ECONOMIC THEORY DISCUSSION PAPERS

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Trade and Almost Common Knowledge

by Stephen Morris*

Department of Economics,
University of Pennsylvania.
and
Department of Applied Economics,
University of Cambridge.

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Abstract: Suppose there is some transaction cost for trading, an upper bound on any individual's conceivable gains from trade and no ex ante gains from trade. Then, for p sufficiently close to one, it cannot be the case that every iterated statement of the form, "everyone believes with probability at least p that everyone believes with probability at least p.... that everyone is prepared to accept the trade," is true. In the language of Monderer and Samet (1989), there can be no common p-belief trade. On the other hand, for any p, it is possible that every iterated statement of the form "A believes with probability at least p that B believes with probability at least p... that everyone is prepared to accept the trade," is true. In the language of this paper, iterated p-belief trade is possible, for any p. These results cast further light on the important question of the robustness of economic models to their common knowledge assumptions.

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existence of bubbles in rational expectations equilibria depends on the depth of knowledge of an information system, the size of the bubble is bounded if there is common p-belief of some information system with a sufficiently low depth of knowledge.

I will present an approximate no trade theorem under common p-belief. Suppose that there is some small but strictly positive transaction cost associated with engaging in a trade. Fix a proposed trade among a group of individuals. If there is common p-belief among this group of individuals that they are prepared to accept the trade, then the transaction cost can be no greater than 1-p times the maximum possible gain from trade of any individual. Sonsino (1993) independently and earlier proved essentially the same result.

But it turns out that the above result depends crucially on the exact definition of "common almost knowledge". Consider the following alternative notion. Say that event E is "iterated p-belief" if statements of the form "i_1 p-believes that i_2 p-believes that.... E" are true, for every finite sequence of individuals, i_1, i_2,.. i_n. This is typically a less stringent requirement than common p-belief, and indeed we show that it is possible to construct examples where an event is iterated p-belief, for any p strictly less than 1, but not even common 1/2-belief. The approximate no trade theorem outlined above does not hold for iterated p-belief. In particular, for any transaction cost strictly less than the maximum possible gain from trade, and for any p strictly less than 1, it is possible to construct an example where there is iterated p-belief of trade.

Given that iterated and common p-belief give different predictions, we have to consider seriously which notion of almost common knowledge to use. For small groups of people, common p-belief seems a natural requirement. But among small groups of people, iterated and common p-belief make similar predictions. Among large groups of people, common p-belief seems like an excessively stringent requirement in many contexts, requiring (among other things) that each individual assign probability p to the event that every single other individual without exception assign probability p to some event.

The paper is organized as follows. Section 2 reviews results about knowledge and common knowledge, emphasizing why "common knowledge" and "iterated knowledge", unlike common p-belief and iterated p-belief, are equivalent. Section 3 reviews Monderer and Samet's work on belief operators and common p-belief, and introduces our notion of iterated p-belief. Section 4 presents Monderer and Samet's approximate agreeing to disagree result, and shows

how it immediately extends to iterated p-belief. Section 5 gives an approximate no trade result for common p-belief, but shows by a counterexample that it does extend not to iterated p-belief.

2: Common and Iterated Knowledge

This section reviews well-known results about common knowledge in a way that will clarify results about beliefs in later sections. Let I be a finite set of individuals, labelled {1,.. , I}, with $I \geq 2$, and let Ω be a finite state space, with typical element ω. For each $i \in I$, let $\mathcal{P}_i$ be a partition of Ω. Write $P_i(\omega)$ for the (unique) element of $\mathcal{P}_i$ containing ω. The partition $\mathcal{P}_i$ is interpreted as individual i's information, so that if the true state is ω, individual i knows only that the true state is an element of $P_i(\omega)$. Thus an individual i knows event C at state ω if $P_i(\omega) \subset C$. Write $K_i(C)$ for the set of states where i knows event C. Thus:-

$$K_i(C) = \{\omega \in \Omega | P_i(\omega) \subset C\}$$

We will consider two natural hierarchical notions of common knowledge. Let f be an indexing function of degree n if f: {1, 2,.. n} $\rightarrow$ I. Write F(n) for the set of indexing functions of degree n. Say that event C is *iterated knowledge* at ω if

$$\omega \in K_{f(1)}(K_{f(2)}..(K_{f(n)}(C))..),\ for\ all\ n \geq 1,\ f \in F(n)$$

