution π that assigns probability $\frac{1}{2}$ both to (x, y) and $(y+k, x+k)$, with $x, y, k \in \mathbb{R}_+$ and $x > y$. The two pairs of outcomes have the same absolute difference, but (x, y) has a higher relative difference. Therefore, (x, y) is overweighted to $(y+k, x+k)$. Since (x, y) favors the row marginal, the DM chooses to be paid according to the row outcome. This reasoning is formalized in the Diminishing Sensitivity axiom.

Axiom 8 (Diminishing Sensitivity) *For every $x > y > 0$, and $k \in \mathbb{R}_+$*

$$\pi = \left((x, y), \frac{1}{2}; (y+k, x+k), \frac{1}{2} \right) \in \Pi.$$

If moreover $\pi \in \hat{\Pi}$ whenever $k \in \mathbb{R}_{++}$, $\hat{\Pi}$ satisfies strict Diminishing Sensitivity.

The following proposition shows that our testable definition of Diminishing Sensitivity corresponds to the original property of BGS.

Proposition 3 *If Π admits a σ-distorted representation, it satisfies strict Diminishing Sensitivity if and only if σ satisfies BGS-Diminishing Sensitivity.*

In particular, it turns out that Diminishing Sensitivity alone is *not* in contrast with the conventional notion of prospect theory. It is a generalization of the property of risk aversion over positive outcomes and risk loving over negative outcomes (cf. also Proposition 6) to decision criteria that are not necessarily transitive. Denote as $\mathbb{E}(p) = \sum_{x \in X} p(x)\, x$ the expected value of the marginal distribution $p \in \Delta(X)$.

Definition 7 Π *satisfies risk aversion (risk loving) for outcomes in* (a,b) *if* $\pi \in \Pi$ *(resp.* $\bar{\pi} \in \Pi$*) for all* $\pi \in \Delta(X)$ *with* $\operatorname{supp}\pi \subseteq (a,b)$ *and such that* π_2 *is a mean preserving spread of* π_1*.*

The previous definition is a translation of the usual risk aversion notion in the language of preference sets: a DM is risk averse over the outcome range (a,b) if she prefers the expected value of a lottery supported over (a,b) to the lottery itself.

Proposition 4 *Let* Π *admit an expected utility representation with a strictly increasing utility function. Then* Π *satisfies Diminishing Sensitivity if and only if* Π *satisfies risk aversion for positive outcomes.*

This result confirms that the BGS-Diminishing Sensitivity of the function σ allows for risk-aversion of the agents in the main specification of the BGS model (Equation (6)) without relying on the more general form[15]

$$\sum_{(x,y)\in X\times X} (u(x)-u(y))\,\sigma(x,y)\,\pi(x,y)\,.$$

Remark 3 *Under the correlation-sensitive representation, risk aversion for positive outcomes always implies Diminishing Sensitivity. However, the following example shows that risk aversion for positive outcomes is a strictly more demanding property. Let the salience function be equal to the leading example in BGS, that is*

$$\sigma(x,y) = \frac{x-y}{x+y+1}. \tag{8}$$

Then σ *satisfies BGS-Ordering and BGS-Diminishing Sensitivity, and by Proposition 4,* Π *satisfies Diminishing Sensitivity. The joint distribution* π *given in the following table is such that the row marginal is a mean preserving spread of the column*

[15] Moreover, a representation where σ satisfies Diminishing Sensitivity and u is concave and differentiable can always be reformulated as $\sum_{(x,y)\in X\times X}(x-y)\,\hat{\sigma}(x,y)\,\pi(x,y) \geq 0$, where $\hat{\sigma}(x,y) = \begin{cases} \frac{\sigma(x,y)[u(x)-u(y)]}{x-y} & x \neq y \\ 0 & x = y \end{cases}$ is a continuous function that satisfies BGS-Diminishing Sensitivity.

marginal:

π	0	1	2
0	0	1/4	0
1	0	1/2	0
2	1/8	0	1/8

Therefore risk aversion in the positive domain would prescribe that $\pi \notin \Pi$*. However,* $\pi \in \Pi$ *for a DM with salience function given by (8) because of the high salience of the realization* $(2,0)$*. Therefore, the preference set of such a DM satisfies Diminishing Sensitivity but not risk aversion for positive outcomes.*

4.3 The Weak Reflexivity Axiom

The last property introduced by BGS is Weak Reflexivity, which captures the symmetry around 0 of the distortions. Again, we provide a testable counterpart of their axiom.

Axiom 9 (Weak Reflexivity) *For every* $x, y, w, z \in \mathbb{R}_+$*, with* $x - y = z - w$

$$\left((x,y), \frac{1}{2}; (w,z), \frac{1}{2}\right) \in \hat{\Pi} \Leftrightarrow \left((-y,-x), \frac{1}{2}; (-z,-w), \frac{1}{2}\right) \in \hat{\Pi}.$$

The axiom is easily seen to be one to one with the corresponding property of the distortion function σ.

Proposition 5 *If* Π *admits a* σ*-distorted representation,* Π *satisfies Weak Reflexivity if and only if* σ *satisfies BGS-Weak Reflexivity.*

So far, we have not attached any specific interpretation to the lotteries' realizations, except that they are expressed in monetary units. In particular, they can represent either the total wealth or gains and losses obtained after realizing some uncertainty. However, the Weak Reflexivity axiom, with the implied role for outcome 0, better suits the latter interpretation. We notice that Weak Reflexivity implies the preference reversal of risk attitudes featured by prospect theory.

Proposition 6 *Suppose that* Π *has an EU representation and satisfies Monotonicity and Weak Reflexivity. Then* Π *is risk-averse (resp. risk-loving) for lotteries with values in* $(a, b) \subseteq \mathbb{R}_+$ *if and only if* $\succsim$ *is risk loving (resp. risk-averse) for lotteries with values in* $(-b, -a)$.

This result sheds light on the observation made in BGS that salience theory can explain the experimental evidence in favor of the fourfold pattern (see, e.g., Bruhin, Fehr-Duda, and Epper 2010). Diminishing Sensitivity would only induce risk aversion in the gain domain. Its combination with Ordering creates the risk aversion for small gains vs. risk loving for large gains, and Weak Reflexivity gives the opposite patterns for losses.

4.4 Complete Characterization of Salience Theory

We now put the pieces together and provide a complete characterization of the salience model. To do so, we need a final continuity axiom.

Axiom 10 (Continuity at Identity) *Let* $x \in X$. *Then, for every* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \downarrow x$, *and for every* $k \in X$ *and* $\varepsilon \in \mathbb{R}_{++}$ *there exists an* $m \in \mathbb{N}$ *such that for all* $n \geq m$

$$((x, x_n), (1 - (x_n - x)); (k + \varepsilon, k), (x_n - x)) \in \Pi.$$

Moreover, for every $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \uparrow x$, *and for every* $k \in X$ *and* $\varepsilon \in \mathbb{R}_{++}$ *there exists an* $m \in \mathbb{N}$ *such that for all* $n \geq m$

$$((x_n, x), (1 - (x - x_n)); (k + \varepsilon, k), (x - x_n)) \in \Pi.$$

The axiom requires that joint realizations with two components that are arbitrarily closed can be almost neglected. More precisely, the weight to these realizations declines more than linearly in their differences when these become sufficiently small, capturing a form of indistinguishability. With this, we have a complete characterization of the salience model.[16]

[16] As shown by the proof of Theorem 2, adding Monotonicity, Continuity in Outcomes and Continuity at Identity to a correlation sensitive representation implies that $\phi(x, y) = (x - y)\sigma(x, y)$ for

Theorem 2 *A preference set* Π *admits a salience representation if and only if* Π *satisfies Completeness, Strong Independence, Archimedean Continuity, Monotonicity, Continuity in Outcomes, Continuity at Identity, Ordering, Diminishing Sensitivity, and Weak Reflexivity.*

4.5 Comparison with Other Models

Relation with Regret Theory Theorem 2 allows us to compare salience theory with regret theory readily. Indeed, recall that the most general version of regret theory, proposed by Loomes and Sugden (1987), requires that the preference set Π of the DM admits a correlation-sensitive representation, it satisfies Monotonicity, and the representing ϕ satisfies Regret Aversion:

$$\phi(x, y) > \phi(x, z) + \phi(z, y) \text{ for all } x > z > y.$$

Corollary 1 *If a preference set* Π *satisfies Completeness, Strong Independence, Archimedean Continuity, and Ordering, the representing* ϕ *satisfies Regret Aversion.*

The two models remain inherently different despite Ordering being a stronger property than Regret Aversion in binary decision problems. First, they have different psychological foundations that imply different behaviors when the DM is given additional information. The behavior of a salience-sensitive DM is the same when only the realization of the chosen marginal is shown and when the counterfactual is announced. Instead, regret theory prescribes an EU consistent behavior in the first scenario but is highly sensitive to correlation in the second.

Second, by making additional assumptions such as Ordering and Diminishing Sensitivity, salience theory delivers a novel set of predictions. This is particularly evident for problems with more than two alternatives, where the salience model predicts the decoy effect, background contrast effects, and other context effects, a phenomenon that we illustrate in the extension of Online Appendix C. As the

some continuous and symmetric $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ such that $\sigma(x, x) = 0$ for all $x \in \mathbb{R}$. Then, by adding Ordering, Diminishing Sensitivity, and Weak Reflexivity, σ is forced to be a salience function.

empirical literature has highlighted the widespread presence of these effects, salience theory seems a better correlation-sensitive model in terms of the performance across different decision environments.[17]

Relation with Reference-Dependent Preferences Another model under for which the correlation between alternatives play a key role is the one of Koszegi and Rabin (2007, 2009). Here, we show how to translate the model in the language of preference set, highlight that it does not fall in the general class of correlation-sensitive preferences we have proposed, and shows that this comes from violations of the Strong Independence axiom.

Koszegi and Rabin (2007, 2009) model endogenous reference-dependent preferences as a (personal) choice-unacclimating equilibrium. More precisely, under their criterion, a lottery p can be chosen if, when alternatives are evaluated taking p as the reference point, p has the highest evaluation for the DM. The reference dependence is endogenous because the reference point is the candidate choice. It is called choice-unacclimating, as when looking from deviations from a candidate, the agent still evaluates them with the candidate as the reference point (as opposed to using the deviation itself as the reference point). When generalized to allow for correlated alternatives (similarly to Sugden 2003) and rephrased in the language of preference sets, their decision criterion says that:

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y):x \geq y} \lambda (x-y) \pi (x,y) + \sum_{(x,y):x<y} (x-y) \pi (x,y) \geq 0$$

for some $\lambda > 1$ that measures how much worse losses are than gains.[18] The interpretation is that the agent prefers the row marginal (i.e., $\pi \in \Pi$) when taking the row marginal as the status quo. Here, the fact that the row marginal is the status quo is

[17] Of course, salience theory also makes some predictions about non-choice behavior that separate it from preexisting models, such as the attention dedicated to each dimension of the alternatives. Thus, the use of additional instruments such as eye-tracking to further investigate is critical. Moreover, Herweg and Muller (2021) argue that the more restrictive form of regret theory presented in Loomes and Sugden (1982) is itself a special case of salience theory.

[18] We focus on their main specification and thank a referee for suggesting this link.

captured by the fact that the gains induced by the column marginal are evaluated with weight 1, while the losses are evaluated with weight λ.

