A Strategic Topology on Information Structures

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a key ingredient of game theory

- the information structure
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- players’ beliefs and higher-order beliefs about the game
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- i.e., what do players believe about the game, what do they believe that others believe, and so on....?
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- the information structure
- players’ beliefs and higher-order beliefs about the game
- i.e., what do players believe about the game, what do they believe that others believe, and so on....?
- the space of all information structures is an interesting mathematical object
we ask: what is the coarsest topology on information structures that generates continuity of equilibrium?
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- important for applied economic questions
an event is approximate common knowledge (Monderer-Samet 89) if (for $p$ close to 1) everyone believes with probability at least $p$ that it is true, everyone believes with probability at least $p$ that everybody believes it with probability at least $p$. Two information structures are close if each assigns high ex ante probability to there being approximate common knowledge (Monderer-Samet 89) that interim (conditional) beliefs are close.
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3. if time:
   3.1 modelling choices
   3.2 applications
   3.3 context / literature
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Setting
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- a base game \( G = (A_i, u_i)_{i \in I} \), where
  - each \( A_i \) is a finite set of actions
  - each \( u_i : A_i \times A_{-i} \times \Theta \rightarrow [-M, M] \) is a bounded payoff function
information: hierarchies of beliefs

- define $(T_i)_{i \in I}$ recursively:
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- given \((\mathcal{T}_i^{m-1})_{i \in I}\) for \(m > 1\), define

\[
\mathcal{T}_i^m \subseteq \mathcal{T}_i^{m-1} \times \Delta \left( I_{\mathcal{T}_i^{m-1}} \times \Theta \right)
\]

as

\[
\left\{ \left( \left( \tau_1^i, \ldots, \tau_i^{m-1} \right), \tau_i^m \right) \middle| \text{marg}_{I_{\mathcal{T}_i^{m-2}} \times \Theta} \tau_i^m = \tau_i^{m-1} \right\}
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  - $\mathcal{T}_i$ is the set of sequences $\tau_i = (\tau_i^m)_m$ such that for $\overline{m} \in \mathbb{N}$, the truncated sequence $(\tau_i^m)_{m \leq \overline{m}}$ belongs to $\overline{\mathcal{T}}_i^m$
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- \(\mathcal{T}_i\) is the set of sequences \(\tau_i = (\tau^m_i)_m\) such that for \(\overline{m} \in \mathbb{N}\), the truncated sequence \((\tau^m_i)_{m \leq \overline{m}}\) belongs to \(\overline{\mathcal{T}}_i\)
- \(\tau_i \in \mathcal{T}_i\) is a type of player \(i\)
versal state space $\Omega = \mathcal{T} \times \Theta$
information: universal state space

- versal state space $\Omega = \mathcal{T} \times \Theta$
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and let $d_\Pi$ be a metric on $\mathcal{T} \times \Theta$ inducing the product topology on $\mathcal{T}$ and discrete topology on $\Theta$
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Mertens-Zamir 85 showed that for each $\tau_i \in \mathcal{T}_i$, there is a unique belief $\tau_i^* \in \Delta (\mathcal{T}_{-i} \times \Theta)$ so that, for all $m \in \mathbb{N}$,

$$\tau_i^m = \text{marg}_{\mathcal{T}_{-i}^{m-1} \times \Theta} (\tau_i^*)$$

and $\tau \rightarrow \tau^*$
common prior assumption: players’ beliefs and higher order beliefs are interim (conditional) beliefs given some common prior distribution of the universal type space
information structures

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- So an information structure is a prior $P \in \Delta (\Omega)$ where there is a version of the conditional probability $P_i : \mathcal{T}_i \rightarrow \Delta (\mathcal{T}_{-i} \times \Theta)$, so that for every $m > 1$,

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now \((G, P)\) is a "game of incomplete information"
we will describe action choices by a decision rule: a measurable map \( \sigma : \mathcal{T} \times \Theta \to \Delta(A) \)
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a decision rule $\sigma$ is $\varepsilon$-obedient if, for each player $i$ and some regular conditional probability $P_i$

$$
\int_{\mathcal{T}_- \times \Theta} \sum_{a_{-i}} \left( u_i (a_i, a_{-i}, \theta) - u_i (a'_i, a_{-i}, \theta) \right) d\sigma \circ P_i (a_i, a_{-i}, \tau_{-i}, \theta | \tau_i) > -\varepsilon \text{ a.s.}
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for all $a_i, a'_i$
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for all $a_i, a'_i$

a decision rule $\sigma$ is belief-invariant if, for each player $i$ and action $a_i \in A_i$, $\sigma (a_i \times A_{-i} | (\tau_i, \tau_{-i}, \theta)) = \sigma (a_i | \tau_i)$ does not depend on $(\tau_{-i}, \theta)$
Definition
a decision rule is an \( \varepsilon \)-belief-invariant correlated equilibrium (\( \varepsilon \)-BIBCE) if it \( \varepsilon \)-obedient and belief invariant