That is, every statement of the form "f(1) knows that f(2) knows that... f(n) knows that C" is true at ω. An alternative definition would be the following. Define an "everyone knows" operator as follows:-

$$K_*(C) = \cap_{i \in I} K_i(C)$$

Now say that event C is *common knowledge* at ω if

$$\omega \in K_*^n(C),\ for\ all\ n \geq 1$$

That is, every statement of the form "everyone knows that everyone knows that.... everyone knows that C" is true at ω. Iterated and common knowledge are actually equivalent,

which is why these definitions are often used interchangeably in motivating common knowledge. It will be useful for our purposes, however, to make clear exactly what property of knowledge this equivalence depends on. The following lemma appeared in Morris, Shin and Postlewaite (1993) and is implicit in standard treatments of common knowledge.

LEMMA 1. *Event C is iterated knowledge at ω if and only if event C is common knowledge at ω.*

PROOF. The proof uses only the following distributive property of the knowledge operator:

$$K_i(A) \cap K_i(C) = K_i(A \cap C)$$

This is true since $\omega \in K_i(A \cap C) \Leftrightarrow P_i(\omega) \subset A \cap C \Leftrightarrow P_i(\omega) \subset A$ and $P_i(\omega) \subset C \Leftrightarrow \omega \in K_i(A)$ and $\omega \in K_i(C)$. The proof then proceeds by induction on m. For m=1, $K_*^m(A) = K_*(A)$, and $K_*(A) = \cap_{i \in I} K_i(A)$. Since F(1) consists of all functions f such that $f(1) \in I$, we have:

$$K_*(A) = \bigcap_{f \in F(1)} K_{f(1)}(A).$$

For the inductive step, assume that the statement of the lemma holds for m-1. Then,

$$\begin{aligned} K_*^m(A) &= K_*(K_*^{m-1}(A)) \\ &= K_*\Big[\bigcap_{f \in F(m-1)} K_{f(m-1)}(\cdots K_{f(1)}(A)\cdots)\Big] \\ &= \bigcap_{i \in I} K_i\Big[\bigcap_{f \in F(m-1)} K_{f(m-1)}(\cdots K_{f(1)}(A)\cdots)\Big] \\ &= \bigcap_{i \in I}\ \bigcap_{f \in F(m-1)} K_i(K_{f(m-1)}(\cdots K_{f(1)}(A)\cdots)), \\ &\qquad \textit{by distributive property of } K \\ &= \bigcap_{f \in F(m)} K_{f(m)}(\cdots K_{f(m-1)}(\cdots K_{f(1)}(A)\cdots)) \end{aligned}$$ ■

Much progress in the study of common knowledge has been possible because of the

realization that both these hierarchical notions are equivalent to a fixed point notion of common knowledge. Say that event D is *evident* if, for all $i \in I$, $D = K_i D$. The following theorem is implicit in Aumann (1976).

THEOREM 1. *Event C is common knowledge at ω if and only if there is an evident event D such that* $\omega \in D \subset C$.

3: Common and Iterated p-Belief

In order to discuss belief rather than merely knowledge, we must introduce beliefs on the state space. Let μ be strictly positive probability measure on the finite state space Ω. We write $\mu(\omega)$ for the probability of the singleton event $\{\omega\}$, and $\mu(A|B)$ for the conditional probability of event A, given event B, for any non-empty event B. Now an individual *p-believes* an event C at state ω if the conditional probability of C, given $P_i(\omega)$, is at least p. Thus,

$$B_i^p(C) = \{\omega \in \Omega | \mu(C|P_i(\omega)) \geq p\}$$

Say that event C is *iterated p-belief* at ω if

$$\omega \in B_{f(1)}^p(B_{f(2)}^p .. (B_{f(n)}^p(C))..), \textit{ for all } n \geq 1, f \in F(n)$$