Interestingly, by letting $\hat{\phi}(x,y) = \lambda(x-y)$, for $x \geq y$ and $\hat{\phi}(x,y) = x - y$ for $x < y$, this criterion admits a representation

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \hat{\phi}(x,y)\,\pi(x,y) \geq 0.$$

However, the loss aversion coefficient makes this $\hat{\phi}$ not skew symmetric. Moreover, by Theorem 1 the reference-dependent model does not admit an alternative correlation-sensitive representation. Indeed, a simple example shows that this criterion violates Strong Independence.[19]

Example 3 *Let* $\pi = \frac{\delta_{(10,1)}}{2} + \frac{\delta_{(1,10)}}{2}$ *and* $\pi' = \delta_{(1.1,1)}$. *Then both the row and column marginals can be chosen under* π, *and only the row marginal can be chosen under* π', *(i.e.,* $\pi \in \Pi\backslash\hat{\Pi}, \pi' \in \hat{\Pi}$*), and Strong Independence would prescribe that* $\alpha\pi + (1-\alpha)\pi' \in \hat{\Pi}$ *for all* $\alpha \in (0,1)$. *However, for* λ *and* α *sufficiently high* $\alpha\pi + (1-\alpha)\pi' \in \Pi\backslash\hat{\Pi}$. *The intuition is simple: a highly loss-averse agent that takes the column marginal as the reference point can stick to it because of the high loss associated with the realization* $(1,10)$.

Therefore, beyond establishing the formal distinction between this model and the class of correlation-sensitive preferences that contain salience and regret, the preference set approach hints at violations of the "strict" part of the Strong Independence axiom as the essential relaxation to allow for status quo biases. Loosely speaking, the reference-dependent model have "too much Completeness" due to loss aversion. Recall that the row marginal is the reference point, so there will be several instances in which both marginals can be chosen if they were the original reference point. This thickness of the indifference curves can lead to violations of Strong Independence, as even if one of the original joint distributions is in the strict preference set, the resulting convex combination may fall in the thick indifference curve part, i.e., it may

[19] A complete axiomatization of reference dependence using preference sets is beyond the scope of this paper; see Masatlioglu and Raymond (2016) for the axiomatization using binary relations.

be in the preference set but not in the strict preference set. Instead, it is easy to see that Completeness, Archimedean Continuity, and the "weak" part of the Strong Independence are still satisfied.

4.6 Identification of the Salience Function

Another advantage of our use of preference sets is that in light of Theorem 1, we can directly test salience theory by first constructing a candidate salience function σ and then checking whether it satisfies the properties imposed by BGS. As a preliminary observation, it is immediate from (6) that if the preferences set Π admits a salience representation with salience function σ, they also admit a salience representation with salience function $k\sigma$ whenever $k \in \mathbb{R}_{++}$. Therefore, to eliminate this degree of freedom, we set $\sigma(1,0) = 1$.

Now, for every $(x, y) \in \mathbb{R}^2$ with $y > x$, if the preference set Π admits a smooth salience representation, by Theorem 2 Π satisfies Completeness, Archimedean Continuity, Strong Independence and Monotonicity, and therefore by Theorem 1 there exists a unique $\alpha_{x,y} \in (0,1)$ such that

$$\alpha_{x,y}\delta_{(x,y)} + (1 - \alpha_{x,y})\,\delta_{(1,0)} \in \Pi \backslash \hat{\Pi}.$$

Therefore, we can define

$$\sigma(x, y) = \frac{(1 - \alpha_{x,y})}{\alpha_{x,y}(y - x)}.$$

It is immediate to check that this is the only possible value for σ. We can use this procedure and the fact that by symmetry $\sigma(x, y) = \sigma(y, x)$ for those $(x, y) \in \mathbb{R}^2$ with $x > y$ to construct the candidate salience function. At this point, checking salience theory boils down to verifying that σ satisfies BGS-Ordering, BGS-Diminishing Sensitivity, and BGS-Weak Reflexivity.

5 Conclusion

This work provides a simple axiomatic characterization of preferences over risky choices when the agent cares about the correlation between the alternatives considered. We proved that when the joint distribution is included in the decision environment, we can pinpoint Transitivity as the EU's relaxation needed for correlation sensitivity. This setting, moreover, allows a cleaner axiomatic comparison of theories such as regret and salience with EU.

As the second payoff of our approach, we obtain a simple axiomatization of the salience model of Bordalo, Gennaioli, and Shleifer (2012) within the realm of these correlation-sensitive preferences. This provides a one-to-one map from the BGS assumptions of Ordering, Diminishing Sensitivity, and Weak Reflexivity to testable counterparts. Our characterization reveals that Ordering is the property that cannot be reconciled with prospect theory, whereas Diminishing Sensitivity paired with Weak Reflexivity corresponds to the usual risk-averse in gains, risk-loving in losses. Moreover, the axiomatization allows for direct comparisons of the different EU axioms relaxed by salience theory, prospect theory, regret theory, and the reference-dependent preferences of Koszegi and Rabin (2007).

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A Main Proofs

Let $\oplus = \left\{(x,y) \in X \times X : \delta_{(x,y)} \in \Pi\right\}$ and $\hat{\oplus} = \left\{(x,y) \in X \times X : \delta_{(x,y)} \in \hat{\Pi}\right\}$.

Proof of Theorem 1

(Necessity of the axioms) Completeness is necessary since the skew symmetry of ϕ guarantees that $\sum_{(x,y)\in X\times X} \pi(x,y)\,\phi(x,y) = -\sum_{(x,y)\in X\times X} \bar{\pi}(x,y)\,\phi(x,y)$. For Strong Independence, let $\pi, \chi \in \Pi$ (resp. $\pi \in \Pi$ and $\chi \in \hat{\Pi}$) and $\lambda \in (0,1)$. Then

$$\begin{aligned} &\sum_{(x,y)\in X\times X} (\lambda\pi + (1-\lambda)\,\chi)(x,y)\,\phi(x,y) \\ = \;& \lambda \sum_{(x,y)\in X\times X} \pi(x,y)\,\phi(x,y) + (1-\lambda) \sum_{(x,y)\in X\times X} \chi(x,y)\,\phi(x,y) \geq \text{(resp. } >)0. \end{aligned}$$

For Archimedean Continuity, let $\pi \in \hat{\Pi}, \chi \notin \Pi$. If we define $K :\, = \sum_{(x,y)\in X\times X} \pi(x,y)\,\phi(x,y) > 0 > \sum_{(x,y)\in X\times X} \chi(x,y)\,\phi(x,y) =: k$, then any $\alpha > \frac{-k}{K-k}$ and $\beta < \frac{-k}{K-k}$ are easily seen to satisfy the requirements.

(Sufficiency of the axioms) We start by establishing some initial claims.

Claim 1 *If* $\operatorname{supp}\pi \subseteq \oplus$, *then* $\pi \in \Pi$.

Proof The claim is proved by induction on the size of $\operatorname{supp}\pi$. The claim is clearly true when $|\operatorname{supp}\pi| = 1$. Suppose the result holds for all the lotteries with support of size $n \in \mathbb{N}$. Let π be such that $|\operatorname{supp}\pi| = n+1$. Choose arbitrarily $(x', y') \in \operatorname{supp}\pi$. Then, we can define $\chi \in \Delta(X \times X)$ as

$$\chi(x,y) = \begin{cases} 0 & \text{if } (x,y) = (x', y') \\ \frac{\pi(x,y)}{1-\pi(x',y')} & \text{otherwise.} \end{cases}$$

Since $|\operatorname{supp}\chi| = n$ and $\operatorname{supp}\chi \subseteq \oplus$, we have $\chi \in \Pi$. Moreover, $\pi = \pi(x', y')\,\delta_{(x',y')} + (1 - \pi(x', y'))\,\chi$ and by Strong Independence, we have $\pi \in \Pi$. □

Claim 2 *Let* $\pi \in \hat{\Pi}$, $\chi \notin \Pi$, *there exists a unique* $\lambda \in (0,1)$ *such that* $\lambda\pi + (1-\lambda)\,\chi \in \Pi \backslash \hat{\Pi}$.

Proof We let $A = \left\{\lambda \in [0,1] : \lambda\pi + (1-\lambda)\chi \in \hat{\Pi}\right\}$, $B = \{\lambda \in [0,1] : \lambda\pi + (1-\lambda)\chi \notin \Pi\}$. By Archimedean Continuity, both A and B have a nonempty intersection with $(0,1)$. Suppose that $\lambda \in A$ and $\mu \in (\lambda, 1]$. Then $\mu\pi + (1-\mu)\chi = \frac{\mu-\lambda}{1-\lambda}\pi + \frac{1-\mu}{1-\lambda}(\lambda\pi + (1-\lambda)\chi)$ and Strong Independence implies that $\mu\pi + (1-\mu)\chi \in \hat{\Pi}$. This, in turn, implies that $\mu \in A$.

Suppose instead that $\lambda \in B$ and $\mu \in [0,\lambda)$. Then by Completeness $\lambda\bar{\pi} + (1-\lambda)\bar{\chi} = \overline{\lambda\pi + (1-\lambda)\chi} \in \hat{\Pi}$ and $\mu\bar{\pi} + (1-\mu)\bar{\chi} = \frac{\lambda-\mu}{\lambda}\bar{\chi} + \frac{\mu}{\lambda}(\lambda\bar{\pi} + (1-\lambda)\bar{\chi})$. Therefore, $\mu\bar{\pi} + (1-\mu)\bar{\chi} \in \hat{\Pi}$ by Strong Independence, and $\mu\pi + (1-\mu)\chi \notin \Pi$. This, in turn, implies that $\mu \in B$.

Summing up, A and B are two intervals in $[0,1]$ with empty intersection. Suppose by contradiction that $A \cup B = [0,1]$. Then, we either have $A = [\lambda^*, 1]$ and $B = [0, \lambda^*)$ or $A = (\lambda^*, 1]$ and $B = [0, \lambda^*]$. In the first case, $\lambda^*\pi + (1-\lambda^*)\chi \in \hat{\Pi}$, $\chi \notin \Pi$, and Archimedean Continuity imply the existence of a $\mu \in [0, \lambda^*)$ such that $\mu\pi + (1-\mu)\chi \in \hat{\Pi}$, a contradiction. Similarly, we can rule out the other case. Therefore, there exists $\lambda^* \in [0,1] \setminus (A \cup B)$, that is, $\lambda^*\pi + (1-\lambda^*)\chi \in \Pi \backslash \hat{\Pi}$.

