

## B Online Appendix

### B.1 Proof of Lemma 4

For each  $i \neq j$ , let  $N_t^{-i}(j)$  denote the (random) number of players in the set  $-i = I \setminus \{i\}$  who, by period  $t$ , have met a player in  $-i$  who met a player in  $-i$  who... met player  $j$ . Letting  $T = Z \log_2 N$ , it suffices to show that there exists a constant  $c > 0$  such that  $\Pr(N_T^{-i}(j) = N - 1 \forall i, j) \geq 1 - \exp(-cZ)$ . The idea of the proof is to show that, with high probability,  $\min_{i,j} N_t^{-i}(j)$  grows exponentially in  $t$  until it reaches a constant fraction of  $N$ , and that subsequently  $N - \min_{i,j} N_t^{-i}(j)$  shrinks exponentially.

We first show that  $\min_{i,j} N_t^{-i}(j)$  grows exponentially until it reaches  $\frac{2}{3}N$ .

**Lemma 5** *There exists  $\bar{\gamma} \in (0, \frac{1}{2}]$  such that, for every  $N$  and  $n \leq \frac{2}{3}N$ ,*

$$\Pr\left(\min_{i,j} N_{t+1}^{-i}(j) \leq (1 + \bar{\gamma}) \min_{i,j} N_t^{-i}(j) \mid \min_{i,j} N_t^{-i}(j) = n\right) \leq N(N-1) \frac{e}{2\pi\bar{\gamma}^{\frac{1}{2}}(1-\bar{\gamma})^{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{\bar{\gamma}n}{2}}.$$

**Proof.** By monotonicity in the number of informed players and symmetry, it suffices to prove that, for each particular  $i \neq j$ ,

$$\Pr(N_{t+1}^{-i}(j) \leq (1 + \bar{\gamma}) N_t^{-i}(j) \mid N_t^{-i}(j) = n) \leq \frac{e}{2\pi\bar{\gamma}^{\frac{1}{2}}(1-\bar{\gamma})^{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{\bar{\gamma}n}{2}}.$$

Fixing  $i \neq j$ , and suppressing  $i$  and  $j$  in the notation, let  $I_t$  be the set of players who received player  $j$ 's message through a path excluding  $i$  by period  $t$ : thus,  $|I_t| = n$ . Note that, for each number  $n' \leq N - n$  with the same parity as  $n$ ,  $\Pr(N_{t+1} = n + n' \mid N_t = n)$  is at most

$$\begin{aligned} & \underbrace{\binom{n}{n'}}_{\substack{\text{who in } I_t \text{ meets} \\ \text{players in } I \setminus (I_t \cup \{i\})}} \times \underbrace{\frac{n-1}{N-1}}_{\substack{\text{"first" player in } I_t \\ \text{meets someone in } I_t}} \times \underbrace{\frac{n-3}{N-3}}_{\substack{\text{"second" (remaining) player in } I_t \\ \text{meets some (remaining) player in } I_t}} \times \cdots \times \frac{n'+1}{N-n+n'+1} \\ &= \binom{n}{n'} \prod_{k=1}^{\frac{n-n'}{2}} \frac{n-2k+1}{N-2k+1}. \end{aligned}$$

This expression is an upper bound, as we neglect the probability that the players in  $I_t$  who are selected to meet someone in  $I \setminus (I_t \cup \{i\})$  actually do so. Similarly, for each  $n'$  with the opposite parity as  $n$ ,  $\Pr(N_{t+1} = n + n' \mid N_t = n)$  is at most

$$\begin{aligned} & \underbrace{\binom{n}{n'}}_{\substack{\text{who in } I_t \text{ meets } i}} \times \underbrace{\frac{1}{N-1}}_{\substack{\text{prob. of meeting } i}} \times \underbrace{\binom{n-1}{n'}}_{\substack{\text{who in } I_t \text{ meets} \\ \text{players in } I \setminus (I_t \cup \{i\})}} \times \underbrace{\frac{n-2}{N-3} \times \cdots \times \frac{n'+1}{N-n+n'}}_{\substack{\text{remaining players in } I_t \\ \text{match with each other}}} \\ &= \binom{n-1}{n'} \prod_{k=1}^{\frac{n-n'+1}{2}} \frac{n-2k+2}{N-2k+1} \leq \binom{n}{n'} \prod_{k=1}^{\frac{n-n'+1}{2}} \frac{n-2k+2}{N-2k+1}. \end{aligned}$$