That is, every statement of the form "f(1) p-believes that f(2) p-believes that... f(n) p-believes that C" is true at ω. Define an "everyone p-believes" operator as follows:-

$$B_*^p(C) = \bigcap_{i \in I} B_i^p(C)$$

Monderer and Samet (1989) introduced the following notion of common p-belief. Say that event C is *common p-belief* at ω if

$$\omega \in [B_*^p]^n(C), \textit{ for all } n \geq 1.$$

That is, every statement of the form "everyone p-believes that everyone p-believes that.... everyone p-believes that C" is true at ω.

The equivalence of iterated and common knowledge does *not*, in general, hold with respect to iterated and common p-belief. This is because the belief operator typically fails to satisfy the distributive property that if event A is believed with probability at least p, and event C is believed with probability at least p, then event $A \cap C$ is believed with probability at least p, so that we may have

$$B_i^p(A) \cap B_i^p(C) \neq B_i^p(A \cap C)$$

The following example shows how iterative and common p-belief can differ.

EXAMPLE 1.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{P}_1 = (\ \{1,2\}, \{3\}, \{4\}, \{5, 6\}\ )$

$\mathcal{P}_2 = (\ \{1, 3, 4\}, \{2, 5, 6\}\ )$

There is a uniform prior on the six states. In this example, the event $E = \{1, 2, 3\}$ is iterated p-belief, but not common p-belief, at state 3 for every $1/2 < p \leq 2/3$. To see why observe that for every p in that interval, $B_1^p(E) = \{1, 2, 3\}$; $B_2^p(E) = \{1, 3, 4\}$; $B_2^p(B_1^p(E)) = \{1, 3, 4\}$; and $B_1^p(B_2^p(E)) = \{3, 4\}$. But now since $B_1^p(\{1, 3, 4\}) = \{3,4\}$ and $B_2^p(\{3,4\}) = \{1, 3, 4\}$, we have that E is iterated p-belief at state 3. However $B_\bullet^p(E) = \{1, 3\}$; so $B_1^p(B_\bullet^p(E)) = \{3\}$ and $B_2^p(B_\bullet^p(E)) = \{1, 3, 4\}$, giving $[B_\bullet^p]^2(E) = \{3\}$. Now $B_1^p([B_\bullet^p]^2(E)) = \{3\}$ and $B_2^p([B_\bullet^p]^2(E)) = \emptyset$, giving $[B_\bullet^p]^3(E) = \emptyset$.[2]

But with only two people, common p-belief and iterated p-belief cannot differ much: if an event is common p-belief for p close to 1, then it is iterated p-belief for some p close to 1. But in the following example with many people, common and iterated p-belief differ

2. Note that with only two individuals, the notions are in fact guaranteed to coincide for the first two levels, i.e., for every event E, $[B_\bullet^p]^2(E) = B_1^p(E) \cap B_2^p(E) \cap B_1^p(B_2^p(E)) \cap B_2^p(B_1^p(E))$. Even this equivalence does not hold for more than two individuals.

substantially. In particular, if we fix any $1 > p \geq r \geq 0.5$, the following example has an event is iterated p-belief, but not common r-belief at some state.

EXAMPLE 2.

Suppose that the economy is either in good condition (G) or in bad condition (B); each event occurs with prior probability 0.5. Choose a probability q such that $q > 0.5$ and $q^2 + (1-q)^2 > p$, and then choose the number of individuals, I, such that $q^I + (1-q)^I < 0.5$. Now suppose each individual hears a good signal (G) or a bad signal (B). The probability that an individual's signal is the same as the condition of the economy is q, and each individual's signal is independent (conditional on the state of the economy) of the others. Formally, we can represent this by a state space,

$$\Omega = \{G,B\}^{I+1}$$

where each state ω is an $I+1$ vector, whose 0th co-ordinate is the state of the economy, and whose ith co-ordinate ($i = 1,I$) is the signal observed by individual i. Write $n(\omega)$ for the number of individuals whose signal is correct, i.e. $n(\omega)$ is the number of elements of $\{\ i \in I \mid \omega_i = \omega_0\ \}$. Then the prior and information partitions are given by:

$$\mu(\omega) = \frac{1}{2}q^{n(\omega)}(1-q)^{I-n(\omega)}$$

$$\mathcal{P}_i = \{G_i, B_i\} = \{\{\omega \in \Omega | \omega_i = G\}, \{\omega \in \Omega | \omega_i = B\}\}$$

