Speed versus Resilience in Contagion

Andrew Koh
MIT

Stephen Morris
MIT

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Motivation

- Threshold contagion: Infected IFF $\geq q$ proportion of my neighbours infected
  - Pretty well-studied (Econ, CS, OR, Sociology...)

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  - Speed (conditional on spreading, how quickly?)

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  - Specific to threshold contagion
  - Exception: Centola and Macy (2007): simple vs complex contagion, tension in re-wiring simulations, will discuss later

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Key intuition: many faraway (weak) links means fewer local (strong) links (in proportion)

- This paper: **simple(st) model** which channels this?

Model

- Develop continuous network model in 1 dimension ($\mathbb{R}$)
- Analytically more tractable (c.f. global games)
- Deterministic dynamics but approximates large random graphs (Lovász)
- Link to Watts-Strogatz, Newman, Newman-Watts etc. models of random graphs (but also important differences. More on this later.)
- Other extensions: many dimensions, less structure on neighborhoods
- I am counting on network people to tell me (i) what generalizations are or aren't important; (ii) how to test

Results

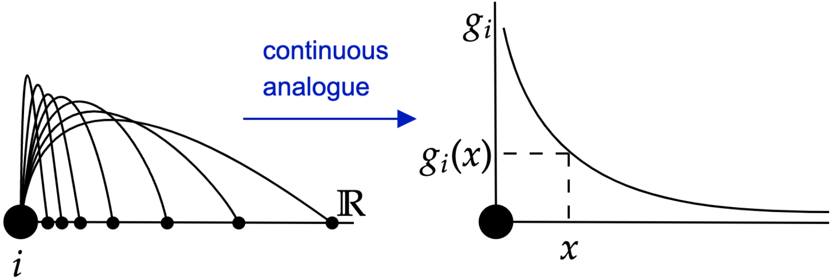
- **Tradeoff**: contagion is **faster** in networks where it is **harder** to initiate contagion

Results

- **Tradeoff**: contagion is **faster** in networks where it is **harder** to initiate contagion
- 3 simple propositions. Tradeoff...
  - **...is stark** when graphs ordered by FOSD: for every threshold, one is harder to initiate contagion & spreads faster;
  - ...**can be absent** for fixed thresholds under 'single-crossing' ordering: worst of both worlds—easier to initiate contagion, but also spreads faster
  - **...always occurs** for any 2 graphs (but potentially different contagion thresholds);

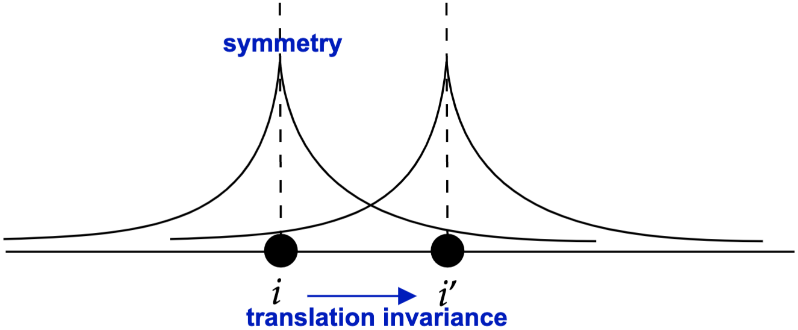
Model

- infinite measure of agents indexed $i \in \mathbb{R}$ sitting on a line. Each agent has links of measure 1 (normalization).
- i's links to the right given by $G_i : \mathbb{R}_+ \to [0, 1/2]$ ('CDF')
  - $G_i(x)$ is the weight i places on agents at location $[i, i + x]$.
- Assume:
  - G_i admits a density $g_i : \mathbb{R}_+ \to [0, 1]$ ('$g(x)$ is the weight placed on agent x away')
  - G_i has decreasing differences i.e., g strictly decreasing on support ('homophily')



Model

- Impose **symmetry** and **translation invariance**
  - **symmetry**: i's links on the left and right distributed identically
  - **translation invariance:** i's distribution of links over $[i, i + x]$ same as that of i' over $[i', i' + x]$



- Work with (single) G directly (sufficient statistic) rather than with explicit description of neighbourhoods

Model

- Binary action space: $\{0, 1\}$ 'uninfected or infected'

q-contagion

Player i takes action 1 if and only if $\geq q$ proportion of her neighbours take action 1.