It only remains to prove uniqueness. Suppose that both λ^* and μ^* have the desired property, and let $\mu^* > \lambda^*$. Then, $\mu^*\pi + (1-\mu^*)\chi = \frac{\mu^*-\lambda^*}{1-\lambda^*}\pi + \frac{1-\mu^*}{1-\lambda^*}(\lambda^*\pi + (1-\lambda^*)\chi)$ and by Strong Independence, $\mu^*\pi + (1-\mu^*)\chi \in \hat{\Pi}$, a contradiction. □

Claim 3 *Let $x, y, z, w, t, v \in X$, $\lambda, \mu, \alpha \in (0,1)$ and $\delta_{(x,y)}, \delta_{(z,w)}, \delta_{(t,v)} \in \hat{\Pi}$ with*

$$\begin{aligned} \lambda\delta_{(x,y)} + (1-\lambda)\delta_{(w,z)} &\in \Pi\backslash\hat{\Pi}, \\ \mu\delta_{(z,w)} + (1-\mu)\delta_{(v,t)} &\in \Pi\backslash\hat{\Pi}, \\ \alpha\delta_{(t,v)} + (1-\alpha)\delta_{(y,x)} &\in \Pi\backslash\hat{\Pi}. \end{aligned}$$

Then

$$\frac{\lambda}{1-\lambda} \cdot \frac{\mu}{1-\mu} \cdot \frac{\alpha}{1-\alpha} = 1.$$

Proof Let

$$\gamma = \frac{\mu}{\mu + 1 - \lambda}$$

and

$$\pi = \gamma\left(\lambda\delta_{(x,y)} + (1-\lambda)\,\delta_{(w,z)}\right) + (1-\gamma)\left(\mu\delta_{(z,w)} + (1-\mu)\,\delta_{(v,t)}\right).$$

By Strong Independence, $\pi \in \Pi\backslash\hat{\Pi}$. Since $\gamma\,(1-\lambda) = (1-\gamma)\,\mu$, by Completeness we have that $\frac{\delta_{(w,z)}+\delta_{(z,w)}}{2} = \frac{\gamma(1-\lambda)\delta_{(w,z)}+(1-\gamma)\mu\delta_{(z,w)}}{\gamma(1-\lambda)+(1-\gamma)\mu} \in \Pi\backslash\hat{\Pi}$. Suppose by way of contradiction that

$$\frac{\gamma\lambda\delta_{(x,y)} + (1-\gamma)\,(1-\mu)\,\delta_{(v,t)}}{\gamma\lambda + (1-\gamma)\,(1-\mu)} \notin \Pi.$$

Then Completeness implies that

$$\frac{\gamma\lambda\delta_{(y,x)} + (1-\gamma)\,(1-\mu)\,\delta_{(t,v)}}{\gamma\lambda + (1-\gamma)\,(1-\mu)} \in \hat{\Pi}$$

and by Strong Independence, $\bar{\pi} \in \hat{\Pi}$. But this leads to the contradiction $\pi \notin \Pi$. Similarly, suppose by contradiction that

$$\frac{\gamma\lambda\delta_{(x,y)} + (1-\gamma)\,(1-\mu)\,\delta_{(v,t)}}{\gamma\lambda + (1-\gamma)\,(1-\mu)} \in \hat{\Pi}.$$

Then, Strong Independence implies that $\pi \in \hat{\Pi}$, another contradiction. Therefore, we have that

$$\frac{\gamma\lambda\delta_{(x,y)} + (1-\gamma)\,(1-\mu)\,\delta_{(v,t)}}{\gamma\lambda + (1-\gamma)\,(1-\mu)} \in \Pi\backslash\hat{\Pi}$$

and by definition of $\hat{\Pi}$

$$\frac{\gamma\lambda\delta_{(y,x)} + (1-\gamma)\,(1-\mu)\,\delta_{(t,v)}}{\gamma\lambda + (1-\gamma)\,(1-\mu)} \in \Pi\backslash\hat{\Pi}.$$

Thus Claim 2 gives $1-\alpha = \frac{\gamma\lambda}{\gamma\lambda+(1-\gamma)(1-\mu)}$ that implies

$$\alpha\mu\lambda = (1-\lambda)\,(1-\mu)\,(1-\alpha)$$

proving the statement. □

Claim 4 *If* $\operatorname{supp}\pi \subseteq \oplus$, *and* $\operatorname{supp}\pi \cap \hat{\oplus} \neq \emptyset$ *then* $\pi \in \hat{\Pi}$.

Proof If $\pi = \delta_{(x,y)}$ for some (x, y), the result holds by definition of $\hat{\oplus}$. Therefore, suppose that π is supported at least on two joint outcome realizations, let $(x', y') \in \operatorname{supp} \pi \cap \hat{\oplus}$, and define

$$\chi(x, y) = \begin{cases} 0 & (x, y) = (x', y') \\ \frac{\pi(x,y)}{1-\pi(x',y')} & \text{otherwise.} \end{cases}$$

By Claim 1, $\chi \in \Pi$. Since

$$\pi = \pi(x', y') \, \delta_{(x',y')} + (1 - \pi(x', y')) \, \chi$$

Strong Independence implies that $\pi \in \hat{\Pi}$. □

Claim 5 *If $\eta, \chi \in \Pi \backslash \hat{\Pi}$, then for all $\rho \in \Delta(X \times X)$*

$$\lambda\rho + (1 - \lambda) \chi \in \Pi \iff \lambda\rho + (1 - \lambda) \eta \in \Pi.$$

Proof By Strong Independence both statements hold if $\rho \in \Pi$. If $\rho \notin \Pi$, then by Completeness $\bar{\rho} \in \hat{\Pi}$, and by assumption $\bar{\eta}, \bar{\chi} \in \Pi$. Therefore, by Strong Independence, both $\lambda\bar{\rho} + (1 - \lambda) \bar{\chi}$ and $\lambda\bar{\rho} + (1 - \lambda) \bar{\eta}$ are in $\hat{\Pi}$. But then, neither $\lambda\rho + (1 - \lambda) \chi \in \Pi$ nor $\lambda\rho + (1 - \lambda) \eta \in \Pi$. □

If for every $x, y \in X$, $\delta_{(x,y)} \in \Pi \backslash \hat{\Pi}$, by Claim 1 every $\pi \in \Delta(X \times X)$ is in $\Pi \backslash \hat{\Pi}$, and the statement of the theorem trivially holds by letting $\phi(x, y) = 0$ for all $x, y \in X$. Therefore, by Completeness, we can assume that there exists $(\hat{x}, \hat{y})$ with $\delta_{(\hat{x},\hat{y})} \in \hat{\Pi}$ and let $\phi(\hat{x}, \hat{y})$ be an arbitrary strictly positive real number. Moreover, let $\phi(x, y) = 0$ for all $\delta_{(x,y)} \in \Pi \backslash \hat{\Pi}$. If $(x, y) \notin \oplus$, by Claim 2, there exists a unique $\lambda \in (0, 1)$ with

$$\lambda\delta_{(\hat{x},\hat{y})} + (1 - \lambda) \, \delta_{(x,y)} \in \Pi \backslash \hat{\Pi}.$$

In this case, let

$$\phi(x, y) = -\phi(\hat{x}, \hat{y}) \frac{\lambda}{(1 - \lambda)}.$$

It only remains to define ϕ when $(x, y) \in \hat{\oplus}$. We set

$$\phi(x, y) = -\phi(y, x) \quad \forall (x, y) \in \hat{\oplus}.$$

We now claim that the previous procedure defines ϕ uniquely up to a positive linear transformation. Given the choice of a particular $(\hat{x}, \hat{y})$, the only degree of freedom is the choice of the (strictly positive) number $\phi(\hat{x}, \hat{y})$, and the values assumed by ϕ on the rest of the domain are linear in $\phi(\hat{x}, \hat{y})$. Suppose instead that we define $\bar{\phi}$ starting from a different $(\bar{x}, \bar{y}) \in \hat{\oplus}$. Since we prove uniqueness only up to a positive linear transformation, we can choose the (strictly positive) value of $\bar{\phi}(\bar{x}, \bar{y})$. In particular, set

$$\bar{\phi}(\bar{x}, \bar{y}) = \phi(\bar{x}, \bar{y}) = \phi(\hat{x}, \hat{y}) \frac{\mu}{(1-\mu)}$$

where

$$\mu \delta_{(\hat{x}, \hat{y})} + (1-\mu) \delta_{(\bar{y}, \bar{x})} \in \Pi \backslash \hat{\Pi}$$

and consider $(x, y) \notin \oplus$. Then, by Claim 2 there exist unique λ_0, λ_1, such that

$$\begin{aligned} \lambda_0 \delta_{(\hat{x}, \hat{y})} + (1-\lambda_0) \delta_{(x,y)} &\in \Pi \backslash \hat{\Pi}, \\ \lambda_1 \delta_{(\bar{x}, \bar{y})} + (1-\lambda_1) \delta_{(x,y)} &\in \Pi \backslash \hat{\Pi}. \end{aligned}$$

Given our definitions,

$$\begin{aligned} \phi(x, y) = \bar{\phi}(x, y) &\iff \phi(\hat{x}, \hat{y}) \tfrac{\lambda_0}{(1-\lambda_0)} = \phi(\bar{x}, \bar{y}) \tfrac{\lambda_1}{(1-\lambda_1)} \\ &\iff \phi(\hat{x}, \hat{y}) \tfrac{\lambda_0}{(1-\lambda_0)} = \phi(\hat{x}, \hat{y}) \tfrac{\mu}{(1-\mu)} \tfrac{\lambda_1}{(1-\lambda_1)} \\ &\iff \tfrac{\lambda_0}{(1-\lambda_0)} = \tfrac{\mu}{(1-\mu)} \tfrac{\lambda_1}{(1-\lambda_1)} \end{aligned}$$

and Claim 3 together with Completeness guarantee that the condition in the last line holds true. Finally, we want to show that

$$\pi \in \Pi \iff \sum_{(x,y) \in \operatorname{supp} \pi} \pi(x, y) \phi(x, y) \geq 0.$$

We will consider three possible cases.

(*First Case*) Suppose $\operatorname{supp}\pi \subseteq \oplus$, then by Claim 1, $\pi \in \Pi$, and by definition of ϕ, $\phi(x,y) \geq 0$ for every $(x,y) \in \operatorname{supp}\pi$.

(*Second Case*) Suppose $\operatorname{supp}\bar{\pi} \subseteq \oplus$, and $\operatorname{supp}\bar{\pi} \cap \hat{\oplus} \neq \emptyset$. Then by Claim 4 $\bar{\pi} \in \hat{\Pi}$ and $\pi \notin \Pi$. By definition of ϕ, $\phi(x,y) \leq 0$ for every $(x,y) \in \operatorname{supp}\pi$, and $\phi(x,y) < 0$ for some $(x,y) \in \operatorname{supp}\pi$.

(*Third Case*) Finally, we show that all the other possibilities can be reduced into one of the first two cases. Fix $t \in X$. Suppose we are not in one of the first two cases, that is, there exist $(x_0,y_0),(x_1,y_1) \in \operatorname{supp}\pi$ with $(x_0,y_0),(y_1,x_1) \in \hat{\oplus}$. Then by Claim 2 there exists a unique $\alpha \in (0,1)$ such that $\alpha\delta_{(x_0,y_0)} + (1-\alpha)\delta_{(x_1,y_1)} \in \Pi \backslash \hat{\Pi}$. By Claim 3 and uniqueness up to a positive linear transformation, $\frac{\alpha}{(1-\alpha)}\phi(x_0,y_0) = \phi(y_1,x_1)$. If $\frac{\alpha}{1-\alpha} = \frac{\pi(x_0,y_0)}{\pi(x_1,y_1)}$, then Claim 5 guarantees that $\pi \in \Pi$ if and only if $\pi' \in \Pi$ where[20]

$$\pi'(x,y) = \begin{cases} \pi(x,y) & (x,y) \notin \{(x_0,y_0),(x_1,y_1),(t,t)\} \\ 0 & (x,y) \in \{(x_0,y_0),(x_1,y_1)\} \\ \pi(t,t) + \pi(x_0,y_0) + \pi(x_1,y_1) & (x,y) = (t,t). \end{cases}$$

Moreover,

$$\pi(x_0,y_0)\phi(x_0,y_0) + \pi(x_1,y_1)\phi(x_1,y_1) = 0 = \phi(t,t)\left(\pi(t,t) + \pi(x_0,y_0) + \pi(x_1,y_1)\right)$$

so that $\sum_{(x,y)\in\operatorname{supp}\pi} \pi(x,y)\phi(x,y) \geq 0 \Leftrightarrow \sum_{(x,y)\in\operatorname{supp}\pi'} \pi'(x,y)\phi(x,y) \geq 0$.