Consider the event $G_0 = \{\ \omega \in \Omega \mid \omega_0 = G\ \}$, i.e. the set of states where the economy really is in the good condition. Each individual assigns probability q to this event when he has observed the good signal, 1-q otherwise. Since $q = q^2 + q(1-q) > q^2 + (1-q)^2 > p \geq r \geq 0.5$, we have:-

$$B_i^p(G_0) = B_i^r(G_0) = G_i \quad and \quad B_\bullet^p(G_0) = B_\bullet^r(G_0) = \bigcap_{i \in I} G_i$$

Each individual i also assigns probability $q^2 + (1-q)^2 > p$ to some other individual j observing the same signal as him, but assigns probability $q^I + (1-q)^I$ to all other individuals observing the same signal as him. By choice of I, we have $q^I + (1-q)^I < 0.5 \leq r$, and thus

$$B_i^p(G_j) = B_i^r(G_j) = G_i \quad and \quad B_*^p\left(\bigcap_{i\in I} G_i\right) = B_*^r\left(\bigcap_{i\in I} G_i\right) = \emptyset$$

By induction, we have

$$B_{f(1)}^p(B_{f(2)}^p..(B_{f(n)}^p(G_0))..) = G_{f(1)} \quad and \quad [B_*^r]^n(G_0) = \emptyset, \quad for\ all\ f \in F(n),\ n \geq 2.$$

Now at the two states with $\omega_i = G$, for all $i \in I$, event G_0 is iterated p belief, but not common r-belief.

The example shows that we will need different analogues of the Aumann result for each of the generalizations to belief. Monderer and Samet gave the analogue for common p-belief. Say that event C is *evident p-belief* if $C \subset B_i^p(C)$, for all individuals $i \in I$.

THEOREM 2 [Monderer and Samet (1989)]. *Event C is common p-belief at ω if and only if there exists an evident p-belief event D such that $\omega \in D \subset B_i^p(C)$, for all $i \in I$.*

We can give a much less satisfactory fixed point characterization of iterated p-belief. Say that collection of events $\mathscr{E}$ is *mutual evident p-belief* if $B_i^p(C) \in \mathscr{E}$, for all individuals i and events $C \in \mathscr{E}$. Notice that if $\mathscr{E}$ consists of exactly all sets containing set C, i.e. $\mathscr{E} = \{D \in 2^\Omega | C \subset D\}$, then $\mathscr{E}$ is mutual evident p-belief if and only if C is evident p-belief.

THEOREM 3. *Event C is iterated p-belief at ω if and only if there exist mutual evident p-belief events $\mathscr{E}$ with $\omega \in D$, for all $D \in \mathscr{E}$, and $B_i^p(C) \in \mathscr{E}$, for all $i \in I$.*

PROOF. (if)

$$B_i^p(C) \in \mathscr{E},\ so\ B_{f(1)}^p(B_{f(2)}^p..(B_{f(n)}^p(C))..) \in \mathscr{E},\ for\ all\ n \geq 1,\ f \in F(n).$$
$$So\ \omega \in B_{f(1)}^p(B_{f(2)}^p..(B_{f(n)}^p(C))..),\ for\ all\ n \geq 1,\ f \in F(n).$$

(only if) Suppose C is iterated p-belief at ω. Then the following collection of events are mutual evident p-belief and satisfy the other properties of the theorem.

$$Let\ \mathscr{E} = \left\{ E \subset \Omega \middle| E = B_{f(1)}^p(B_{f(2)}^p..(B_{f(n)}^p(C))..)\ ,\ for\ some\ n \geq 1,\ f \in F(n) \right\} \quad \blacksquare$$

We can show that common p-belief implies iterated p-belief.