- We will study the **evolution** of the set of infected agents.
- Let $I_t \subseteq \mathbb{R}$ be the set of agent infected at t
- i is infected at time $t + 1$ if and only if

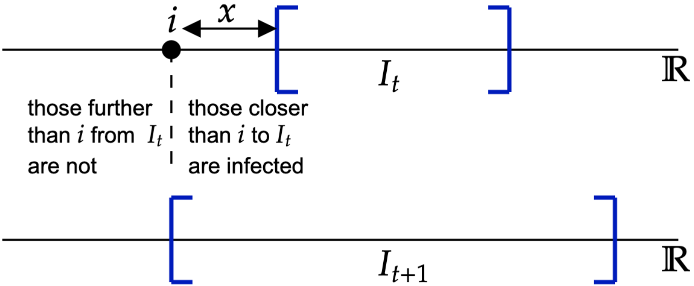
$$\int_{-\infty}^{+\infty} 1[i - x \in I_t] g(x) dx \geq q$$

  - Kind of unwieldy carrying an infinite-dimensional object around

Key simplification → interval contagion

- Key simplifications: I_0 interval + g strictly decreasing
- Implies $\{I_t\}_t$ are intervals (immediately from induction)

$$G(\mu(I_0) + x) - G(x) = q$$



- Discussion:
  - I_0 is not minimal to induce contagion. (but can get LB)
  - Homophily (at least in expectation) seems natural.

Defining resilience & speed

- Let's keep track of measure of infected agents rather than sets.
  - **Define** $m_t = \mu(I_t) \leftarrow$ measure of infected agents at t
  - **Define** $a_t = m_t - m_{t-1} \leftarrow$ change in measure from $t-1$ to t

Definition (Contagion occuring)

Contagion occurs if $\lim_{t\to+\infty} m_t = +\infty$

Definition (Resilience)

$m_0(G,q) := \inf m_0 \quad s.t. \quad \text{contagion occurs}$

- Note: could also fix m_0 and look at max threshold q

Definition ((Limit) Speed)

$a_\infty(G,q) := \lim_{t\to\infty} a_t$

Expression for m_0

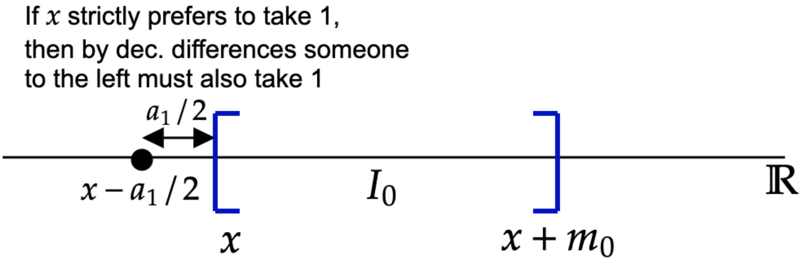
- Tuple (G, m_0, q) sufficient to pin down contagion dynamics.

Lemma

Contagion occurs from (G, m_0, q) if and only if $G(m_0) > q$

- Sketch: ($\Longleftarrow$)Take $I_0 = [x, x + m_0]$. WLOG b/c translation invariance. Let $a_1/2$ be additional measure infected at $t = 1$ on the left of I_0 and conjecture that > 0
- We know that the guy $x - a_1/2$ must be ‘indifferent’:

$$G(a_1/2 + m_0) - G(a_1/2) = q < G(m_0)$$



Expression for m_0

Lemma

Contagion occurs from (G, m_0, q) if and only if $G(m_0) > q$

- Sketch: ($\Longrightarrow$) Consider any $i < x$ and define $\epsilon := x - i$. If the condition on G doesn't hold,

$$q \geq G(m_0) > G(\epsilon + m_0) - G(\epsilon) \text{ because} g \text{ is decreasing.}$$

- But then any $i < x$ is not infected in period 1. So $m_{t+1} \leq m_t$ and $\lim_{t\to\infty} m_t \leq m_0 < +\infty$
- **Useful expression from the lemma:**