- a BIBCE is a 0-BIBCE
belief-invariant Bayes correlated equilibrium: definition

Definition

A decision rule is an **ε-belief-invariant correlated equilibrium** (ε-BIBCE) if it ε-obedient and belief invariant

- A BIBCE is a 0-BIBCE.
- Subtlety: A belief-invariant decision rule may induce correlation between \(a\) and \(\theta\), but \(a_i\) alone provides \(i\) with no additional information about \((\tau_{-i}, \theta)\).
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- Equivalent to standard "Bayes Nash equilibrium" (BNE) but players can observe correlating devices that are not individually informative about the state and others’ beliefs...
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  1. existence of BIBCE is guaranteed by Stinchcombe (2011) while it is well known that BNE do not without additional restrictions
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- will postpone full motivation of solution concept but note two relevant properties:
  1. existence of BIBCE is guaranteed by Stinchcombe (2011) while it is well known that BNE do not without additional restrictions
  2. allowing information structures with "redundancies" - i.e., multiple types with the same beliefs and higher-order beliefs - makes no difference to the set of equilibrium outcomes
Main Result
universal state space $\Omega = \mathcal{T} \times \Theta$

$p$-belief operator: for every $p \in [0, 1]$, event $E \subseteq \Omega$, define

$$B^p(E) = \{(\tau, \theta) \mid \forall i, \tau_i^*(E_{-i}) \geq p\}$$

where $E_{-i}$ is projection of $E$ on $\mathcal{T}_{-i} \times \Theta$
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- for all $m \in \mathbb{N}$, $[B^p]^m (E)$ is the $m$-fold application of $B^p$
- the set of states where the event $E$ is common $p$-belief is

$$C^p (E) = \cap_{m \in \mathbb{N}} [B^p]^m (E)$$
interim beliefs are close

- an event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true
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- An event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true.
- An information structure is minimal if there is no non-trivial belief-closed subset.
- Observation: Any two distinct minimal information structures are disjoint.
- Nonetheless, we want to talk about whether interim (conditional) beliefs are close across perhaps minimal information structures is a little subtle.
interim beliefs are close

- neighborhood of state:

\[ \mathcal{N}_\varepsilon (\omega) = \{ \omega' \in \Omega \mid d_\Pi (\omega, \omega') < \varepsilon \} \]
interim beliefs are close

- Neighborhood of state:

\[ \mathcal{N}_\varepsilon (\omega) = \{ \omega' \in \Omega \mid d_{\Pi} (\omega, \omega') < \varepsilon \} \]

- \(\varepsilon\)-support of information structures:

\[
\text{supp}_\varepsilon (P) = \bigcup_{\omega \in \Omega : P(\mathcal{N}_\varepsilon (\omega)) > 0} \mathcal{N}_\varepsilon (\omega)
\]
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- **\(\varepsilon\)-support of information structures:**

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- **intersection of information structures’ \(\varepsilon\)-supports:**

\[ \hat{T}_\varepsilon(P, P') = \text{supp}_\varepsilon(P) \cap \text{supp}_\varepsilon(P') \]
Interim beliefs are close

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- Intersection of information structures’ \( \varepsilon \)-supports:
  \[ \hat{T}_\varepsilon(P, P') = \text{supp}_\varepsilon(P) \cap \text{supp}_\varepsilon(P') \]

- This is the set of states where interim beliefs are close
approximate common knowledge topology

- approximate common knowledge distance:

\[
d^{ACK}(P, P') = \inf \left\{ \varepsilon \geq 0 \mid P \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) \geq 1 - \varepsilon, P' \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) \geq 1 - \varepsilon \right\}
\]
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  \[ P' \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) \geq 1 - \varepsilon \]