LEMMA 2. *If event C is common p-belief at ω, then event C is iterated p-belief at ω.*

PROOF. Observe first that if $A \subset C$, then $B_i^p(A) \subset B_i^p(C)$. Thus for any event D and individual i, $B_*^p(D) \subset B_i^p(D)$

$$So\ [B_*^p]^n C \subset B_{f(1)}^p(B_{f(2)}^p..(B_{f(n)}^p(C))..),\ for\ all\ n \geq 1,\ f \in F(n) \quad \blacksquare$$

In the example, we saw that in order to create a large difference between common and iterative p-belief, a large number of individuals was required. It is presumably possible to bound the difference between common and iterative p-belief as a function of the number of individuals.

4: Agreeing To Disagree

Monderer and Samet proved an approximate version of Aumann's agreeing to disagree result. It is useful to first present these results, and an iterated p-belief extension, because

agreeing to disagree results are closely related to no trade results. The connection is explored in Geanakoplos (1989) and Rubinstein and Wolinsky (1990).

Fix an event $C \subset \Omega$. Then individuals' posterior beliefs about event C are defined by

$$X_i(\omega) = \mu(C|P_i(\omega))$$

Then the event $D = \{ \omega \in \Omega \mid X_i(\omega) = r_i \}$ represents the set of states where each individual assigns probability r_i to event C. Posteriors r of C are common p-belief (knowledge) at ω if the event D is common p-belief (knowledge) at ω. Then we have:-

THEOREM 4 [Aumann (1976)]. *If the posteriors of an event are common knowledge, then they are the same.*

THEOREM 5 [Monderer and Samet (1989), Neeman (1993b)]. *If the posteriors of an event are common p-belief, then any two posteriors can differ by at most 1-p.*[3]

In fact, because the condition on posteriors makes only pairwise comparisons between individuals, theorem 5 extends easily to iterated p-belief.

COROLLARY. *If the posteriors of an event are iterated p-belief, then any two posteriors can differ by at most 1-p.*

PROOF. Suppose the posteriors of all individuals are iterated p-belief among all individuals. Then for any subset of individuals, posteriors of those individuals are iterated p-belief among that subset. Thus, for any pair i and j, their posteriors are iterated p-belief among the pair. But by the corollary to lemma 3, posteriors are thus common p-belief among that pair. But then by theorem 5, they differ by at most 1-p. But this is true for any pair i and j, so the corollary holds

■

3. Monderer and Samet first proved the result for a bound of 2(1-p). Neeman improved the bound to 1-p.

However, this equivalence of common p-belief and iterated p-belief is merely a consequence of the bilateral nature of the condition on posteriors. By contrast, "no trade" results allow for more complex interactions among individuals, and the distinction will become important.

5. Trade

Our purpose here is to give a no trade analogue of Monderer and Samet's agreeing to disagree result and explore the difference between iterated and common p-belief in this context.

Suppose now that individuals must decide whether to accept a trade. Assume the individuals are risk-neutral. A trade will consist of a collection of random variables, $\mathbf{X} = \{X_i\}_{i \in I}$, with each $X_i: \Omega \rightarrow \mathbb{R}$, and, for all $\omega \in \Omega$,

$$\sum_{i \in I} X_i(\omega) \leq 0$$

We will write E (Y | C) for the expectation, under μ, of random variable Y, conditional on event C. Thus

$$E(Y|C) = \frac{\sum_{\omega \in C} \mu(\omega) Y(\omega)}{\sum_{\omega \in C} \mu(\omega)}$$

Then the set of states where individual i would be (strictly) prepared to accept the trade would be:

$$A_i = \{ \omega \in \Omega \mid E(X_i|P_i(\omega)) > 0 \}$$

The event that everyone accepts trade will thus be

$$A = \bigcap_{i \in I} A_i = \{ \omega \in \Omega \mid E(X_i|P_i(\omega)) > 0, \textit{ for all } i \in I \}$$

THEOREM 6. *There is no common knowledge acceptance of trade (i.e. the event A is nowhere common knowledge).*

This is a many person version of the no common knowledge trade result of Sebenius and Geanakoplos (1983). Milgrom and Stokey's (1982) result captures the same essential idea with possibly risk averse individuals. No finite level of knowledge is sufficient for this result. In particular, for any N, it is possible to construct an example where there is Nth order knowledge (but not common knowledge) of acceptance of trade. The following example is closely related to an example in Geanakoplos (1992).