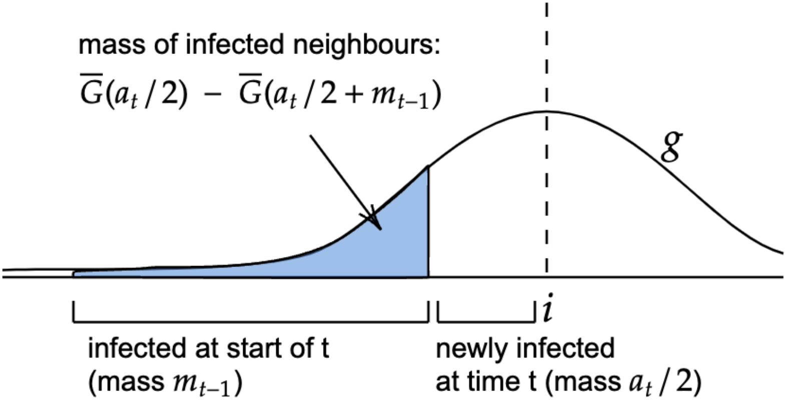
$$\boxed{m_0(G, q) = G^{-1}(q)} \quad \textbf{(Min-Seed)}$$

- Interpretation: **Local links** matter for resilience
  - $G(x)$ is the 'CDF' of links: 'what proportion of my links are less than x distance away from me?'
  - $G^{-1}(q)$: 'what's the distance of guys around me which needs to be infected before q proportion of my neighbours are infected?'

Expression for a_∞

- Recall: $a_t = m_t - m_{t-1}$
- Define $\overline{G} := 1/2 - G$ as the anti-CDF
- If infection occurs, a_t solve the nonlinear diff. eqn.

$$\overline{G}(a_t/2) - \overline{G}(a_t/2 + m_{t-1}) = q$$



Expression for a_∞

$$\boxed{\overline{G}(a_t/2) - \overline{G}(a_t/2 + m_{t-1}) = q}$$

- Path $\{a_t\}_t$ will, in general, depend on G.
- Observe that if contagion occurs, a_t **will be increasing.**
  - from decreasing differences since $m_{t-1} = \sum_0^{t-1} a_s$ is growing
- But will converge to a limit:

$$\begin{aligned} a_\infty &:= \lim_{t\to\infty} a_t \\ &= \boxed{2(\overline{G})^{-1}(q)} \quad \text{since} \quad \lim_{m_{t-1}\to\infty} \overline{G}(a_t/2 + m_{t-1}) = 0. \end{aligned}$$

(Lim-Speed)

- Interpretation: **Distant links** matter for speed.
  - $\overline{G}^{-1}(q)$ is the distance 'from the interval stretching to infinity' required to have q proportion of neighbours
  - Contrast with **Min-Seed:** $\boxed{m_0(G, q) = G^{-1}(q)}$

Tradeoff is **stark** when FOSD-ordered

Proposition

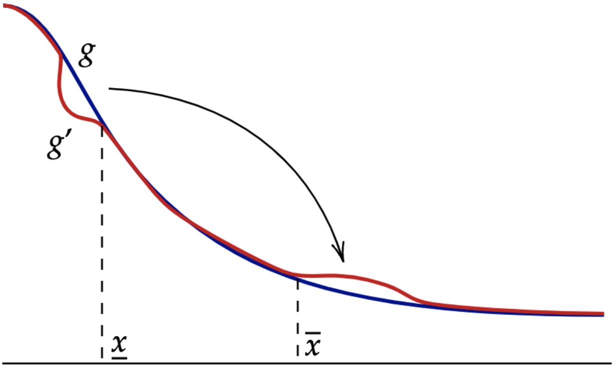
If $G, G' \in \mathcal{G}$ are such that $G \leq G'$, then for all $q \in (0, 1/2)$,

(i) *G is more resilient than G' i.e., $m_0(G, q) \geq m_0(G', q)$; and*

(ii) *G has a quicker limit speed than G' i.e., $a_\infty(G, q) \geq a_\infty(G', q)$.*

Tradeoff is **stark** when FOSD-ordered

- Moving from G to G' as if we're 'shifting mass' from nearby links (closer than $\underline{x}$) further away (beyond $\overline{x}$).