[20] To see this, apply Claim 5 with $\eta = \delta_{(t,t)}$, $\chi = \alpha\delta_{(x_0,y_0)} + (1-\alpha)\delta_{(x_1,y_1)}$, $\lambda = 1 - \pi(x_0,y_0) - \pi(x_1,y_1)$, and $\rho(x,y) = \frac{\pi(x,y)}{1-\pi(x_0,y_0)-\pi(x_1,y_1)}$ if $(x,y) \notin \{(x_0,y_0),(x_1,y_1)\}$ and $\rho(x,y) = 0$ otherwise.

If $\frac{\alpha}{1-\alpha} > \frac{\pi(x_0,y_0)}{\pi(x_1,y_1)}$, Claim 5 guarantees that $\pi \in \Pi$ if and only $\pi' \in \Pi$ where[21]

$$\pi'(x,y) = \begin{cases} \pi(x,y) & (x,y) \notin \{(x_0,y_0),(x_1,y_1),(t,t)\} \\ 0 & (x,y) = (x_0,y_0) \\ \pi(x_1,y_1) - \frac{1-\alpha}{\alpha}\pi(x_0,y_0) & (x,y) = (x_1,y_1) \\ \pi(t,t) + \pi(x_0,y_0) + \frac{1-\alpha}{\alpha}\pi(x_0,y_0) & (x,y) = (t,t). \end{cases}$$

Moreover,

$$\begin{aligned} & \pi(x_0,y_0)\phi(x_0,y_0) + \pi(x_1,y_1)\phi(x_1,y_1) + \phi(t,t)\pi(t,t) \\ = \; & -\pi(x_0,y_0)\frac{1-\alpha}{\alpha}\phi(x_1,y_1) + \pi(x_1,y_1)\phi(x_1,y_1) + 0 \\ = \; & \left(\pi(x_1,y_1) - \frac{1-\alpha}{\alpha}\pi(x_0,y_0)\right)\phi(x_1,y_1) + 0 \\ = \; & \pi'(x_1,y_1)\phi(x_1,y_1) + \phi(t,t)\pi'(t,t) \end{aligned}$$

so that

$$\sum_{(x,y)\in \operatorname{supp}\pi} \pi(x,y)\phi(x,y) \geq 0 \Leftrightarrow \sum_{(x,y)\in \operatorname{supp}\pi'} \pi'(x,y)\phi(x,y) \geq 0.$$

A similar equivalence can be obtained if $\frac{\alpha}{1-\alpha} < \frac{\pi(x_0,y_0)}{\pi(x_1,y_1)}$. In every instance, the resulting π' has strictly fewer elements in the support that do not belong to $\Pi \backslash \hat{\Pi}$ than the original π. Since the support is finite, by repeating this procedure a finite number of times, we will obtain a $\hat{\pi} \in \Delta(X \times X)$ that falls in one of the first two

[21] To see this, apply Claim 5 with $\eta = \delta_{(t,t)}$, $\chi = \alpha\delta_{(x_0,y_0)} + (1-\alpha)\delta_{(x_1,y_1)}$, $\lambda = 1 - \pi(x_0,y_0) - \frac{1-\alpha}{\alpha}\pi(x_0,y_0)$, and

$$\rho(x,y) = \begin{cases} \frac{\pi(x,y)}{1-\pi(x_0,y_0)-\frac{1-\alpha}{\alpha}\pi(x_0,y_0)} & (x,y) \notin \{(x_0,y_0),(x_1,y_1)\} \\ 0 & (x,y) = (x_0,y_0) \\ \frac{\pi(x_1,y_1)-\frac{1-\alpha}{\alpha}\pi(x_0,y_0)}{1-\pi(x_0,y_0)-\frac{1-\alpha}{\alpha}\pi(x_0,y_0)} & (x,y) = (x_1,y_1). \end{cases}$$

cases, and such that $\pi \in \Pi \Leftrightarrow \hat{\pi} \in \Pi$ and

$$\sum_{(x,y)\in \operatorname{supp}\pi} \pi(x,y)\,\phi(x,y) \geq 0 \Leftrightarrow \sum_{(x,y)\in \operatorname{supp}\hat{\pi}} \hat{\pi}(x,y)\,\phi(x,y) \geq 0$$

concluding the proof. ■

Proof of Proposition 1 We establish Proposition 1 by proving the following more general result.[22]

Claim 6 *If* Π *admits a correlation-sensitive representation then the following are equivalent:*

1. Π *satisfies Transitivity;*
2. $\succsim^{\Pi}$ *satisfies Classic Completeness;*
3. $\succsim^{\Pi}$ *satisfies Classic Completeness, Classic Transitivity, Classic Archimedean Continuity, and Classic Strong Independence;*
4. $\succsim^{\Pi}$ *admits an expected utility representation.*

Proof We define the binary relation $\geq$ over outcomes as $x \geq y \Leftrightarrow \delta_{(x,y)} \in \Pi$. We will be interested in whether ϕ is modular with respect to this binary relation, i.e.,[23]

$$\forall x, x', y, y' \in X \quad \phi((x,y) \vee (x',y')) + \phi((x,y) \wedge (x',y')) = \phi(x,y) + \phi(x',y'). \quad (9)$$

The claim is proved by showing that each of the different conditions in the statement is equivalent to Equation (9). Notice that since positive linear transformations preserve modularity, it does not matter which representing ϕ we consider.

[22] We use the adjective Classic for the conventional versions of Completeness, Transitivity, Archimedean Continuity, and Strong Independence for binary relations. The definition for these standard notions are in Online Appendix B.

[23] Note the slight abuse of terminology here, as $\geq$ defined as above is not in general antisymmetric (although it is in our application to salience theory with monetary outcomes) and therefore the join and meet of two elements of the set may be not well defined. In that case everything works even with indifferencies with the understanding that $(x,y) \vee (x',y')$ is any pair (z,w) where $z \in \{x,x'\}$ $z \geq x$, $z \geq x'$ and $w \in \{y,y'\}$ $w \geq y$, $w \geq y'$ and $(x,y) \wedge (x',y')$ is any pair (z,w) where $z \in \{x,x'\}$ $z \leq x$, $z \leq x'$ and $w \in \{y,y'\}$ $w \leq y$, $w \leq y'$.

Equation (9) $\Rightarrow$ 4. Let $x_0 \in X$. Define $u(z)$ as $\phi(z, x_0)$. Fix a pair (z, w), with $z \geq w$. There are three cases:

- $z \geq w \geq x_0$. Applying (9) with $x = z$, $y = x' = x_0$ and $y' = w$ we have:

$$\begin{aligned} \phi(z,w) + \phi(x_0, x_0) &= \phi(z, x_0) + \phi(x_0, w) \Leftrightarrow \\ \phi(z,w) &= \phi(z, x_0) - \phi(w, x_0) \Leftrightarrow \phi(z,w) = u(z) - u(w) \end{aligned}$$

  where the first implication follows from the skew symmetry of ϕ.

- $z \geq x_0 \geq w$. Applying (9) with $x = z$, $y = w$ and $x_0 = y' = x'$ we have:

$$\phi(z, x_0) + \phi(x_0, w) = \phi(z,w) + \phi(x_0, x_0) \Leftrightarrow \phi(z,w) = u(z) - u(w)$$

  where the implication follows from the skew symmetry of ϕ and the definition of u.

- $x_0 \geq z \geq w$. Applying (9) with $x = z$, $y = x' = x_0$ and $y' = w$ we have:

$$\phi(x_0, x_0) + \phi(z,w) = \phi(z, x_0) + \phi(x_0, w) \Leftrightarrow \phi(z,w) = u(z) - u(w)$$

  where the implication follows from the skew symmetry of ϕ and the definition of u.

This proves that $\phi(z,w) = u(z) - u(w)$ whenever $z \geq w$. If $w > z$, by skew-symmetry of ϕ, $\phi(z,w) = -\phi(w,z) = -(u(w) - u(z)) = u(z) - u(w)$ proving that the equality $\phi(z,w) = u(z) - u(w)$ holds for every $z, w \in X$. Therefore, we have $\pi \in \Pi$ if and only if

$$\begin{aligned} \sum_{(x,y)\in X\times X} \pi(x,y)\,\phi(x,y) \geq 0 \quad &\Longleftrightarrow \quad \sum_{(x,y)\in X\times X} \pi(x,y)\,(u(x) - u(y)) \geq 0 \\ &\Longleftrightarrow \quad \sum_{x\in X} \pi_1(x)\,u(x) \geq \sum_{x\in X} \pi_2(x)\,u(x) \end{aligned}$$

proving that Π admits an EU representation.

4⇒ *Equation (9)* If Π admits an EU representation then

$$\pi \in \Pi \Longleftrightarrow \sum_{(x,y)\in X\times X} \pi(x,y)\left(u(x) - u(y)\right) \geq 0.$$

Therefore, if we define $\phi(z,w) = (u(z) - u(w))$, modularity holds: let $x, y, x', y' \in X$

$$\begin{aligned} & \phi\left((x,y) \vee (x',y')\right) + \phi\left((x,y) \wedge (x',y')\right) \\ = \; & u(x \vee x') - u(y \vee y') + u(x \wedge x') - u(y \wedge y') \\ = \; & u(x) + u(x') - u(y) - u(y') = \phi(x,y) + \phi(x',y'). \end{aligned}$$

$3 \Leftrightarrow 4$ is a version of the vN-M EU theorem.
$4 \Rightarrow 1$ is straightforward given the representation.
$4 \Rightarrow 2$ holds trivially.
$2 \Rightarrow$*Equation (9)* and $1 \Rightarrow$*Equation (9)* are proved by contradiction. Suppose that there exist $x, y, x', z' \in X$ such that

$$\phi\left((x,y) \vee (x',y')\right) + \phi\left((x,y) \wedge (x',y')\right) > \phi(x,y) + \phi(x',y')$$

with $(x \vee x') = x$ and $(y \vee y') = y'$. Then the inequality reads

$$\phi(x,y') + \phi(x',y) > \phi(x,y) + \phi(x',y'). \tag{10}$$

Choose $(z,w) \in (X \times X)$ and $\alpha \in [0,1]$ such that

$$\alpha\phi(z,w) + (1-\alpha)\left(\frac{\phi(x,y') + \phi(x',y)}{2}\right) > 0 > \alpha\phi(z,w) + (1-\alpha)\left(\frac{\phi(x,y) + \phi(x',y')}{2}\right).$$

The existence of such (z,w) and α is guaranteed by (10). Then

$$\alpha\delta_{(z,w)} + \frac{(1-\alpha)\,\delta_{(x,y')}}{2} + \frac{(1-\alpha)\,\delta_{(x',y)}}{2} \in \hat{\Pi} \text{ and } \alpha\delta_{(z,w)} + \frac{(1-\alpha)\,\delta_{(x,y)}}{2} + \frac{(1-\alpha)\,\delta_{(x',y')}}{2} \notin \Pi. \tag{11}$$

We now show that (11) implies that neither Classic Completeness of $\succsim^{\Pi}$ nor Transitivity of Π holds. For Classic Completeness notice that (11) implies that neither

$$\alpha\delta_z + \frac{(1-\alpha)\,\delta_x}{2} + \frac{(1-\alpha)\,\delta_{x'}}{2} \succsim^{\Pi} \alpha\delta_w + \frac{(1-\alpha)\,\delta_{y'}}{2} + \frac{(1-\alpha)\,\delta_y}{2}$$

nor

$$\alpha\delta_w + \frac{(1-\alpha)\,\delta_{y'}}{2} + \frac{(1-\alpha)\,\delta_y}{2} \succsim^{\Pi} \alpha\delta_z + \frac{(1-\alpha)\,\delta_x}{2} + \frac{(1-\alpha)\,\delta_{x'}}{2}$$

holds, and $\succsim^{\Pi}$ does not satisfy Classic Completeness.