Suppose a number ω between 1 and N is chosen (with equal likelihood). Two individuals observe a noisy version of that number. Individual 1 is told the true value of ω if ω is odd, and is told ω + 1 if ω is even. Individual 2 is told the true value of ω if ω is even, and is told ω + 1 if ω is odd. This gives the following information structure (for N even).

$$\Omega = \{1,2,\ldots,N\}$$
$$\mu = \left\{\frac{1}{N},\frac{1}{N},\ldots,\frac{1}{N}\right\}$$
$$\mathcal{P}_1 = (\{1\},\{2,3\},\ldots,\{N-2,N-1\},\{N\})$$
$$\mathcal{P}_2 = (\{1,2\},\{3,4\},\ldots,\{N-1,N\})$$

Now suppose that a trade specifies that if ω is odd, individual 2 must pay individual 1 ω; whereas if ω is even, individual 1 must pay individual 2 ω. Thus the trade can be written as vectors as follows.

$$X_1 = (1,-2,3,\ldots,-N)$$
$$X_2 = (-1,2,-3,\ldots,N)$$

Now individual 2 will always want to accept the trade ($A_2 = \Omega$) and individual 1 will want to accept the trade unless the state is N ($A_1 = \Omega/\{N\}$). Thus for N very large we can have arbitrarily high order knowledge of acceptance of trade. Specifically, we have

$$K_*^m(A) = \{1,2,\ldots,N-m-1\}, \quad so \bigcap_{m=1}^{N-2} K_*^m(A) = \{1\}$$

Thus there is always (N-2)th order knowledge of acceptance of trade at state 1.

It is also straightforward to construct examples where there is common p-belief of trade (see, for example, Neeman (1993a)). But such examples have the characteristic that either the possible gains from trade must become arbitrarily large as p approaches one, or the expected gains from trade must become arbitrarily small. We can use this property to provide an analogue to Monderer and Samet's no agreeing to disagree approximate common knowledge result. Suppose that each individual had to pay a transaction cost or tax, t > 0, if he accepted the trade. Then the set of states where individual i would be prepared to accept the trade would be:

$$A_i = \{\, \omega \in \Omega \mid E(X_i|P_i(\omega)) \geq t \,\}$$

The event that everyone accepts trade will thus be

$$A = \bigcap_{i \in I} A_i = \{\, \omega \in \Omega \mid E(X_i|P_i(\omega)) \geq t, \textit{ for all } i \in I \,\}$$

Write X^* for the maximum possible value of trade to any individual, i.e.

$$X^* = \max_{i \in I,\, \omega \in \Omega} X_i(\omega) \geq 0$$

THEOREM 7. *If it is common p-belief that trade X is accepted with transaction cost t, then $t \leq (1-p)X^*$.*

Thus for any fixed transaction cost and bound on the gains from trade, there exists some p^* strictly less than one such that there cannot ever be common p-belief of acceptance of trade, for any $p \geq p^*$. The following proof uses a very similar logic to the Monderer and Samet's agreeing to disagree result (theorem 5).