Tradeoff is **stark** when FOSD-ordered

- **Proof immediate from expressions...**
- Resilience: Fix any $q \in (0, 1/2)$. $G \leq G'$ implies $G^{-1} \leq G'^{-1}$. Hence from our expression for m_0

$$\begin{aligned} m_0(G, q) &= G^{-1}(q) \\ &\leq G'^{-1}(q) = m_0(G', q). \end{aligned}$$

- Speed: If contagion doesn't occur, speed is identically zero. If it does, $G \leq G'$ implies $\overline{G} \geq \overline{G}'$ and so $(\overline{G})^{-1} \geq (\overline{G}')^{-1}$ and from our expression for a_∞,

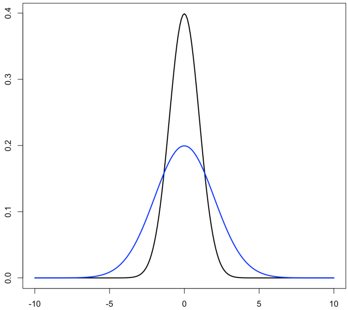
$$a_\infty(G, q) = 2(\overline{G})^{-1}(q) \geq 2(\overline{G'})^{-1}(q) = a_\infty(G', q).$$

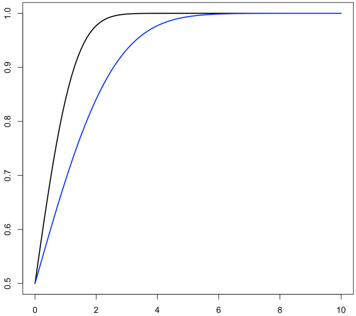
Example: normal distributions

- Suppose that i's links are 'normally distributed' (i is sitting in the middle so i's links are $\sim N(i, \sigma^2)$)
- $G_\sigma(x) = \Phi(x/\sigma) - \frac{1}{2}$
- Invert and rearrange...

$$\boxed{m_0(G_\sigma, q) = \sigma \cdot \Phi(q + \frac{1}{2})} \quad \boxed{a_\infty(G_\sigma, q) = 2\sigma \cdot \Phi(1 - q)}$$

- High σ: more mass on faraway links.





Other ways to shift mass...

- Increasing the s.d. of normal is quite special...
- We've seen that local links matter for resilience; faraway links matter for speed.
- These two things can coexist by shifting 'middle links':
  - Closer: more local links, less resilient
  - Further: more tail links, quicker speed

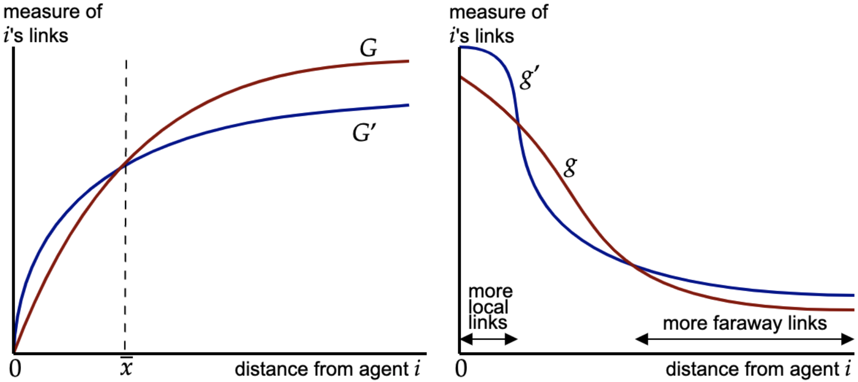
Proposition

For $G, G' \in \mathcal{G}$, suppose that there exists some $\bar{x} \in (0, +\infty)$ such that for all $x' \leq \bar{x}$, $G(x') \leq G'(x')$ and for all $x'' \geq \bar{x}$, $G(x'') \geq G'(x'')$. Then

(i) *for sufficiently low values of q, G is both more resilient than G' as well as has slower limit speeds; and*

(ii) *for sufficiently high values of q, G is both less resilient than G' as well as has quicker limit speeds.*

Other ways to shift mass...



- Here $G' \geq G$ before $\bar{x}$, and the opposite after $\bar{x}$. Implies that G' has more local links but also fatter tails
- Note that q controls 'how local' and 'how far away' the links need to be for them to matter for resilience and speed
  - lower $q \rightarrow$ more remote tails matter, more local neighbourhoods matter

Other ways to shift mass...