For any  $\gamma \in (0, \frac{1}{2}]$ , if  $n' \leq \gamma n$ , Stirling's formula gives

$$\binom{n}{n'} \leq \frac{en^{n+\frac{1}{2}}e^{-n}}{2\pi(\gamma n)^{\gamma n+\frac{1}{2}}e^{-\gamma n}((1-\gamma)n)^{(1-\gamma)n+\frac{1}{2}}e^{-(1-\gamma)n}} \leq \frac{e}{2\pi(\gamma)^{\gamma n+\frac{1}{2}}(1-\gamma)^{(1-\gamma)n+\frac{1}{2}}}.$$

We also have

$$\prod_{k=1}^{\frac{n-n'}{2}} \frac{n-2k+1}{N-2k+1} \leq \left(\frac{n-1}{N-1}\right)^{\frac{n-n'}{2}} \leq \left(\frac{n}{N-1}\right)^{\frac{(1-\gamma)n}{2}}, \text{ and}$$

$$\prod_{k=1}^{\frac{n-n'+1}{2}} \frac{n-2k+2}{N-2k+1} \leq \left(\frac{n}{N-1}\right)^{\frac{n-n'+1}{2}} \leq \left(\frac{n}{N-1}\right)^{\frac{(1-\gamma)n}{2}}.$$

Therefore, for any  $\gamma \in (0, \frac{1}{2}]$  and  $n' \leq \gamma n$ , we have

$$\Pr(N_{t+1} = n + n' | N_t = n) \leq \frac{e}{2\pi(\gamma)^{\gamma n+\frac{1}{2}}(1-\gamma)^{(1-\gamma)n+\frac{1}{2}}} \left(\frac{n}{N-1}\right)^{\frac{(1-\gamma)n}{2}},$$

and hence

$$\begin{aligned} \Pr(N_{t+1} \leq n + \gamma n | N_t = n) &\leq \frac{e(\gamma n + 1)}{2\pi(\gamma)^{\gamma n+\frac{1}{2}}(1-\gamma)^{(1-\gamma)n+\frac{1}{2}}} \left(\frac{n}{N-1}\right)^{\frac{(1-\gamma)n}{2}} \\ &= \frac{e}{2\pi\gamma^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}} \left(\frac{(\gamma n + 1)^{\frac{2}{\gamma n}}}{\gamma^2(1-\gamma)^{2\frac{1-\gamma}{\gamma}}} \left(\frac{n}{N-1}\right)^{\frac{1-\gamma}{\gamma}}\right)^{\frac{\gamma n}{2}} \\ &\leq \frac{e}{2\pi(\gamma)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}} \left(\frac{e^2}{\gamma^2(1-\gamma)^{2\frac{1-\gamma}{\gamma}}} \left(\frac{2}{3}\frac{N}{N-1}\right)^{\frac{1-\gamma}{\gamma}}\right)^{\frac{\gamma n}{2}}. \end{aligned} \quad (11)$$

Fix  $\bar{\gamma} \in (0, \frac{1}{2}]$  such that

$$\frac{e^2}{\bar{\gamma}^2(1-\bar{\gamma})^{2\frac{1-\bar{\gamma}}{\bar{\gamma}}}} \left(\frac{8}{9}\right)^{\frac{1-\bar{\gamma}}{\bar{\gamma}}} < \frac{1}{2}.$$

Such a  $\bar{\gamma}$  exists as the left-hand side of this inequality goes to 0 as  $\bar{\gamma} \rightarrow 0$ . Since  $N \geq 4$ , we have  $\frac{2}{3}\frac{N}{N-1} \leq \frac{8}{9}$ . Hence, substituting  $\gamma = \bar{\gamma}$  in (11), we have, for every  $N$  and  $n \leq \frac{2}{3}N$ ,

$$\Pr(N_{t+1} \leq n + \bar{\gamma} n | N_t = n) \leq \frac{e}{2\pi(\bar{\gamma})^{\frac{1}{2}}(1-\bar{\gamma})^{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{\bar{\gamma} n}{2}},$$

as desired. ■

Fix  $\bar{\gamma}$  satisfying the conditions of Lemma 5. Let  $n^*(N)$  satisfy

$$N(N-1) \frac{e}{2\pi\bar{\gamma}^{\frac{1}{2}}(1-\bar{\gamma})^{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{\bar{\gamma} n^*(N)}{2}} = \frac{1}{4}.$$

Note that

$$n^*(N) = \hat{c}(\log_2 N + \log_2(N - 1)),$$

where  $\hat{c} > 0$  is a constant independent of  $N$ . The following lemma is an immediate consequence of Lemma 5.