- if \( d^{ACK}(P, P') \) is small, there is high probability under both information structures that there is approximate common knowledge that interim beliefs are close
Definition
the approximate common knowledge topology is generated by open sets

\[ \left\{ P' \mid d^{ACK}(P, P') \leq \varepsilon \right\} \]

thus \( P^k \rightarrow_{ACK} P \) if \( d^{ACK}(P^k, P) \rightarrow 0 \)
Definition

the approximate common knowledge topology is generated by open sets

\[ \left\{ P' \mid d^{ACK}(P, P') \leq \epsilon \right\} \]

- thus \( P^k \rightarrow^{ACK} P \text{ if } d^{ACK}(P^k, P) \rightarrow 0 \)
- this is a metric topology (shown by constructing a variant of the distance)
our object of interest is the set of *outcomes* in \( \nu \in \Delta (A \times \Theta) \) that can arise in BIBCE.
strategic topology: outcomes

- our object of interest is the set of outcomes in $\nu \in \Delta (A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE decision rule $\sigma : \mathcal{T} \times \Theta \rightarrow \Delta (A)$ and an information structure $P$ will induce an extended outcome $\sigma \circ P \in \Delta (A \times \mathcal{T} \times \Theta)$
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- In particular, a BIBCE decision rule $\sigma : \mathcal{T} \times \Theta \to \Delta (A)$ and an information structure $P$ will induce an extended outcome $\sigma \circ P \in \Delta (A \times \mathcal{T} \times \Theta)$.
- Our outcome of interest is the marginal of $\sigma \circ P \in \Delta (A \times \mathcal{T} \times \Theta)$ on $(A \times \Theta)$ which we write as $\nu_{\sigma} \in \Delta (A \times \Theta)$. 

Writing $\text{BIBCE}((G, P))$ for the set of BIBCE of $(G, P)$, the set of BIBCE outcomes is $O((G, P)) = \nu_{\sigma} \Delta (A \times \Theta)$ such that $\nu = \nu_{\sigma}$.
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writing $BIBCE (\mathcal{G}, P)$ for the set of BIBCE of $(\mathcal{G}, P)$, the set of BIBCE outcomes is

$$
\mathcal{O} (\mathcal{G}, P) = \left\{ \nu \in \Delta (A \times \Theta) \left| \exists \sigma \in BIBCE (\mathcal{G}, P) \text{ such that } \nu = \nu_\sigma \right. \right\}
$$
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in particular, a BIBCE decision rule $\sigma : T \times \Theta \rightarrow \Delta (A)$ and an information structure $P$ will induce an extended outcome $\sigma \circ P \in \Delta (A \times T \times \Theta)$

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$$O (G, P) = \left\{ \nu \in \Delta (A \times \Theta) \left| \exists \sigma \in BIBCE (G, P) \text{ such that } \nu = \nu_\sigma \right\} \right.$$ 

we want to say that if information structures are close their BIBCE outcomes are close in all games
our object of interest is the set of outcomes in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.

in particular, a BIBCE decision rule $\sigma: \mathcal{T} \times \Theta \to \Delta(A)$ and an information structure $P$ will induce an extended outcome $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$

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we want to say that if information structures are close their BIBCE outcomes are close in all games

but what do we mean by BIBCE outcomes being close?
strategic topology: outcomes

- recall that the set of outcomes induced by BIBCE of \((\mathcal{G}, P)\)

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\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \mid \exists \sigma \in \text{BIBCE}(\mathcal{G}, P) \text{ such that } \nu = \nu_\sigma \right\}
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- recall that the set of outcomes induced by BIBCE of \((G, P)\)

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\]

- define \(\mathcal{O}_\varepsilon(G, P)\) as the set of outcomes that are \(\varepsilon\)-close to \(\varepsilon\)-BIBCE outcomes of \((G, P)\):

\[
\mathcal{O}_\varepsilon(G, P) = \left\{ \nu \in \Delta(A \times \Theta) \left| \exists \sigma \in \text{BIBCE}_\varepsilon(G, P) \text{ such that } \|\nu_\sigma, \nu\| \leq \varepsilon \right. \right\}
\]
recall that the set of outcomes induced by BIBCE of \((\mathcal{G}, P)\)

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\]

note the two forms of approximation
the strategic distance between two information structures $P$ and $P'$ is given by

$$d^* \left( P, P' \mid G \right) = \inf \left\{ \varepsilon \geq 0 \mid \mathcal{O} \left( G, P \right) \subseteq \mathcal{O}_{\varepsilon} \left( G, P' \right) \right\} ,$$
we first show that if two information structures are close (in the ACK topology), then they nearby equilibrium outcomes in all games.