- **Sketch**: Choose $q' = G(\overline{x})$ and note that by the condition of single crossing at $\overline{x}$ in the proposition, for any $q \leq q'$,

$$m_0(G, q) = G^{-1}(q) \\ \geq G'^{-1}(q) = m_0(G', q).$$

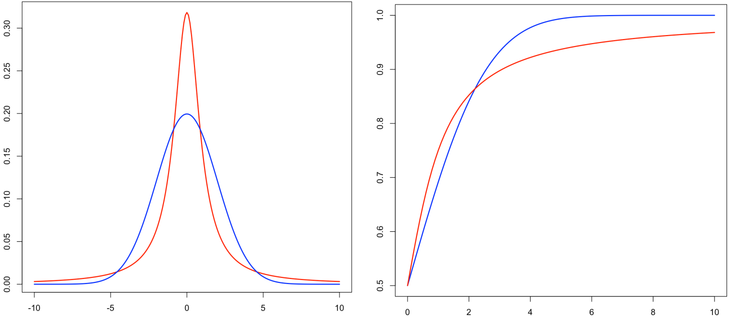
with the reverse equality for $q \geq q'$. choose $q'' = \overline{G}(\overline{x})$ and for $q \leq q''$ by the condition in the proposition,

$$a_\infty(G, q) = 2\overline{G}^{-1}(q) \\ \leq 2\overline{G'}^{-1}(q) = a_\infty(G', q)$$

with the reverse equality for $q \geq q''$. Part (i) follows for thresholds $q \leq q' \wedge q''$; part (ii) follows for thresholds $q \geq q' \vee q''$.

Other ways to shift mass...

- Cauchy: $G_{C,\gamma}(x) = \frac{1}{\pi}\text{arctan}(x/\gamma)$
  - $m_0(G_{C,\gamma}, q) = \gamma \cdot \tan(q\pi)$, $a_\infty(G_{C,\gamma}, q) = 2\gamma \cdot \tan((\frac{1}{2} - q)\pi)$



- Cauchy tails decay polynomially ($\propto 1/x^2$) hence $a_\infty \simeq 1/q^2$. Subgaussian: $a_\infty \lesssim (\log(1/q))^{1/2}$
- More generally, always have freedom to control:
  - tails: sub-exponential, heavy tailed, polynomial decay etc.
  - local 'peakedness'

Tradeoff obtains for any pair of networks

- Note: need $q < 1/2$ for contagion to occur; same logic as Morris (2000)

Proposition

For $G, G' \in \mathcal{G}$, if $G \neq G'$ then there exists $q, q' \in (0, 1/2)$ such that one is more resilient than the other under q, but has a quicker limit speed than the other under q'.

- Any two graphs exhibit the tradeoff for some contagion thresholds
- E.g., can find $q = 0.3,\ q' = 0.1$ so that
  - $m_0(G, 0.1) > m_0(G', 0.1) \leftarrow$ G is more resilient than G'
  - $a_\infty(G, 0.3) > a_\infty(G', 0.3) \leftarrow$ G spreads faster than G'

Tradeoff obtains for **any pair** of networks

- **Sketch pf.**: let's assume WLOG that $G(x) < G'(x)$ for some $x \in [0, \infty)$. This implies that there exists $y \in (G(x), G'(x))$ such that $G^{-1}(y) > x > G'^{-1}(y)$. Now set $q = y < 1/2$ and by the expressions

$$\begin{aligned} m_0(G, q) = G^{-1}(q) &= G^{-1}(y) \\ &> G'^{-1}(y) = G'^{-1}(q) = m_0(G', q). \end{aligned}$$

- Next, recall we defined $\overline{G} = 1/2 - G$. There exists $z \in (\overline{G}'(x), \overline{G}(x))$ such that $\overline{G'}^{-1}(z) < x < \overline{G}^{-1}(z)$ and setting $q' = z < 1/2$, we have

$$\begin{aligned} a_\infty(G, q') = \overline{G}^{-1}(q') &= \overline{G}^{-1}(z) \\ &> \overline{G'}^{-1}(z) = \overline{G'}^{-1}(q') = a_\infty(G', q'). \end{aligned}$$

Taking stock

- Tradeoff...
  - **...is stark** when graphs ordered by FOSD: for every threshold, one is harder to initiate contagion & spreads faster;
  - ...**can be absent** for fixed thresholds under 'single-crossing' ordering: worst of both worlds—easier to initiate contagion, but also spreads faster
  - **...always occurs** for any 2 graphs (but potentially different contagion thresholds);
- Extension 1: What is the link with random (discrete) graphs?
  - Scale & truncate model so that bounded measure. Graphons approximate contagion dynamics of discrete random graph sampled from it Lovász (2012)Erol et al. (2020)
- Extension 2: Higher dimensions vs 1D
  - analytically quite ugly, but some results go through. We may simulate the rest....