As for Transitivity, let

$$\begin{aligned}
\pi &= \alpha\delta_{(z,z)} + \frac{(1-\alpha)\,\delta_{(x,x)}}{2} + \frac{(1-\alpha)\,\delta_{(x',x')}}{2}, \\
\chi &= \alpha\delta_{(z,w)} + \frac{(1-\alpha)\,\delta_{(x,y')}}{2} + \frac{(1-\alpha)\,\delta_{(x',y)}}{2}, \\
\rho &= \alpha\delta_{(z,w)} + \frac{(1-\alpha)\,\delta_{(x,y)}}{2} + \frac{(1-\alpha)\,\delta_{(x',y')}}{2}.
\end{aligned}$$

Completeness of Π implies that $\pi \in \Pi$, and (11) gives $\chi \in \Pi$, $\rho \notin \Pi$. However, since $\pi_1 = \rho_1$, $\pi_2 = \chi_1$, and $\chi_2 = \rho_2$, Transitivity of Π does not hold. Similar arguments can be used to obtain contradictions for other violations of modularity. ■

Proof of Remark 1 (If) Let $x, y, z \in X$ with $x > y$. By assumption, we have $\phi(x,z) > \phi(y,z)$. Therefore, for every $\alpha \in (0,1)$, $z \in X$, and $\pi \in \Delta(X)$

$$\begin{aligned}
\alpha\delta_{(y,z)} + (1-\alpha)\,\pi \in \Pi &\iff \alpha\phi(y,z) + (1-\alpha) \sum_{(x',y') \in X \times X} \pi(x',y')\,\phi(x',y') \geq 0 \\
&\Rightarrow \alpha\phi(x,z) + (1-\alpha) \sum_{(x',y') \in X \times X} \pi(x',y')\,\phi(x',y') > 0 \\
&\iff \alpha\delta_{(x,z)} + (1-\alpha)\,\pi \in \hat{\Pi}.
\end{aligned}$$

(Only if) We first prove that ϕ is strictly increasing in the first argument. Let $x_1, x_2, y \in X$, $x_1 > x_2$. Under a correlation-sensitive representation, we have $\frac{\delta_{(x_2,y)}}{2} + \frac{\delta_{(y,x_2)}}{2} \in \Pi$. Then by Monotonicity $\frac{\delta_{(x_1,y)}}{2} + \frac{\delta_{(y,x_2)}}{2} \in \hat{\Pi}$ and given the correlation-sensitive representation this implies $\phi(x_1,y) > \phi(x_2,y)$. To see that ϕ is strictly decreasing in the second argument, notice that by skew symmetry:

$$\phi(x_1, y) > \phi(x_2, y) \Rightarrow -\phi(y, x_1) > -\phi(y, x_2) \Rightarrow \phi(y, x_1) < \phi(y, x_2)$$

concluding the proof. ■

Proof of Remark 2 (Only if) If ϕ is always equal to 0 the claim is obvious. Therefore, suppose there exists $z, w \in X$ with $\phi(z, w) > 0$. We prove continuity in the first argument; continuity in the second argument follows from skew-symmetry. Let $(x_n)_{n\in\mathbb{N}} \to x$, and suppose that there exists $y \in X$ such that $\phi(x_n, y) \nrightarrow \phi(x, y)$. There are two cases:

i) There exists an infinite subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ and an $\varepsilon > 0$ such that $\phi(x_{n_k}, y) \geq \phi(x, y) + \varepsilon$ for all $k \in \mathbb{N}$. If $\phi(x, y) \geq -\varepsilon$ notice that we have

$$\begin{aligned}
& \forall k \in \mathbb{N} \quad \frac{\phi(z, w)}{\phi(x, y) + \varepsilon + \phi(z, w)} \phi(x_{n_k}, y) + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(z, w)} \phi(w, z) \geq 0 \\
\Leftrightarrow \quad & \forall k \in \mathbb{N} \quad \frac{\phi(z, w)}{\phi(x, y) + \varepsilon + \phi(z, w)} \delta_{(x_{n_k}, y)} + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(z, w)} \delta_{(w,z)} \in \Pi \\
\Longrightarrow \quad & \frac{\phi(z, w)}{\phi(x, y) + \varepsilon + \phi(z, w)} \delta_{(x,y)} + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(z, w)} \delta_{(w,z)} \in \Pi \\
\Leftrightarrow \quad & \phi(x, y) \geq \phi(x, y) + \varepsilon
\end{aligned}$$

a contradiction. If $\phi(x, y) < -\varepsilon$ notice that we have

$$\begin{aligned}
& \forall k \in \mathbb{N} \quad \frac{\phi(w, z)}{\phi(x, y) + \varepsilon + \phi(w, z)} \phi(x_{n_k}, y) + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(w, z)} \phi(z, w) \geq 0 \\
\Leftrightarrow \quad & \forall k \in \mathbb{N} \quad \frac{\phi(w, z)}{\phi(x, y) + \varepsilon + \phi(w, z)} \delta_{(x_{n_k}, y)} + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(w, z)} \delta_{(z,w)} \in \Pi \\
\Longrightarrow \quad & \frac{\phi(w, z)}{\phi(x, y) + \varepsilon + \phi(w, z)} \delta_{(x,y)} + \frac{\phi(x, y) + \varepsilon}{\phi(x, y) + \varepsilon + \phi(w, z)} \delta_{(z,w)} \in \Pi \\
\Leftrightarrow \quad & \phi(x, y) \geq \phi(x, y) + \varepsilon
\end{aligned}$$

a contradiction.

ii) There exists an infinite subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ and an $\varepsilon > 0$ such that

$\phi(x_{n_k}, y) \leq \phi(x,y) - \varepsilon$ for all $k \in \mathbb{N}$. If $\phi(x,y) \geq \varepsilon$ notice that we have

$$\begin{aligned}
& \forall k \in \mathbb{N} \quad \frac{\phi(z,w)}{\phi(x,y) - \varepsilon + \phi(z,w)} \phi(x_{n_k}, y) + \frac{\phi(x,y) - \varepsilon}{\phi(x,y) - \varepsilon + \phi(z,w)} \phi(w,z) \leq 0 \\
\Leftrightarrow \quad & \forall k \in \mathbb{N} \quad \frac{\phi(z,w)}{\phi(x,y) - \varepsilon + \phi(z,w)} \delta_{(y, x_{n_k})} + \frac{\phi(x,y) - \varepsilon}{\phi(x,y) - \varepsilon + \phi(z,w)} \delta_{(z,w)} \in \Pi \\
\Longrightarrow \quad & \frac{\phi(z,w)}{\phi(x,y) - \varepsilon + \phi(z,w)} \delta_{(y,x)} + \frac{\phi(x,y) - \varepsilon}{\phi(x,y) + \varepsilon + \phi(z,w)} \delta_{(z,w)} \in \Pi \\
\Leftrightarrow \quad & \phi(x,y) - \varepsilon \geq \phi(x,y).
\end{aligned}$$

a contradiction. The case $\phi(x,y) \leq \varepsilon$ is proved along the same lines.

(If) Trivial. ■

A.1 Salience Characterization

Proof of Proposition 2 Let Π admit a correlation-sensitive representation, $x, y \in \mathbb{R}$, and $\alpha, \beta \in [0,1]$ with $x > y$ and $\beta > \alpha$ with at least one between α and β in $(0,1)$. We have $\left((x,y), \frac{\beta-\alpha}{1+\beta-\alpha}; (\alpha x + (1-\alpha) y, \beta x + (1-\beta) y), \frac{1}{1+\beta-\alpha}\right) \in \hat{\Pi}$ if and only if $\frac{\beta-\alpha}{1+\beta-\alpha}\phi(x,y) + \frac{1}{1+\beta-\alpha}\phi(\alpha x + (1-\alpha) y, \beta x + (1-\beta) y) > 0$ that by skew symmetry of ϕ is equivalent to

$$\phi(x,y)(\beta - \alpha) > \phi(\beta x + (1-\beta) y, \alpha x + (1-\alpha) y). \tag{12}$$

Now, let Π admit a σ-distorted representation. We show that Ordering of Π implies BGS-Ordering of σ, the other direction is trivial.

We first show that if $x \geq z > w \geq y$ with $[y,x] \supset [w,z]$, then $\sigma(x,y) > \sigma(w,z)$. Define $\alpha = \frac{w-y}{x-y}$ and $\beta := \frac{z-y}{x-y}$ and notice that $0 \leq \alpha < \beta \leq 1$ with at least one of the two inequalities being strict. Therefore, (12) implies that

$$\begin{aligned}
(\beta-\alpha)(x-y)\sigma(x,y) &> (\beta-\alpha)(x-y)\sigma(\alpha x + (1-\alpha) y, \beta x + (1-\beta) y) \\
&= (\beta-\alpha)(x-y)\sigma(w,z),
\end{aligned}$$

and $\sigma(x,y) > \sigma(w,z)$.

Next, let $z = w$, with $x \geq w \geq y$ and at least one of the two inequalities strict, say $x > w \geq y$. Suppose by way of contradiction that $\sigma(w, w) \geq \sigma(x, y)$. By continuity of σ, there exists an $\varepsilon < \frac{x-w}{2}$ with $\sigma(w + \varepsilon, w) > \sigma(x, y)$. But this is a contradiction with what was proved in the previous paragraph. ■

Proof of Proposition 3 Let Π admit a σ-distorted representation and satisfy strict Diminishing Sensitivity. Fix $x > y > 0$, $k > 0$, we have that $\left((x, y), \frac{1}{2}; (y + k, x + k), \frac{1}{2}\right) \in \hat{\Pi}$. Given the σ-distorted representation, this is equivalent to $(x - y) \sigma(x, y) + (y - x) \sigma(y + k, x + k) > 0$. The previous inequality holds if and only if $\sigma(x, y - k) > \sigma(y, x + k) = \sigma(x + k, y)$ proving that σ satisfies BGS-Diminishing Sensitivity. All the steps are reversible. ■

Proof of Proposition 4 (If) Let $x \geq y \geq 0$ and $k \geq 0$. Consider the two marginal distributions $p = \left(x, \frac{1}{2}; y + k, \frac{1}{2}\right)$ and $q = \left(x + k, \frac{1}{2}; y, \frac{1}{2}\right)$. Notice that q is a mean-preserving spread of p, since q can be obtained by further randomizing each realization z of p with the additional random term h_z defined as $h_x = \left(k, \frac{(x-y)}{(x-y)+k}; (y - x), \frac{k}{(x-y)+k}\right)$ and $h_{y+k} = \left((x - y), \frac{k}{(x-y)+k}; -k, \frac{(x-y)}{(x-y)+k}\right)$. Therefore, as risk-averse expected utility DMs dislike mean-preserving spreads:

$$\sum_{z \in X} p(z) u(z) \geq \sum_{z \in X} q(z) u(z).$$

Rearranging the terms $\frac{1}{2}(u(x) - u(y)) + \frac{1}{2}(u(y + k) - u(x + k)) \geq 0$ or

$$\left((x, y), \frac{1}{2}; (y + k, x + k), \frac{1}{2}\right) \in \Pi$$

and Diminishing Sensitivity holds.