**Lemma 6** For every  $n$  satisfying  $n^*(N) \leq n \leq \frac{2}{3}N$ ,

$$\Pr\left(\min_{i,j} N_{t+1}^{-i}(j) \leq (1 + \bar{\gamma}) \min_{i,j} N_t^{-i}(j) \mid \min_{i,j} N_t^{-i}(j) = n\right) \leq \frac{1}{4}.$$

We now consider the case where  $n \leq n^*(N)$ , considering first the subcase where  $n \geq 12$ .

**Lemma 7** There exists  $\bar{N}_1$  such that, for every  $N \geq \bar{N}_1$  and  $n$  satisfying  $12 \leq n \leq n^*(N)$ ,

$$\Pr\left(\min_{i,j} N_{t+1}^{-i}(j) \leq \frac{3}{2} \min_{i,j} N_t^{-i}(j) \mid \min_{i,j} N_t^{-i}(j) = n\right) \leq \frac{1}{4}.$$

**Proof.** Taking  $\gamma = \frac{1}{2}$  in (11), we have

$$\begin{aligned} & \Pr\left(\min_{i,j} N_{t+1}^{-i}(j) \leq \frac{3}{2} \min_{i,j} N_t^{-i}(j) \mid \min_{i,j} N_t^{-i}(j) = n\right) \\ & \leq N(N-1) \frac{e}{2\pi\gamma^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}} \left(\frac{e^2}{\gamma^2(1-\gamma)^{2\frac{1-\gamma}{\gamma}}} \left(\frac{n}{N-1}\right)^{\frac{1-\gamma}{\gamma}}\right)^{\frac{\gamma n}{2}} \\ & = N(N-1) \frac{e}{2\pi\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}}} \left(\frac{e^2}{\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2} \frac{n}{N-1}\right)^{\frac{n}{4}}. \end{aligned}$$

Since  $12 \leq n \leq \hat{c}(\log_2 N + \log_2(N - 1))$ , this is at most

$$N(N-1) \frac{e}{\pi} \left(16e^2 \frac{\hat{c}(\log_2 N + \log_2(N - 1))}{N-1}\right)^3,$$

which is less than  $\frac{1}{4}$  for sufficiently large  $N$ . ■

The next lemma addresses the subcase with fewer than 12 informed players.

**Lemma 8** There exists  $\bar{N}_2$  such that, for every  $N \geq \bar{N}_2$ ,

$$\Pr\left(\min_{i,j} N_{t+6}^{-i}(j) \leq 12 \mid \min_{i,j} N_t^{-i}(j) \geq 1\right) \leq \frac{1}{4}.$$

**Proof.** Fix  $i \neq j$ , and suppose  $N_{t+6}^{-i}(j) \leq 12$ . Since  $\min_{i,j} N_t^{-i}(j) \geq 1$  and  $12 < 2^4$ , this is possible only if  $N_{t'+1}^{-i}(j) = 2N_{t'}^{-i}(j)$  for at most 3 out of the 6 periods  $t' \in \{t+1, \dots, t+6\}$ . That is, in at least 3 out of these 6 periods, some player in  $I_{t'}^{-i}(j)$  must meet someone in  $I_{t'}^{-i}(j) \cup \{i\}$ . Since by hypothesis  $N_{t'}^{-i}(j) \leq 12$  for each such period  $t'$ , the probability of this event is at most  $\binom{6}{3} \times 12 \times \left(\frac{12}{N-1}\right)^3$ . Hence,

$$\Pr\left(\min_{i,j} N_{t+6}^{-i}(j) \leq 12 \mid \min_{i,j} N_t^{-i}(j) \geq 1\right) \leq N(N-1) \frac{20 \times 12^4}{(N-1)^3},$$

which is less than  $\frac{1}{4}$  for sufficiently large  $N$ . ■

In total, since  $\bar{\gamma} \leq \frac{1}{2}$ , we have the following lemma:

**Lemma 9** For every  $N \geq \max\{\bar{N}_1, \bar{N}_2\}$ ,

1. For any  $(N_t^{-i}(j))_{i,j}$  such that  $\min_{i,j} N_t^{-i}(j) \geq 1$ , we have

$$\Pr\left(\min_{i,j} N_{t+6}^{-i}(j) \leq 12 \mid (N_t^{-i}(j))_{i,j}\right) \leq \frac{1}{4}.$$

2. For any  $(N_t^{-i}(j))_{i,j}$  such that  $\min_{i,j} N_t^{-i}(j) = n$  satisfies  $12 \leq n \leq \frac{2}{3}N$ , we have

$$\Pr\left(\min_{i,j} N_{t+1}^{-i}(j) \leq (1 + \bar{\gamma}) \min_{i,j} N_t^{-i}(j) \mid (N_t^{-i}(j))_{i,j}\right) \leq \frac{1}{4}.$$

We now provide a symmetric bound for the case where  $N_t^{-i}(j)$  is “large” for each  $i \neq j$ . Let  $M_t^{-i}(j) = N - 1 - N_t^{-i}(j)$  be the number of players  $-i$  who have not yet received player  $j$ ’s message through a path excluding  $i$ ; and let  $J_t^{-i}(j)$  be the set of such players.

**Lemma 10** There exists  $\bar{N}_3$  such that, for each  $N \geq \bar{N}_3$ ,

1. For any  $(M_t^{-i}(j))_{i,j}$  such that  $\max_{i,j} M_t^{-i}(j) \leq 12$ , we have

$$\Pr\left(\max_{i,j} M_{t+6}^{-i}(j) > 0 \mid (M_t^{-i}(j))_{i,j}\right) \leq \frac{1}{4}.$$

2. For any  $(M_t^{-i}(j))_{i,j}$  such that  $\max_{i,j} M_t^{-i}(j) = n$  satisfies  $12 \leq n \leq \frac{1}{3}N$ , we have

$$\Pr\left(\max_{i,j} M_{t+1}^{-i}(j) \geq (1 - \bar{\gamma}) \max_{i,j} M_t^{-i}(j) \mid \max_{i,j} M_t^{-i}(j) = n\right) \leq \frac{1}{4}.$$

**Proof.** Lemmas 5–9 provide an upper bound for the probability that fraction  $\bar{\gamma}$  of players in  $I_t^{-i}(j)$  do not meet players outside of  $I_t^{-i}(j) \cup \{i\}$ . The current lemma provides an upper bound for the probability that fraction  $\bar{\gamma}$  of players in  $J_t^{-i}(j)$  do not meet players outside of  $J_t^{-i}(j) \cup \{i\}$ . The argument is symmetric. ■

We now combine Lemmas 9 and 10 to prove Lemma 4. We first assume  $N \geq \max\{\bar{N}_1, \bar{N}_2, \bar{N}_3\}$ .

We have the following properties. First, if  $\min_{i,j} N_t^{-i}(j) < 12$ , then  $\min_{i,j} N_{t+6}^{-i}(j) \geq 12$  with probability at least  $\frac{3}{4}$ . Second, if  $12 \leq \min_{i,j} N_t^{-i}(j) \leq \frac{2}{3}N$ , then  $\min_{i,j} N_{t+1}^{-i}(j) \geq (1 + \bar{\gamma}) \min_{i,j} N_t^{-i}(j)$  with probability at least  $\frac{3}{4}$ . (And note that  $\log_{(1+\bar{\gamma})} \frac{2}{3}N$  “increases” by a factor of  $(1 + \bar{\gamma})$  suffice to raise  $\min_{i,j} N_t^{-i}(j)$  to  $\frac{2}{3}N$ .) Third, if  $\frac{2}{3}N \leq \min_{i,j} N_t^{-i}(j) \leq N - 13$ —or equivalently  $12 \leq \max_{i,j} M_t^{-i}(j) \leq \frac{1}{3}N$ —then  $\max_{i,j} M_{t+1}^{-i}(j) \leq (1 - \bar{\gamma}) \max_{i,j} M_t^{-i}(j)$  with probability at least  $\frac{3}{4}$ . (Note that  $\log_{(1-\bar{\gamma})} 3\frac{1}{N}$  “decreases” suffice to reduce  $\max_{i,j} M_t^{-i}(j)$  to 12.) Finally, if  $\max_{i,j} M_t^{-i}(j) \leq 12$ , then  $\min_{i,j} N_{t+6}^{-i}(j) = N - 1$  (equivalently  $\max_{i,j} M_t^{-i}(j) = 0$ ) with probability at least  $\frac{3}{4}$ .