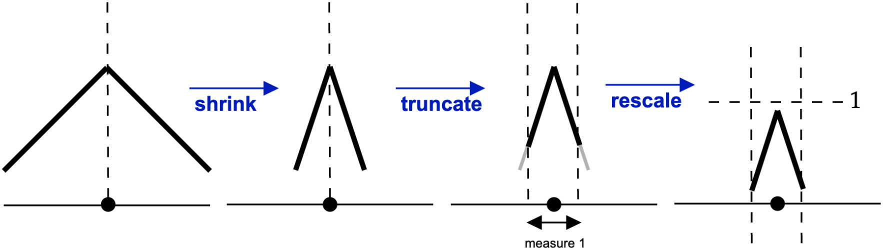
(Informal) Link to finite mass population random discrete graphs

- So far we worked on $\mathbb{R}$. Allows us to speak of 'limit speed', work with canonical full-support distributions etc.
- Now: Unit measure of agents $i \in [0, 1]$.
  - Allows us to link contagion results to discrete random graphs: sample uniformly from $[0, 1]$ (see Lovász (2012))
- Define $\mathcal{G}$ as was our space of graphs on $\mathbb{R}$. Define $\mathcal{G}^T$ as the space of graphs over the unit circle $[0, 1]$ with similar conditions (density exists, homophily etc.)
- **Goal**: define a transformation $\mathcal{G} \to \mathcal{G}^T$ which preserves contagion dynamics (& tradeoffs) studied in $\mathbb{R}$.

Link to random discrete graphs

Goal: find map ψ_s: $\mathcal{G} \to \mathcal{G}^T$ which 'preserves contagion dynamics'

- Here's the map we use:



- Let ψ_s be this map, where s controls the shrinkage factor: (in 1st step $G_s(sx) = G(x)$)

$\lim_{t\to\infty} \lim_{s\to 0} a_t(\psi_s(G), q) = s \cdot a_\infty(G, q)$	$\lim_{s\to 0} m_0(\psi_s(G), q) = s \cdot m_0(G, q)$

Link to random discrete graphs

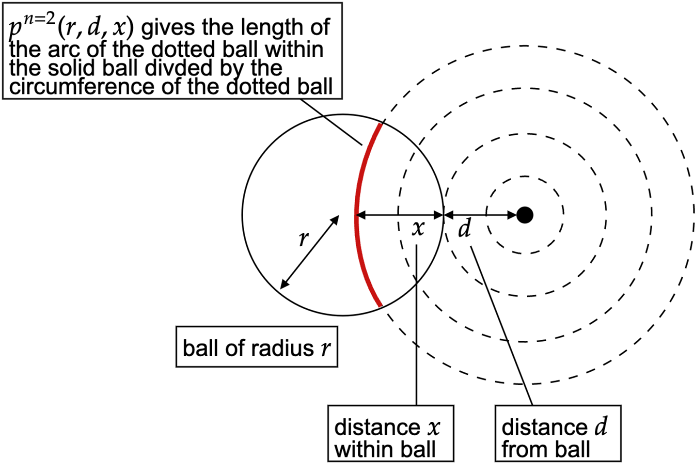
- For small but finite s, exhibit the same tradeoffs.
  - Now with avg speed (up to full infected) rather than limit speed.
  - Could work directly on $[0,1]$ but uglier.
- Taking stock:

$$\boxed{\infty \text{ measure}} \underbrace{\longrightarrow}_{\checkmark} \boxed{\text{Unit measure}} \underbrace{\longrightarrow}_{\text{now}} \boxed{\text{Random graphs}}$$

- Graphs in $\mathcal{G}^T$ are graphons: $W : [0,1]^2 \to [0,1]$.
- $W(i,j)$: weight that i puts on j. In our setting: $W(i,j) = g(|i-j|) = W(j,i)$.
- Graphons approximate random graphs:
  - Sample S from $[0,1]$ uniformly at random.
  - Let's say $S = \{i,j\}$. Then on the random graph, i and j are connected with probability $W(i,j)$.
- Recent paper in JET by Erol, Parise, and Teytelboym (2020): contagion on graphons approximate contagion on sampled graph

Higher dimensions

- General idea is to work on the Euclidian ball in $\mathbb{R}^n$. Analog of translation invariance and symmetry s.t. graph can once again be summarized by a single CDF
- But now the dimension, size of the ball, and distance all matter!



Conclusion

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Conclusion

- **Tradeoff**: contagion is **faster** in networks where it is **harder** to initiate contagion
- clear empirical implications
- special model but clearly generalizes: what is the right way to do so to make it compelling to network theorists?

Thanks!

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