PROOF. Since A is common p-belief, there exists an evident p-belief event C with $\omega \in C$, $C \subset B_i^p(C)$, and $C \subset B_i^p(A) \subset B_i^p(A_i)$ for all $i \in I$. Observe that $\omega' \in P_i(\omega)$ and $\omega' \in A_i$ implies $\omega \in A_i$. Thus $P_i(\omega) \cap A \neq \emptyset$ implies $\omega \in A_i$. Since $\omega \in B_i^p(A)$ implies $P_i(\omega) \cap A \neq \emptyset$, we have $C \subset B_i^p(A) \subset A_i$. Now $\omega \in B_i^p(C)$ implies $P_i(\omega) \cap C \neq \emptyset$ implies $P_i(\omega) \cap A_i \neq \emptyset$. Thus $B_i^p(C) \subset A_i$ and so $E(X_i \mid B_i^p(C)) \geq t$. Now define, for each i in I,

$$x_i = E(X_i|C) \text{ and } \alpha_i = \frac{\mu(C)}{\mu(B_i^p(C))} \geq p$$

$$\text{Then } \mu(B_i^p(C))E(X_i|B_i^p(C)) = \mu(C)x_i + \mu(B_i^p(C)\backslash C)E(X_i|B_i^p(C)\backslash C)$$

$$\begin{aligned}\text{and thus } t \leq E(X_i|B_i^p(C)) &= \frac{\mu(C)}{\mu(B_i^p(C))}x_i + \frac{\mu(B_i^p(C)\backslash C)}{\mu(B_i^p(C))}E(X_i|B_i^p(C)\backslash C) \\ &\leq \alpha_i x_i + (1-\alpha_i)X^* \\ &\leq x_i + (1-p)X^*\end{aligned}$$

$$\text{But now } \sum_{i\in I} X_i(\omega) \leq 0, \text{ for all } \omega\in\Omega \quad \Rightarrow \quad \sum_{i\in I} x_i \leq 0 \quad \Rightarrow \quad t \leq (1-p)X^* \qquad \blacksquare$$

This theorem does not generalize to iterated p-belief. In particular, we will show that for any $t < X^*$ and $0.5 < p < 1$, there exist situations where there is iterated p-belief of acceptance of trade. Without loss of generality, consider trades with $X^* = 1$. Let us describe the example verbally first, and then formally. Each individual may observe two types of signals. First, with probability α, all individuals will observe a *general* signal. Second, with independent probability p, each individual may observe a *personal* signal. Assume that each individual knows only the number of signals he has observed. He does not know what signals others observed and, if he observed one signal, he does not know whether it was general or personal. Choose the number of individuals, I, and α such that

$$p^{I-1}(I+1) < \frac{1-t}{2} \quad \text{and} \quad \frac{(1-p)\alpha}{p(1-\alpha)+(1-p)\alpha} < \frac{1-t}{2(I+1)}$$

If we let ω_i be the number of signals that individual i has observed, a state can be described by the vector $\omega = (\omega_1,\ldots,\omega_I)$. The example then has the following formal representation.

$$\Omega = \left\{ \omega\in\{0,1,2\}^I \;\middle|\; \begin{array}{l} \text{Either } \omega_i \neq 2, \text{ for all } i\in I \\ \text{Or } \omega_i \neq 0, \text{ for all } i\in I \end{array} \right\}$$

Write $n_k(\omega)$ for the number of elements of the set $\{ i\in I \mid \omega_i = k\}$. Then the prior is given by:-

$$\mu(\omega) = \begin{cases} \alpha p^{n_2(\omega)}(1-p)^{1-n_2(\omega)}, & \text{if } n_2(\omega) \neq 0 \\ \alpha(1-p)^I + (1-\alpha)p^I, & \text{if } n_1(\omega) = I \\ (1-\alpha)p^{n_1(\omega)}(1-p)^{I-n_1(\omega)}, & \text{if } n_0(\omega) \neq 0 \end{cases}$$

Individual i's information partition is given by:-

$$\mathcal{P}_i = \{S_i^0, S_i^1, S_i^2\} = \{ \{\omega\in\Omega|\omega_i=0\}, \{\omega\in\Omega|\omega_i=1\}, \{\omega\in\Omega|\omega_i=2\} \}$$

We will be interested in the following trade.