**Proposition 1 (Sufficiency):** for every game $\mathcal{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ so that if $d^{ACK}(P, P') < \delta$, then $d^*(P, P' | \mathcal{G}) < \varepsilon$
we then show that if two information structures are not close (in the ACK topology), then equilibrium outcomes are not close in some game.

**Proposition 2 (Necessity):** for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $G$ such that $d^*(P, P'|G) \geq \varepsilon$.
Theorem: The ACK topology is the coarsest topology generating continuity of strategic outcomes.
Proof Sketch
proof sketch of sufficiency

**Proposition 1 (Sufficiency):** for every game $G$ and $\varepsilon > 0$, there exists $\delta > 0$ so that if $d^{ACK}(P, P') < \delta$, then $d^*(P, P'|G) < \varepsilon$

- i.e., must show that for all $G$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if (i) $\sigma$ is a BIBCE of $(G, P)$ and (ii) $d^{ACK}(P, P') \leq \delta$, then there exists $\sigma'$, a $\varepsilon$–BIBCE of $(G, P')$, such that $\|\nu_{P,\sigma}, \nu_{P',\sigma'}\| \leq \varepsilon$
extension of decision rule

Let $\sigma$ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d^{ACK}(P, P') < \delta$. If $\omega/2 \supset \operatorname{supp}(P)$, let play at $\omega$ be an average of play on the overlap of $\operatorname{supp}(P)$ and an $\delta$-ball around $\omega$. I write $b_{\sigma}$ for that extension of $\sigma$ to $\operatorname{supp}(\delta(P))$. 
let $\sigma$ be any BIBCE of $(G, P)$ and suppose $d^{ACK}(P, P') < \delta$

we will continuously extend $\sigma$ from $\text{supp}(P)$ to $\text{supp}_\delta(P)$

and thus to $\hat{T}_\delta(P, P')$ and $C^{1-\delta}\left(\hat{T}_\delta(P, P')\right)$
extension of decision rule

- let $\sigma$ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d_{ACK}^{ACK}(P, P') < \delta$
- we will continuously extend $\sigma$ from $\text{supp}(P)$ to $\text{supp}_\delta(P)$ and thus to $\widehat{T}_\delta(P, P')$ and $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$
- if $\omega \notin \text{supp}(P)$, let play at $\omega$ be an average of play on the overlap of $\text{supp}(P)$ and an $\delta$-ball around $\omega$
let $\sigma$ be any BIBCE of $(G, P)$ and suppose $d^{ACK}(P, P') < \delta$

we will continuously extend $\sigma$ from $\text{supp}(P)$ to $\text{supp}_\delta(P)$ and thus to $\hat{T}_\delta(P, P')$ and $C^{1-\delta}\left(\hat{T}_\delta(P, P')\right)$

if $\omega \notin \text{supp}(P)$, let play at $\omega$ be an average of play on the overlap of $\text{supp}(P)$ and an $\delta$-ball around $\omega$

write $\hat{\sigma}$ for that extension of $\sigma$ to $\text{supp}_\delta(P)$
extension of decision rule

- consider modified version of game \((\mathcal{G}, P')\): force players to follow \(\hat{\sigma}\) on the event \(C^{1-\delta} \left( \hat{T}_\delta(P, P') \right)\) (which will overlap with \(\text{supp}(P')\))
consider modified version of game \((\mathcal{G}, P')\): force players to follow \(\hat{\sigma}\) on the event \(C^{1-\delta} \left( \hat{T}_\delta(P, P') \right)\) (which will overlap with \(\text{supp}(P')\))

find a BIBCE \(\sigma'\) of this modified game
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find a BIBCE \(\sigma'\) of this modified game

now because \(\hat{\sigma}\) was mandated on the common \((1-\delta)\)-belief event, \(\sigma'\) is an \(\epsilon\)-BIBCE of the unmodified game \((\mathcal{G}, P')\), where \(\epsilon\) depends on \(\delta\) and \(\mathcal{G}\)
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- now because \(\hat{\sigma}\) was mandated on the common \((1-\delta)\)-belief event, \(\sigma'\) is an \(\varepsilon\)-BIBCE of the unmodified game \((G, P')\), where \(\varepsilon\) depends on \(\delta\) and \(G\)