Combining these properties, we see that  $\Pr(\min_{i,j} N_T^{-i}(j) = N - 1)$  is lower-bounded by the probability that, out of  $T/6$  Bernoulli random variables with parameter  $\frac{3}{4}$ , the realizations of at

least  $2 + \log_{(1+\bar{\gamma})} \frac{2}{3}N + \log_{(1-\bar{\gamma})} 3\frac{1}{N}$  of them equal 1. By Hoeffding's inequality, this probability is at least

$$1 - \exp\left(-2\left(\frac{3}{4} - \frac{2 + \log_{(1+\bar{\gamma})} \frac{2}{3}N + \log_{(1-\bar{\gamma})} 3\frac{1}{N}}{\frac{T}{6}}\right)^2 \frac{T}{6}\right).$$

If  $T = Z \log_2 N$ , then

$$\begin{aligned} \frac{2 + \log_{(1+\bar{\gamma})} \frac{2}{3}N + \log_{(1-\bar{\gamma})} 3\frac{1}{N}}{\frac{Z \log_2 N}{6}} &< \frac{2 + (\log_2 N) \left(\frac{1}{\log_2(1+\bar{\gamma})} - \frac{1}{\log_2(1-\bar{\gamma})}\right)}{\frac{Z \log_2 N}{6}} \\ &< \frac{6}{Z} \left(2 + \frac{1}{\log_2(1+\bar{\gamma})} - \frac{1}{\log_2(1-\bar{\gamma})}\right). \end{aligned}$$

Hence, there exists  $\bar{Z}_1 > 0$  such that if  $Z > \bar{Z}_1$  then, for all  $N \geq \max\{\bar{N}_1, \bar{N}_2, \bar{N}_3\}$ , we have

$$\frac{2 + \log_{(1+\bar{\gamma})} \frac{2}{3}N + \log_{(1-\bar{\gamma})} 3\frac{1}{N}}{\frac{Z \log_2 N}{6}} < \frac{1}{4},$$

and hence

$$\Pr\left(\min_{i,j} N_T^{-i}(j) = N - 1\right) \geq 1 - \exp\left(-2\left(\frac{1}{2}\right)^2 \frac{Z \log_2 N}{6}\right) \geq 1 - \exp\left(-\frac{1}{12}Z\right).$$

Finally, for the case  $N < \max\{\bar{N}_1, \bar{N}_2, \bar{N}_3\}$ , Hoeffding's inequality implies

$$\Pr\left(\min_{i,j} N_T^{-i}(j) = N - 1\right) \geq 1 - N(N-1) \exp\left(-2\left(\frac{1}{N-1}\right)^2 T\right).$$

Hence, there exist  $c_1 > 0$  and  $\bar{T} > 0$  such that, for all  $N < \max\{\bar{N}_1, \bar{N}_2, \bar{N}_3\}$  and  $T > \bar{T}$ , we have

$$\Pr\left(\min_{i,j} N_T^{-i}(j) = N - 1\right) \geq 1 - \exp(-c_1 T).$$

Taking  $c = \min\left\{\frac{1}{12}, c_1\right\}$  and  $\bar{Z} = \max\{\bar{Z}_1, \bar{T}\}$  completes the proof.

## B.2 Proof of Theorem 3

Fix  $\alpha \in (0, 1 - \varepsilon)$  and  $\eta \in (0, \alpha/2)$  throughout the proof. We first describe a protocol for the community to circulate messages. This protocol has the feature that, with high probability, the number of periods required for everyone to learn the message is of order  $\log N$ ; moreover, no single player can prevent the rest of the community from learning. We then use this protocol as a building block in the construction of a block belief-free equilibrium.

### B.2.1 Protocol for Players to Circulate Message $m$

Suppose each player  $i$  wishes to disseminate a message  $m_i$  throughout the community, where each  $m_i$  is an element of some finite set  $M_i$ . We say that *players circulate message  $m = (m_i)_i$  for  $T$  periods* if the players obey the following protocol for  $T$  periods:

In each period  $t \in \{1, \dots, T\}$ , all players take action  $D$ , while sending cheap-talk messages.