$$X_i(\omega) = \begin{cases} 1, & \begin{array}{l}\text{if either } \omega_i = 2 \text{ and } n_2(\omega) \neq I \\ \quad \text{or } \omega_i = 1 \text{ and } n_0(\omega) \neq 0\end{array} \\ -I, & \begin{array}{l}\text{if either } \omega_i = 1 \text{ and } n_0(\omega) = 0 \\ \quad \text{or } \omega_i = 0\end{array} \\ 0, & \text{if } n_2(\omega) = I \end{cases}$$

Clearly this trade is feasible, since whenever some individual receives 1, at least one other ndividual is losing I. Individuals will be prepared to accept the trade if they have observed 1 r 2 signals, i.e. $A_i = S_i^1 \cup S_i^2$. Observe that when individual i observes one signal, his osterior probability that the general signal has been observed is

$$q = \frac{(1-p)\alpha}{p(1-\alpha) + (1-p)\alpha} < \frac{1-t}{2(I+1)}$$

hus his posterior that all other individuals have observed at least one signal is $q + (1-q)p^{I-1}$. lis expected value of the trade is then

$$\begin{aligned}(1-q-(1-q)p^{I-1})(1) + (q+(1-q)p^{I-1})(-I) &= 1 - (1-q)p^{I-1}(I+1) - q(I+1) \\ &\geq 1 - \frac{1-t}{2} - \frac{1-t}{2} \\ &= t\end{aligned}$$

Similarly, when individual i observes two signals, his posterior probability that someone as not observed the second signal $1-p^{I-1}$. His expected value of the trade is

$$(1-p^{I-1})(1) \geq t$$

Now let us check where there is iterated p-belief and common p-belief of acceptance. irst observe that

$$A = \bigcap_{i \in I} A_i = \{\omega \in \Omega | \omega_i \neq 0, \textit{ for all } i \in I\}$$

Each individual assigns this event probability one when he observes 2 signals, but only robability $q + (1-q)p^{I-1} < p$ when he observes one signal. Thus $B_i^p(A) = S_i^2$, for all $i \in I$. Now if individual j observes 2 signals, he knows that the general signal has been observed, and so assigns probability p to any other individual i observing 2 signals. Thus $B_j^p(B_i^p(A)) = S_j^2$ and, by induction,

$$B^p_{f(1)}(B^p_{f(2)} \cdots (B^p_{f(n)}(A))\ldots) = S^2_{f(1)}, \textit{ for all } n \geq 1, f \in F(n)$$

Thus acceptance of trade is iterated p-belief when all individuals observe two signals, i.e. $\omega_i = 2$, for all $i \in I$.

6. Conclusion

The results in this paper were not in an explicitly strategic framework. We asked whether an individual was prepared to accept a trade on a given information set without taking into account the optimal behavior of other traders. Neeman (1993a) studied the consequences of weakening common knowledge to common p-belief in an explicitly strategic framework. He showed that it was possible to construct trades and strategies for players in a trading game with incomplete information, where, for any given $\epsilon > 0$ and $p < 1$, the ex ante probability of one player's strategy not being a best response to the other's is less than ϵ, and in equilibrium there is common p-belief that each player is rationally accepting the trade.

The results of this paper, however, would extend to such an explicitly strategic framework. When in the framework of this paper, there is "common p-belief of trade", we can re-interpret this as common p-belief of rationality, where each player's strategy is to always accept trade. Then "acceptance of trade" in this paper would correspond, in the explicitly strategic framework, to rationally accepting the trade, while non-acceptance would correspond to irrationally accepting the trade. It is also possible in examples to ensure that the ex ante probability of "irrationality" becomes arbitrarily small. Thus in our example where there was iterated p-belief of trade, for any $p < 1$, the ex ante probability of "not accepting" became arbitrarily small as p tended to one. Thus it is consistent with requiring that the ex ante probability of irrationality be arbitrarily small in an explicitly strategic framework.

The results of this and other papers suggest that the sensitivity of results about financial markets to different kinds of higher order uncertainty is little understood. The key challenge now is to develop applied frameworks for understanding financial markets which (unlike standard models) allow for such subtleties (see Shin (1993) for one such attempt).

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