- because \(C^{1-\delta} \left( \hat{T}_\delta(P, P') \right)\) has probability at least \(1 - \delta\) under both \(P\) and \(P'\), and \(\hat{\sigma}\) was a continuous extension of \(\sigma\), the outcomes induced by \(\sigma\) and \(\sigma'\) are close.
proof sketch of necessity

**Proposition 2 (Necessity):** for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $G$ such that $d^*(P, P'|G) \geq \varepsilon$ we will establish contra-positive....
proof sketch of necessity

**Proposition 2 (Necessity):** for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $G$ such that $d^*(P, P'|G) \geq \varepsilon$ we will establish contra-positive........if $d^{ACK}(P, P') > \varepsilon$, we will show the existence of base game $G$ and a BIBCE $\sigma$ of $(G, P)$ generating outcome $\nu_\sigma$, such that every $\varepsilon$-BIBCE of $(G, P)$ generates an outcome that is far from $\nu_\sigma$
proof sketch of necessity

- now $d^{ACK}(P, P') > \varepsilon$ implies either

  $$P \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) < 1 - \varepsilon$$

  or

  $$P' \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) < 1 - \varepsilon$$

  or both
proof sketch of necessity

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  or
  
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  or both

- let’s assume
  
  $$P \left( C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \right) < 1 - \varepsilon$$
proof sketch of necessity

it would be enough to construct a binary action game where

1. action 1 is chosen by all players on the event

$$\text{supp}(P) \setminus C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right)$$

in any $\varepsilon$-BIBCE of $(G, P)$
proof sketch of necessity

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2. action 0 is chosen by all players in some BIBCE of \((G, P')\)
   - in this case, action 1 would be played on an event of probability at least \(\varepsilon\) in \((G, P)\) and probability 0 in \((G, P')\)
an "email game" argument

- consider a coordination game with two actions, 0 and 1
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- consider a coordination game with two actions, 0 and 1
- action 1 is the unique $\varepsilon$-best response only if you attach probability at least $\varepsilon$ to some other player choosing action 1
- suppose payoffs are always given by this coordination game except on the event

$$D_\varepsilon = \text{supp}(P) \setminus \hat{T}_\varepsilon(P, P')$$

when players have a dominant strategy to play action 1
figure 3
an "email game" argument

since $D_\varepsilon$ is disjoint from the $\text{supp}(P')$, there is an equilibrium where 0 is always played in $(G, P')$
an "email game" argument

- since $D_\varepsilon$ is disjoint from the $\text{supp}(P')$, there is an equilibrium where 0 is always played in $(G, P')$
- but in game $(G, P)$, for action 0 to be played, there must be common $(1 - \varepsilon)$-belief that the state is not in $D_\varepsilon$
an "email game" argument

- since $D_\varepsilon$ is disjoint from the $\text{supp}(P')$, there is an equilibrium where 0 is always played in $(\mathcal{G}, P')$
- but in game $(\mathcal{G}, P)$, for action 0 to be played, there must be common $(1 - \varepsilon)$-belief that the state is not in $D_\varepsilon$
- so 1 must be played on the event

$$\text{supp}(P) \setminus C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right)$$
flaw in the argument

- we assumed that we could make it a dominant strategy to play action 1 on the event $D_\varepsilon$. 
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- we assumed that we could make it a dominant strategy to play action 1 on the event $D_\epsilon$
- but we can’t do this: payoffs have to be measurable with respect to payoff states
flaw in the argument

- we assumed that we could make it a dominant strategy to play action 1 on the event $D_\varepsilon$
- but we can’t do this: payoffs have to be measurable with respect to payoff states
- let’s see if we can correct the flaw in the argument....
make players participate in a second game where they report their \( m \)th order beliefs.
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make it a finite action game by only asking them to report nearest $m$th order belief on a grid
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make it uniquely rationalizable to truthfully report the closest $m$th order in the grid
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make it a finite action game by only asking them to report nearest $m$th order belief on a grid
make it uniquely rationalizable to truthfully report the closest $m$th order in the grid
(is this possible? yes, see iterated scoring rule game in Dekel-Fudenberg-Morris 06)
figure 4
now choose \( m \) and grid of \( m \)th order beliefs so that there is a set of reports \( R \) sent by players in

\[
D_\varepsilon = \text{supp}(P) / \left( \hat{T}_\varepsilon(P, P') \right)
\]

and never sent by players in \( \text{supp}(P') \)
corrected argument

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maintain that 0 is a best response to 0 on the event $C^{1-\varepsilon} \left( \widehat{T}_\varepsilon(P, P') \right)$
corrected argument