(Only If) Let $x_0 \geq y_0 \geq 0$. By Diminishing Sensitivity

$$\left(\left(\frac{x_0 + y_0}{2}, y_0\right), \frac{1}{2}; \left(\frac{x_0 + y_0}{2}, x_0\right), \frac{1}{2}\right) \in \Pi$$

that is $u\left(\frac{x_0+y_0}{2}\right) \geq \frac{u(x_0)+u(y_0)}{2}$ proving the midpoint concavity of u on the set of positive real numbers. Since u is strictly increasing, it is measurable. Since the Sierpinski theorem implies that a midpoint concave and measurable function is concave, the DM is risk-averse on that range. ■

Proof of Proposition 5 Let $x, y, w, z \in \mathbb{R}_+$, with $x - y = z - w > 0$. Under a σ-distorted representation

$$\left((x, y), \frac{1}{2}; (w, z), \frac{1}{2}\right) \in \hat{\Pi} \Leftrightarrow \left((-y, -x), \frac{1}{2}; (-z, -w), \frac{1}{2}\right) \in \hat{\Pi}$$

is tantamount to

$$(x - y)\,\sigma(x, y) > (z - w)\,\sigma(w, z) \Leftrightarrow (x - y)\,\sigma(-x, -y) > (z - w)\,\sigma(-w, -z)$$

which is equivalent to $\sigma(x, y) > \sigma(w, z) \Leftrightarrow \sigma(-x, -y) > \sigma(-w, -z)$. The case in which $x - y = z - w < 0$ is completely analogous, and the one in which $x - y = z - w = 0$ immediately follows from the fact that for all $x, w \in \mathbb{R}_+$, $\left((x, x), \frac{1}{2}; (w, w), \frac{1}{2}\right) \in \Pi$ and $\left((-x, -x), \frac{1}{2}; (-w, -w), \frac{1}{2}\right) \in \Pi$. ■

Proof of Proposition 6 We will prove only the case in which Π is risk-averse in (a, b) as the other case is analogous. Let u be a vN-M utility index representing Π such that $u(0) = 0$, and suppose that Π is risk-averse for lotteries with values in $(a, b) \subseteq \mathbb{R}_+$. Let $-b < -x \leq -y < -a$, since u is concave on (a, b), we have $u(x) - u\left(\frac{x+y}{2}\right) \leq u\left(\frac{x+y}{2}\right) - u(y)$ that is $\left(\left(x, \frac{x+y}{2}\right), \frac{1}{2}; \left(y, \frac{x+y}{2}\right), \frac{1}{2}\right) \notin \hat{\Pi}$. By Weak Reflexivity, this means that $\left(\left(-\frac{x+y}{2}, -x\right), \frac{1}{2}; \left(-\frac{x+y}{2}, -y\right), \frac{1}{2}\right) \notin \hat{\Pi}$ or $u\left(-\frac{x+y}{2}\right) \leq \frac{u(-x)+u(-y)}{2}$. This shows that u is mid-point convex on $(-a, -b)$. Since it is also increasing, it is measurable, and by the Sierpinski theorem it is convex on $(-a, -b)$, proving the statement. ■

Proof of Theorem 2 (Only If) Given a smooth salience representation, let $\phi(x, y) = \sigma(x, y)(x - y)$. By the symmetry axiom for σ, we have $\phi(x, y) = \sigma(x, y)(x - y) = \sigma(y, x)(x - y) = -\sigma(y, x)(y - x) = -\phi(y, x)$ proving that ϕ is skew-symmetric. Then Π satisfies Completeness, Strong Independence, and Archimedean Continuity by Theorem 1. It satisfies Ordering, Diminishing Sensitivity, and Weak Reflexivity by Propositions 2, 3, and 5. Since σ satisfies BGS-Ordering, Π satisfies Monotonicity by Remark 1. To see it, suppose that $y \geq x > x'$. Then $\phi(x,y) = \sigma(x,y)(x-y) \geq \sigma(x',y)(x'-y) = \phi(x,y)$ where the inequality is due to $0 \leq \sigma(x,y) \leq \sigma(x,y')$ with at least one of the two inequalities being strict that in turns is a consequence of BGS Ordering and the fact that a salience function takes positive values by definition. The case $x > x' \geq y$ is proved similarly, and $x > x'$, $y \in (x', x)$ follows immediately from $\phi(x,y) = \sigma(x,y)(x-y) > 0 \geq \sigma(x',y)(x'-y) = \phi(x,y)$. Moreover, Π satisfies Continuity in Outcomes by Remark 2 and since $\phi(y,x)$ is the product of two jointly continuous functions. Finally, let $x \in X$, $(x_n)_{n\in\mathbb{N}}$ be such that $x_n \downarrow x$, $k \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{++}$. Then

$$\begin{aligned}
& ((x,x_n),(1-(x_n-x));(k+\varepsilon,k),(x_n-x)) \in \Pi \\
\Leftrightarrow\; & \phi(x,x_n)(1-(x_n-x)) \geq \phi(k,k+\varepsilon)(x_n-x) \\
\Leftrightarrow\; & \sigma(x,x_n)(x-x_n)(1-(x_n-x)) \geq \sigma(k+\varepsilon,k)\,\varepsilon\,(x-x_n) \\
\Leftarrow\; & \sigma(x,x_n)(1-(x_n-x)) \leq \sigma(k+\varepsilon,k)\,\varepsilon
\end{aligned}$$

where the last inequality holds for sufficiently large n by continuity of σ, proving that Π satisfies the first condition of Continuity at Identity. An analogous argument establishes the second part.

(If) Since Π satisfies Completeness, Strong Independence, and Archimedean Continuity by Theorem 1 it admits the representation

$$\pi \in \Pi \Longleftrightarrow \sum_{(x,y)\in X\times X} \phi(x,y)\,\pi(x,y) \geq 0.$$

Define σ by

$$\sigma(x,y) = \frac{\phi(x,y)}{x-y} \qquad \forall x \neq y$$

and $\sigma(x,x) = 0$ for all $x \in X$. We have that σ maps $X \times X$ into positive real numbers by Monotonicity and Remark 1. It is immediate that

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} (x-y)\,\sigma(x,y)\,\pi(x,y) \geq 0.$$

Propositions 2, 3, and 5 guarantee that σ satisfies respectively BGS-Ordering, BGS-Diminishing Sensitivity, and BGS-Weak Reflexivity. We now check that σ satisfies symmetry and it is continuous. First, σ satisfies symmetry since ϕ is skew symmetric. Moreover since ϕ is continuous by Remark 2 σ is continuous at every (x,y) such that $x \neq y$. We now show that it is continuous at each $(x,x) \in \mathbb{R}\times\mathbb{R}$. We show that $x_n \downarrow x$ implies $\sigma(x_n,x) \to 0$, the proof for the case in which $x_n \uparrow x$ is completely analogous. Without loss of generality, we can assume that $x_n \neq x$ for all $n \in \mathbb{N}$. By Continuity at Identity, for all $k \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{++}$, there exists an $m \in \mathbb{N}$ such that for all $n \geq m$,

$$\begin{aligned}
& ((x,x_n),(1-(x_n-x));(k+\varepsilon,k),(x_n-x)) \in \Pi \\
\Leftrightarrow\ & \phi(x,x_n)(1-(x_n-x)) \geq \phi(k,k+\varepsilon)(x_n-x) \\
\Leftrightarrow\ & \sigma(x,x_n)(x-x_n)(1-(x_n-x)) \geq \sigma(k+\varepsilon,k)\,\varepsilon\,(x-x_n) \\
\Leftrightarrow\ & \sigma(x,x_n)(1-(x_n-x)) \leq \sigma(k+\varepsilon,k)\,\varepsilon.
\end{aligned}$$

Since the ε can be chosen arbitrarily small ε and $\sigma(k+\varepsilon,k)$ is decreasing in ε by the BGS-Ordering property established above, this proves that $\sigma(x,x_n)(1-(x_n-x))$ is converging to 0. This concludes the proof. ■

Proof of Corollary 1 Let $x > z > y$. Then, there exists $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y = z$. Applying Ordering and Proposition 2 with $\alpha = \lambda$ and $\beta = 1$ we get $\phi(x,y)(1-\lambda) > \phi(x,z)$. Applying Ordering and Proposition 2 with $\beta = \lambda$ and $\alpha = 0$ we get $\phi(x,y)\lambda > \phi(z,y)$. By summing the two inequalities, we get the desired result. ■

B Binary Relations and Preference Sets

Lemma 1 *For every binary relation* $\succeq$*, we have* $\succsim^{\Pi_{\succeq}} = \succeq$.

We collect the definitions of some standard axioms for binary relations over $\Delta(X)$.

Axiom 11 (Classic Completeness) *For all* $p, q \in \Delta(X)$*, either* $p \succsim q$ *or* $q \succsim p$ *or both.*

Classic Completeness requires that the DM can (weakly) rank all the marginal lotteries. Our analysis highlights why Classic Completeness may fail: the comparison of some pairs of lotteries may depend on their correlation. The following lemma shows that Completeness of the preference set weakens Classic Completeness. The example discussed in the introduction shows why it may be a strictly weaker requirement.

Lemma 2

1. *Let* $\succsim$ *be a binary relation. If* $\succsim$ *satisfies Classic Completeness, then* $\Pi_{\succsim}$ *satisfies Completeness.*
2. *Let* Π *be a preference set. If* $\succsim^{\Pi}$ *satisfies Classic Completeness, then* Π *satisfies Completeness.*

That is, the preference set derived from a complete binary relation satisfies Completeness. Moreover, the binary relation induced by a preference set is complete only if the preference set satisfies Completeness.

Axiom 12 (Classic Transitivity) *For all* $p, q, r \in \Delta(X)$*, if* $p \succsim q$ *and* $q \succsim r$*, then* $p \succsim r$*. Moreover, if either* $p \succ q$ *or* $q \succ r$*, then* $p \succ r$*.*

Classic Transitivity is the other central tenet of rationality.

Axiom 13 (Classic Strong Independence) *For all* $p, q, r \in \Delta(X)$ *and* $\alpha \in (0,1)$*,*

$$p \succsim q \Leftrightarrow \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r.$$

Classic Strong Independence is the axiom usually paired to Classic Completeness, Classic Transitivity, and Classic Archimedean Continuity to derive the expected utility representation. Since we often work without Classic Transitivity in this paper, we also need to consider an alternative and stronger form of independence.

Axiom 14 (Classic Strong B-Independence) *For all* $p, q, r, s \in \Delta(X)$ *and* $\alpha \in (0,1)$,

$$p \succsim q, r \succsim s \Rightarrow \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) s.$$

Moreover, if $p \succ q$, *then* $\alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) s$.

Classic Strong B-Independence says that convex combinations of preferred alternatives are preferred to the convex combination of the alternatives they dominate. It implies Classic Strong Independence, and the two axioms coincide under Classic Transitivity. The following remark proved in the Online Appendix clarifies that the usual approach that assumes Classic Strong Independence for a binary relation implicitly imposes our notion of Strong Independence for preference sets.

Remark 4 *If a binary relation* $\succsim$ *satisfies Classic Strong B-Independence, then* $\Pi_{\succsim}$ *satisfies Strong Independence.*

The next axiom is a standard and weak form of continuity imposed on preferences defined over a convex set.

Axiom 15 (Classic Archimedean Continuity) *For all* $p, q, r \in \Delta(X)$ *such that* $p \succ q$ *and* $q \succ r$, *there exist* $\alpha, \beta \in (0,1)$ *such that* $\alpha p + (1-\alpha) r \succ q$ *and* $q \succ \beta p + (1-\beta) r$.