- now action 1 chosen in \((G, P)\) any \(\varepsilon\)-BIBCE on the event

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now action 1 chosen in \((G, P)\) any \(\varepsilon\)-BIBCE on the event \(
\text{supp}(P) / \left(\hat{T}_\varepsilon(P, P')\right)\)

in any BIBCE of \((G, P')\), action 0 is always chosen
the game is "minimal"

- we needed (something like) a binary action coordination game and an iterated scoring rule game
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- the binary action coordination game ensures that tails of higher-order beliefs matter
  - play in the iterated scoring rule game depends only of a finite number of levels of beliefs
- the iterated scoring rule game is required to identify when
  \[ C^{1-\varepsilon} \left( \hat{T}_\varepsilon(P, P') \right) \] is not true
Literature 1
Monderer-Samet 89 introduced idea of common p-belief and showed its importance for equilibrium behavior.
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Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes.
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Many (related) minor differences: these papers considered ad hoc, countable (and different) spaces of information structures, BNE, payoff continuity instead of outcome continuity.
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Big difference:
- We look at universal state space and distinguish higher-order beliefs and first order beliefs about payoff states.
- This makes both directions harder and leads to the need for the continuous extension of the decision rule and the \( m \)th level scoring rule.
literature 2:

- use rationalizability as a solution concept
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  - we could have started with their stronger notion of closer hierarchies, and would have got the same topology on information structures
  - but under their stronger (and more complicated) notion of closeness of hierarchies, the common p-belief desideratum would have been for free
denseness of simple information structure

- an information structure is \textit{finite} if there are a finite set of states

\textbf{Lemma}

\textit{finite information structures are dense in the ACK topology}
denseness of simple information structure

- an information structure is \textit{finite} if there are a finite set of states
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Denseness of simple information structure

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- an information structure is \textit{first order belief} if all types in the support have distinct first order beliefs
- an information structure is \textit{simple} if it is finite first order belief

Lemma

\textit{finite information structures are dense in the ACK topology}
Properties
BNE and embedding correlation devices

- a game is rich if, for every action profile \( a \in A \), there exists a state \( \theta_a \) such that, for all players \( i \),

\[
    u_i (a_i, a_{-i}, \theta_a) - u_i (a'_i, a_{-i}, \theta_a)
\]

Lemma

Suppose \( |\Theta| \geq 2 \). For any rich base game \( \mathcal{G} \) and any information structure \( P \),

\[
    \lim_{\varepsilon \downarrow 0} \bigcup_{P': d^*(P, P') \leq \varepsilon} \mathcal{O}^{BNE} (\mathcal{G}, P') = \mathcal{O} (\mathcal{G}, P)
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- construct a (non-canonical) information structure that replicates \( \sigma' \) as an \( \varepsilon \)-BNE.
BNE and embedding correlation devices

- A game is rich if, for every action profile \( a \in A \), there exists a state \( \theta_a \) such that, for all players \( i \),
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- Perturb the information structure to make it canonical.
BNE and embedding correlation devices

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Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure $P$,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : \mathcal{O}^* (P, P') \leq \varepsilon} \mathcal{O}^{BNE} (G, P') = \mathcal{O} (G, P)$$

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- find a nearby simple information structure $P'$ (by denseness) and an $\varepsilon$-BIBCE $\sigma'$ of $(G, P')$ inducing a nearby outcome.
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information design

>a designer has a continuous (in the Hausdorff topology) objective function

\[ V : 2^{\Delta(A \times \Theta)} \setminus \emptyset \to \mathbb{R} \]

Theorem

Now

\[ \sup_{P \in \mathcal{P}^{\text{SIMPLE}} \cap \mathcal{P}^*} V(\mathcal{O}(\mathcal{G}, P)) \leq \sup_{P \in \mathcal{P}^*} V(\mathcal{O}(\mathcal{G}, P)) \]

and if \( \mathcal{G} \) satisfies strong richness

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information design

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\[ V : 2^{\Delta(A \times \Theta)} \setminus \emptyset \rightarrow \mathbb{R} \]

- for an open subset \( \mathcal{P}^* \subseteq \mathcal{P} \) and base game \( \mathcal{G} \), the designer chooses \( \mathcal{P} \) with objective

\[ \sup_{\mathcal{P} \in \mathcal{P}^*} V(\mathcal{O}(\mathcal{G}, \mathcal{P})) \]

**Theorem**

*Now*

\[ \sup_{\mathcal{P} \in \mathcal{P}^{SIMPLE} \cap \mathcal{P}^*} V(\mathcal{O}(\mathcal{G}, \mathcal{P})) \leq \sup_{\mathcal{P} \in \mathcal{P}^*} V(\mathcal{O}(\mathcal{G}, \mathcal{P})) \]

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information design examples

- suppose the designer has a utility function \( u_0 : A \times \Theta \rightarrow \mathbb{R} \)
- standard information design

\[
V(X) = \max_{\nu \in X} \sum_{a, \theta} \nu(a, \theta) u_0(a, \theta)
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information design examples

- suppose the designer has a utility function $u_0 : A \times \Theta \rightarrow \mathbb{R}$
- standard information design

$$V(X) = \max_{\nu \in X} \sum_{a, \theta} \nu(a, \theta) u_0(a, \theta)$$

- adversarial information design

$$V(X) = \min_{\nu \in X} \sum_{a, \theta} \nu(a, \theta) u_0(a, \theta)$$
take homes

1. canonical information structures are natural and workable
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   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them
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   - we use and need both
Old Slides
our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
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- equilibrium version of interim correlated rationalizability (ICR) [Liu 15]
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- but we will discuss this and argue that our topology remains the relevant one
strategic topology in more detail

so two canonical information structures are $\varepsilon$-close in the strategic topology if, for every BIBCE under one information structure, there is an $\varepsilon$-BIBCE under the other information structure inducing an outcome that is $\varepsilon$-close
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an event is "approximate common knowledge" or "common \((1 - \varepsilon)\)-belief" if everyone believes it with probability at least \(1 - \varepsilon\), everybody believes with probability \(1 - \varepsilon\) that everyone believes it with probability at least \(1 - \varepsilon\), and so on....[Monderer and Samet 89]
approximate common knowledge topology in more detail

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- so two canonical information structures are ε-close in the ACK topology if each assigns probability at least 1 − ε to there being common (1 − ε)-belief that belief hierarchies being within ε,.
properties

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3. the set of BIBCE outcomes for a given canonical information structure = the set of BNE outcomes of all nearby (general) information structures
literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
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we require novel proof, as I will review
literature: (rationalizability) strategic topology on belief hierarchies

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- but then it turns out that approximate common knowledge is for free!
talk outline

- setting
- main result
- proof sketch
- properties
a principled justification for belief-invariant Bayes correlated equilibrium

first consider rationalizability

▶ in complete information games, we often study correlated rationalizable actions (instead of independent rationalizable actions of Bernheim 82 and Pearce 82).
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  - usual justification: we don't think that a correlation device is observed; there is just an equivalence between (i) correlated rationalizability; (ii) surviving iterated deletion of strictly dominated strategies; and (iii) consistent with common knowledge of rationality.
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  - can make same response: I don't need to know source of correlation, we just know that there is an equivalence between (i) ICR; (ii) surviving iterated deletion of (interim) strictly dominated strategies; and (iii) consistent with common knowledge of rationality.
a principled justification for belief-invariant Bayes correlated equilibrium

now consider equilibrium, i.e., impose common prior assumption and take ex ante perspective

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- in complete information games, correlated equilibrium is the equilibrium analogue of correlated rationalizability
- in incomplete information games, BIBCE is the equilibrium analogue of ICR
pragmatic justifications for belief-invariant Bayes correlated equilibrium

- BIBCE is exactly the solution concept where the set of outcomes induced by BIBCE on a canonical information structure equals the set of outcomes induced by BIBCE on any information structure
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pragmatic justifications for belief-invariant Bayes correlated equilibrium

- BIBCE is exactly the solution concept where the set of outcomes induced by BIBCE on a canonical information structure equals the set of outcomes induced by BIBCE on any information structure.
- Canonical information structures can be separated by BIBCE play in some game, but other information structures cannot.
- We will also argue that if you are interested in Bayes Nash equilibrium (or any solution concept between BNE and BIBCE) you should still be interested in our topology.