A slightly more demanding version of Classic Archimedean Continuity is needed when dealing with nontransitive and incomplete preferences.

Axiom 16 (Classic Archimedean B-Continuity) *For all* $p, q, r, s \in \Delta(X)$ *such that* $p \succ q$ *and* $r \not\succsim s$, *there exist* $\alpha, \beta \in (0,1)$ *such that*

$$\alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) s \text{ and } \beta p + (1-\beta) r \not\succsim \beta q + (1-\beta) s.$$

Therefore, under Classic Completeness, Classic Archimedean Continuity is the particular case of Classic Archimedean B-Continuity in which $s = q$.

Axiom 17 (Classic Sequential Continuity) *For each pair of sequences* $(p_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *in* $\Delta(X)$ *such that* $(p_n)_{n\in\mathbb{N}} \to p_0$ *and* $(q_n)_{n\in\mathbb{N}} \to q_0$

$$p_n \succsim q_n \text{ for all } n \in \mathbb{N} \implies p_0 \succsim q_0.$$

Classic Sequential Continuity implies Classic Archimedean B-Continuity under Classic Completeness.

Lemma 3 *If* $\succsim$ *satisfies Classic Sequential Continuity and Classic Completeness, then* $\succsim$ *satisfies Classic Archimedean B-Continuity.*

The following remark shows that the usual approach that assumes Classic Archimedean B-Continuity for a binary relation implicitly imposes our notion of Archimedean Continuity for preference sets.

Remark 5 *If* $\succsim$ *satisfies Classic Archimedean B-Continuity, then* $\Pi_{\succsim}$ *satisfies Archimedean Continuity.*

The next result verifies the asserted link between Classic Transitivity and Transitivity of the preference sets.

Lemma 4

1. *If* $\succsim$ *satisfies Classic Transitivity,* $\Pi_{\succsim}$ *satisfies Transitivity.*
2. *If* Π *satisfies Transitivity, then* $\succsim^{\Pi}$ *satisfies Classic Transitivity.*
3. $\succsim^{\Pi}$ *satisfies Classic Transitivity and Classic Completeness if and only if* Π *satisfies Transitivity and Completeness.*

Proposition 1 can be used to highlight an additional benefit of our preference sets approach: we obtain a new set of axioms that are one to one with expected utility theory.

Theorem 3 *For every $\Pi \subseteq \Delta(X \times X)$ the following are equivalent:*

1. *$\succsim_\Pi$ satisfy Classic Completeness, Classic Strong B-Independence, and Classic Archimedean B-Continuity;*

2. *$\succsim_\Pi$ satisfies Classic Completeness, Classic Transitivity, Classic Archimedean Continuity, and Classic Strong Independence;*

3. *$\succsim_\Pi$ admits an expected utility representation.*

Proof It immediately follows by combining Lemmata 1, 2, and 4, Remarks 4 and 5, Claim 6 and the vN-M expected utility theorem. ■

C Choice from arbitrary sets

We now turn to an important question left open by the previous analysis: how the DM chooses from a finite set A of more than two alternatives. In general, since the correlation-sensitive decision criterion is intransitive, it is possible that, given a choice set A, no element of A is (weakly) preferred to all the other options.

To describe the decision maker's preferences when multiple alternatives are available, we will need to generalize the concept of choice rule. In particular, let $\Delta(X^n)$ be the joint distribution over the n dimensional outcomes, and $\Delta = \bigcup_{n\in\mathbb{N}} \Delta(X^n)$ be the set of all the joint distribution over a finite number of outcomes. A choice rule is a $\mathcal{C} : \Delta \rightrightarrows \mathbb{N}$ such that $\pi \in \Delta(X^n)$ implies that $\emptyset \neq \mathcal{C}(\pi) \subseteq \{1, ..., n\}$. The interpretation is that the choice rule takes as an input a $\pi \in \Delta(X^n)$ that describes the joint distribution over outcomes of n alternatives, and gives back the subset of these alternatives preferred by the DM given the correlation structure.

A natural question is whether, given a preference set Π that satisfies Completeness, Strong Independence, and Archimedean Continuity, there always exists a consistent choice rule. Formally, given a preference set Π that admits a correlation-sensitive representation with skew symmetric function ϕ a choice rule $\mathcal{C}$ is *consistent* with Π if for all $n \in \mathbb{N}$, for all $\pi \in \Delta(X^n)$, if $i \in \mathcal{C}(\pi)$, then

$$\forall j \in \{1, ..., n\} \qquad \sum_{(x,y)} \pi_{i,j}(x, y)\, \phi(x, y) \geq 0$$

where $\pi_{i,j}$ is the marginal distribution over alternatives i and j. That is, if the DM prefers to be paid according to the i alternative, then the preference set Π deems i preferable to every other j in their pairwise comparison.

It is immediate that if Π admits an expected utility representation, then there exists a choice function consistent with Π. Unfortunately, given the more general criterion's intransitivity, this may not be the case for some preference sets that satisfy Completeness, Strong Independence, and Archimedean Continuity, as shown in the example below.

However, in some situations, the DM may be able to randomize over the alternatives with a randomization device independent of the alternative under consideration. A stochastic choice rule is a $\mathcal{S} : \Delta \to \Delta(\mathbb{N})$ such that $\pi \in \Delta(X^n)$ implies that $\operatorname{supp} \mathcal{S}(\pi) \subseteq \{1, ..., n\}$. Since the randomization performed by the DM does not introduce any additional correlation, we extend the notion of consistency in a linear way. Given a preference set Π that admits a correlation-sensitive representation with skew symmetric function ϕ a stochastic choice rule $\mathcal{S}$ is *consistent* with Π if for all $n \in \mathbb{N}$, for all $\pi \in \Delta(X^n)$, if $\mathcal{S}(\pi) = \nu$, then

$$\forall \nu' \in \Delta(\{1, ..., n\}) \qquad \sum_{i,j \in \{1,...,n\}} \nu(i)\, \nu(j) \sum_{(x,y)} \pi_{i,j}(x, y)\, \phi(x, y) \geq 0.$$

Fortunately, we can extend a result by Kreweras (1961) to show that a consistent stochastic choice rule always exists.

Proposition 7 *If Π satisfies Completeness, Strong Independence and Archimedean Continuity, then there exists a stochastic choice rule $\mathcal{S}$ that is consistent with Π.*

Proof By Theorem 1 Π admits a correlation-sensitive representation with skew symmetric function ϕ. Let $n \in \mathbb{N}$ and $\pi \in \Delta(X^n)$. We have that

$$\begin{aligned}
& \max_{\nu \in \Delta(\{1,...,n\})} \min_{\nu' \in \Delta(\{1,...,n\})} \sum_{i,j \in \{1,...,n\}} \sum_{(x,y)} \nu(i)\, \nu'(j)\, \pi_{i,j}(x,y)\, \phi(x,y) \\
= & \min_{\nu' \in \Delta(\{1,...,n\})} \max_{\nu \in \Delta(\{1,...,n\})} \sum_{i,j \in \{1,...,n\}} \nu(i)\, \nu'(j) \sum_{(x,y)} \pi_{i,j}(x,y)\, \phi(x,y) \\
= & \min_{\nu' \in \Delta(\{1,...,n\})} \max_{\nu \in \Delta(\{1,...,n\})} \sum_{i,j \in \{1,...,n\}} \nu(i)\, \nu'(j) \left(-\sum_{(x,y)} \pi_{j,i}(x,y)\, \phi(x,y) \right) \\
= & -\max_{\nu \in \Delta(\{1,...,n\})} \min_{\nu' \in \Delta(\{1,...,n\})} \left(\sum_{i,j \in \{1,...,n\}} \sum_{(x,y)} \nu(i)\, \nu'(j)\, \pi_{i,j}(x,y)\, \phi(x,y) \right)
\end{aligned}$$

where the first equality follows from von Neumann's min-max theorem, the second by skew symmetry of ϕ, and the last by simple algebra. Therefore,

$$\max_{\nu \in \Delta(\{1,...,n\})} \min_{\nu' \in \Delta(\{1,...,n\})} \sum_{i \in \{1,...,n\}} \sum_{j \in \{1,...,n\}} \sum_{(x,y)} \nu(i)\, \nu'(j)\, \pi_{i,j}(x,y)\, \phi(x,y) = 0,$$

that is there exists $\nu_\pi \in \Delta(\{1,...,n\})$ such that for all $\nu' \in \Delta(\{1,...,n\})$,

$$\sum_{i \in \{1,...,n\}} \sum_{j \in \{1,...,n\}} \sum_{(x,y)} \nu_\pi(i)\, \nu'(j)\, \pi_{i,j}(x,y)\, \phi(x,y) \geq 0.$$

The result then follows by letting $\mathcal{S}(\pi) = \nu_\pi$ for all $\pi \in \Delta$. ■

Example 4 (The effect of salience on random choice) *Suppose that the preference set* Π *admits a salience representation with salience function* $\sigma(x,y) = |x-y|$. *The DM faces three symmetric alternatives:* $\pi \in \Delta(X^3)$ *with* $\pi(3,1,2) = \pi(2,3,1) = \pi(1,2,3) = \frac{1}{3}$. *Here, choosing to be deterministically paid according to a single alternative is not consistent with* Π, *since for such a salience-sensitive DM* $\pi_{1,2}, \pi_{2,1}, \pi_{3,1} \in \hat{\Pi}$. *However, it is easy to see that the unique randomization consistent with* Π *sees the DM randomizing uniformly over the three acts. Next, suppose that the DM faces the joint distribution* $\pi' \in \Delta(X^4)$ *with* $\pi'(3,1,2,5) = \pi'(2,3,1,0) = \pi'(1,2,3,0) = \frac{1}{3}$. *Notice that for this salience-sensitive DM* $\pi'_{1,4}, \pi'_{3,4} \in \hat{\Pi}$, *but* $\pi'_{4,2} \in \Pi$ *since when the* *second alternative is compared to the fourth, the realization where the fourth alternative pays* 5 *and the second pays* 1 *results sufficiently salient to tilt the comparison in favor of the fourth alternative. It is easy to see that when faced with the choice set* π'*, uniform randomization over the first three alternatives is no longer optimal for the agent and that the unique optimal randomization is* $\left(\frac{1}{2}, 0, \frac{1}{6}, \frac{1}{3}\right)$*. Here, the fourth alternative plays a "stochastic decoy" effect: the probability of the other three alternatives are distorted to favor the ones that perform better in the salient state in which the decoy pays* 5*.* ▲

D Analysis of the Rank-Based Version

In this section, we analyze the relative weaknesses of the alternative rank-based salience theory proposed in BGS. First, note that every function $\sigma : X \times X \to \mathbb{R}$ induces a rank on the support of π. More precisely, if for all $(x, y) \in \operatorname{supp} \pi$ we let

$$\hat{\sigma}_\pi (x, y) = |\{(x', y') \in \operatorname{supp} \pi : \sigma (x', y') > \sigma (x, y)\}|,$$

we obtain $|\operatorname{supp} \pi| > \hat{\sigma}_\pi (x, y) \geq 0$ with $\hat{\sigma}_\pi (x, y) = 0$ for the most salient pair of outcomes. Given these definitions, we can say when a preference relation admits a rank-based salience theory representation.

Definition 8 *A preference set* Π *admits a rank-based salience representation if there exist a salience function* $\sigma : \mathbb{R}^2 \to \mathbb{R}$ *and* $\beta \in (0, 1]$ *such that*

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y) \in \operatorname{supp} \pi} (x - y) \beta^{\hat{\sigma}_\pi (x,y)} \pi (x, y) \geq 0. \tag{13}$$

Since $\beta \leq 1$, and $\hat{\sigma}_\pi$ is decreasing in the salience of a pair of outcomes, the decision criterion is overweighting the most salient joint realizations. Therefore, this criterion has the advantage of suggesting the main features of a salience-sensitive DM: she probabilistically aggregates the difference between what is paid by the two alternatives, with additional weight given to salient pairs of rewards. Notice that if $\beta = 1$, the agent is a risk-neutral EU maximizer.

D.1 Weakness

Rank-based salience theory captures the idea that the distortion in evaluating an event depends only on its relative salience. Hence, small perturbations in the amount paid in a state can dramatically change its evaluation. As outlined above, this decision criterion is intransitive, and it does not satisfy the weaker axiom of Transitive Consistency.[24] For a joint distribution π define the conditional row distribution of π given $y \in \operatorname{supp} \pi_2$ as

$$\pi_y(x) = \frac{\pi(x, y)}{\sum_{x \in X} \pi(x, y)}.$$

Axiom 18 (Transitive Consistency) *Let π, χ be such that $\pi_2 = \chi_2$ and for all $y \in \operatorname{supp} \pi_2$*

$$\pi_y \geq_{FOSD} \chi_y$$

then $\chi \in \Pi$ implies $\pi \in \Pi$.

Transitive Consistency is a minimum rationality requirement imposed on an intransitive DM. The underlying idea is that, under the joint distribution π, the row marginal has been improved *conditional* on every possible realization of the column marginal. This implies that $\pi_1 \geq_{FOSD} \chi_1$, and Transitive Consistency is satisfied both by regret theory and the smooth salience theory.

The following example illustrates the possible transitive inconsistencies of rank-based salience theory.

Example 5 *Let π and χ be*

π	5	11.5
7	1/3	0
9	0	2/3

χ	5	11.5
7	1/3	0
8.8	0	1/3
9	0	1/3

[24] For an in-depth analysis of Transitive Consistency, see Cerreia-Vioglio and Ok (2018), and Nishimura and Ok (2018). For examples of intransitive binary relations satisfying this axiom, see Cerreia-Vioglio, Giarlotta, Greco, Maccheroni, and Marinacci (2020). Se also Kontek (2016) for a related critique of the rank-based model.

Suppose that we use the leading example of salience function proposed in BGS

$$\sigma(x,y)=\frac{|x-y|}{|x|+|y|}$$

and we set $\beta=1/2$. *We obtain* $\sigma(7,5)>\sigma(9,11.5)$. *Therefore* $\pi\notin\Pi$ *since*

$$\frac{1}{3}[7-5]+\beta\frac{2}{3}[9-11.5]=\frac{1}{3}\cdot 2-\frac{1}{2}\cdot\frac{2}{3}\cdot 2.5<0.$$

On the other hand, $\chi\in\Pi$ *since*

$$\sigma(7,5)>\sigma(8.8,11.5)>\sigma(9,11.5).$$

and

$$\begin{aligned}&\frac{1}{3}[7-5]+\beta\frac{1}{3}[8.8-11.5]+\beta^2\frac{1}{3}[9-11.5]\\ =\;&\frac{1}{3}\cdot 2-\frac{1}{2}\cdot\frac{1}{3}\cdot 2.7-\frac{1}{4}\cdot\frac{1}{3}\cdot 2.5>0.\end{aligned}$$

E Minor Proofs

Proof of Lemma 1 Let $p\succeq q$. Then, if $\pi\in\Delta(X\times X)$ and $(\pi_1,\pi_2)=(p,q)$, $\pi\in\Pi_{\succeq}$ by definition of $\Pi_{\succeq}$. However, since π was an arbitrary joint lottery with marginals p and q, by definition of $\succsim^{\Pi_{\succeq}}$, we have $p\succsim^{\Pi_{\succeq}}q$.

Let $p\succsim^{\Pi_{\succeq}}q$. Then, by definition of $\succsim^{\Pi_{\succeq}}$, $p\times q\in\Pi_{\succeq}$. But by definition of $\Pi_{\succeq}$ this means that $p\succeq q$. ■

Proof of Lemma 2 (1) Let $\pi\in\Delta(X\times X)$. Since $\succsim$ satisfies Classic Completeness, at least one between $\pi_1\succsim\pi_2$ and $\pi_2\succsim\pi_1$ holds. By definition of $\Pi_{\succsim}$ this implies that at least one between $\pi\in\Pi$ and $\bar{\pi}\in\Pi$ holds.

(2) Let $\pi\in\Delta(X\times X)$. Since $\succsim^{\Pi}$ satisfies Classic Completeness at least one between $\pi_1\succsim^{\Pi}\pi_2$ and $\pi_2\succsim^{\Pi}\pi_1$ holds, and this implies that at least one between $\pi\in\Pi$ and $\bar{\pi}\in\Pi$ holds. ■

Proof of Remark 4 Let $\pi, \chi \in \Pi_{\succsim}$ (resp. $\chi \in \hat{\Pi}_{\succsim}$) and $\lambda \in (0,1)$. By definition of $\Pi_{\succsim}$, $\pi_1 \succsim \pi_2$ and $\chi_1 \succsim \chi_2$ (resp. $\chi_1 \succ \chi_2$). Since $\succsim$ satisfies Classic Strong B-Independence, $\lambda\pi_1 + (1-\lambda)\chi_1 \succsim \lambda\pi_2 + (1-\lambda)\chi_2$ (resp. $\lambda\pi_1 + (1-\lambda)\chi_1 \succ \lambda\pi_2 + (1-\lambda)\chi_2$), and by definition of $\Pi_{\succsim}$, we have $\lambda\pi + (1-\lambda)\chi \in \Pi_{\succsim}$ (resp. $\lambda\pi + (1-\lambda)\chi \in \hat{\Pi}_{\succsim}$). ■

Proof of Lemma 3 Let $p, q, r, s \in \Delta(X)$ be such that $p \succ q$ and $r \not\succsim s$. We first show that there exists $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) s$. Define $r_n = \left(1 - \frac{1}{n}\right) p + \frac{1}{n} r$ and $q_n = \left(1 - \frac{1}{n}\right) q + \frac{1}{n} s$. If $r_n \succ q_n$ for some $n \in \mathbb{N}$, the result follows by setting $\alpha = 1 - \frac{1}{n}$. Otherwise, by Classic Completeness of $\succsim$, we have $q_n \succsim r_n$ for all $n \in \mathbb{N}$, but by Classic Sequential Continuity this implies that $\lim_n q_n = q \succsim \lim_n r_n = p$, a contradiction.

The existence of $\beta \in (0,1)$ such that $\beta p + (1-\beta) r \not\succsim \beta q + (1-\beta) s$ follows from the first part and noticing that under Classic Completeness $r \not\succsim s \iff s \succ r$ and $\beta p + (1-\beta) r \not\succsim \beta q + (1-\beta) s \iff \beta q + (1-\beta) s \succ \beta p + (1-\beta) r$. ■

Proof of Remark 5 Let $\pi \in \hat{\Pi}_{\succsim}$ and $\chi \notin \Pi_{\succsim}$. By definition of $\Pi_{\succsim}$, this means that $\pi_1 \succ \pi_2$ and $\chi_1 \not\succsim \chi_2$. But then, by Classic Archimedean B-Continuity, there exists $\alpha \in (0,1)$ and $\beta \in (0,1)$ such that $\alpha\pi_1 + (1-\alpha)\chi_1 \succ \alpha\pi_2 + (1-\alpha)\chi_2$ and $\beta\pi_1 + (1-\beta)\chi_1 \not\succsim \beta\pi_2 + (1-\beta)\chi_2$. By definition of $\Pi_{\succsim}$, this means that $\alpha\pi + (1-\alpha)\chi \in \hat{\Pi}_{\succsim}$ and $\beta\pi + (1-\beta)\chi \notin \Pi_{\succsim}$. ■

Proof of Lemma 4

(1) Let $\pi, \chi, \rho \in \Delta(X \times X)$, with $\pi_2 = \chi_1$, $\rho_1 = \pi_1$, and $\rho_2 = \chi_2$, and $\pi \in \Pi_{\succsim}, \chi \in \Pi_{\succsim}$. By definition of $\Pi_{\succsim}$, we have $\rho_1 = \pi_1 \succsim \pi_2 = \chi_1$ and $\chi_1 \succsim \chi_2 = \rho_2$. Since $\succsim$ is transitive, this implies that $\rho_1 \succsim \rho_2$, and by definition of $\Pi_{\succsim}$, we have $\rho \in \Pi_{\succsim}$.

(2) Let $p, q, r \in \Delta(X)$ with $p \succsim^{\Pi} q$ and $q \succsim^{\Pi} r$. Let $\pi = p \times q$, $\chi = q \times r$, and let ρ be such that $\rho_1 = p$ and $\rho_2 = r$. Then π,$\chi \in \Pi$ by definition of $\succsim^{\Pi}$, and $\rho \in \Pi$ by Transitivity of Π. Since ρ was chosen arbitrarily among the joint lotteries with marginals p and r, $p \succsim^{\Pi} r$, and the result follows.

(3) ($\succsim^{\Pi}$ *satisfies Classic Transitivity and Classic Completeness*$\Rightarrow$ Π *satisfies Transitivity and Completeness*) Let $\pi, \chi, \rho \in \Delta(X \times X)$, with $\pi_2 = \chi_1$, $\rho_1 = \pi_1$,

and $\rho_2 = \chi_2$, and $\pi \in \Pi, \chi \in \Pi$. Then, Classic Completeness of $\succsim^\Pi$ implies that $\pi_1 \succsim^\Pi \pi_2 \succsim^\Pi \chi_2$, and Classic Transitivity of $\succsim^\Pi$ implies $\rho_1 \succsim^\Pi \rho_2$, and the definition of $\succsim^\Pi$ implies $\rho \in \Pi$, that is Π satisfies Transitivity. Moreover, Π satisfies Completeness by Lemma 2.

(Π *satisfies Transitivity and Completeness*$\Rightarrow \succsim^\Pi$ *satisfies Classic Transitivity and Classic Completeness*) That $\succsim^\Pi$ satisfies Classic Transitivity follows from the part (2). For Classic Completeness, let $p, q \in \Delta(X)$. Define π as the product measure $\pi = p \times q \in \Delta(X \times X)$. By Completeness of Π, either $\pi \in \Pi$ or $\bar{\pi} \in \Pi$. If $\pi \in \Pi$, let $\rho \in \Delta(X \times X)$ be an arbitrary element of $\Delta(X \times X)$ such that $\rho_1 = p$ and $\rho_2 = q$, and define $\chi = q \times q$. By Completeness, $\chi \in \Pi$, and by Transitivity $\pi \in \Pi$ and $\chi \in \Pi$ together imply that $\rho \in \Pi$. Since ρ was chosen arbitrarily among the joint lotteries with marginals p and q, $p \succsim^\Pi q$. Suppose $\bar{\pi} = q \times p \in \Pi$. Let $\rho \in \Delta(X \times X)$ be an arbitrary element of $\Delta(X \times X)$ such that $\rho_1 = q$ and $\rho_2 = p$, and define $\chi = p \times p$. By Completeness, $\chi \in \Pi$, and by Transitivity $\bar{\pi} \in \Pi$ and $\chi \in \Pi$ together imply that $\rho \in \Pi$. Since ρ was chosen arbitrarily among the joint lotteries with marginals q and p, $q \succsim^\Pi p$. Therefore, $\succsim^\Pi$ satisfies Classic Completeness. ■