Each player  $j$  has a “state”

$$\left( \zeta_{j,t}^{I,-i}, \zeta_{j,t}^{M,-i} \right)_{i \neq j} \subseteq \times_{i \neq j} (I \times (\times_{n \neq i} M_n)).$$

Intuitively,  $\zeta_{j,t}^{I,-i}$  is the set of players  $k$  whose message player  $j$  has heard (directly or indirectly) via a path that excludes  $i$ , and  $\zeta_{j,t}^{M,-i}|_k \subseteq M_k$  is the set of messages reported to  $j$  as having been sent by  $k$  via a path that excludes  $i$ .<sup>32</sup> Note that  $\zeta_{j,t}^{M,-i}$  does not include player  $i$ 's message since it is infeasible to share player  $i$ 's message via a path that excludes  $i$ .

Formally, for each player  $j$  and  $i \neq j$ ,  $(\zeta_{j,1}^{I,-i}, \zeta_{j,1}^{M,-i}) = (\{j\}, (\emptyset, \dots, \emptyset, \{m_j\}, \emptyset, \dots, \emptyset))$ . In each period  $t$ , given  $(\zeta_{j,t}^{I,-i}, \zeta_{j,t}^{M,-i})_{i \neq j}$ , if player  $j$  meets player  $k$ , player  $j$  sends message  $(\zeta_{j,t}^{I,-i}, \zeta_{j,t}^{M,-i})_{i \notin \{j,k\}}$ . That is, player  $j$  passes all of his information to player  $k$ , except for the “ $-k$ ” information being circulated by players  $-k$ . Given his opponent's message  $(\hat{\zeta}_{k,t}^{I,-i}, \hat{\zeta}_{k,t}^{M,-i})_{i \notin \{j,k\}}$ , for each  $i \notin \{j,k\}$ , player  $j$ 's next-period state is given by  $\zeta_{j,t+1}^{I,-i} = \zeta_{j,t}^{I,-i} \cup \hat{\zeta}_{k,t}^{I,-i}$  and  $\zeta_{j,t+1}^{M,-i}|_n = \zeta_{j,t}^{M,-i}|_n \cup \hat{\zeta}_{k,t}^{M,-i}|_n$  for all  $n \neq i$  (recall that  $n \neq i$  since  $\zeta_{j,t}^{M,-i} \subseteq \times_{n \neq i} M_n$ ). For each  $i \in \{j,k\}$ , let  $(\zeta_{j,t+1}^{I,-i}, \zeta_{j,t+1}^{M,-i}) = (\zeta_{j,t}^{I,-i}, \zeta_{j,t}^{M,-i})$ . That is, for each player  $n \neq i$ , player  $j$  adds  $\hat{\zeta}_{k,t}^{M,-i}|_n$  to the set of messages reported to him as having been sent by  $n$  (note that  $k \neq i$  by definition: only player  $k \neq i$  hears a message via a path that excludes  $i$ ). (Throughout, we use hatted variables to denote messages.)

At the end of period  $T$ , for each  $i \neq j$ , if  $\zeta_{j,T}^{I,-i} = -i$  and  $|\zeta_{j,T}^{M,-i}|_n = 1$  for each  $n \neq i$ , we say player  $j$  *infers* message  $m_{-i}(j) \in \times_{k \neq i} M_k$ , where  $m_{-i}(j)|_n$  is equal to the unique element of  $\zeta_{j,T}^{M,-i}|_n$ , for each  $n$ . Otherwise, we say player  $j$  infers  $m_{-i}(j) = \mathbf{error}$ .<sup>33</sup> We also say the match realization is *erroneous* if there exists disjoint players  $i \neq j \neq k \neq i$  such that, by period  $T$ , player  $i$  has not met a player in  $-k$  who met a player in  $-k$  who... met player  $j$ . Otherwise, the match is *regular*.

Note that, if all players follow the protocol, then at the end of period  $T$  either the match is erroneous or  $m_{-i}(j) = m_{-i}$  for all  $i \neq j$ . Moreover, if  $T = Z \log_2 N$ , by Lemma 4 the probability that the match is erroneous decreases exponentially in  $Z$ . We thus have

**Lemma 11** *Let  $T = Z \log_2 N$ . There exist  $c > 0$  and  $\bar{Z} > 0$  such that, for all  $Z > \bar{Z}$  and all  $l$ , we have*

$$\Pr(m_{-i}(j) = m_{-i} \forall i \neq j) \geq 1 - \exp(-cZ).$$

Note also that whether or not the event  $\{m_{-i}(j) = m_{-i}\}$  obtains is independent of player  $i$ 's behavior.

## B.2.2 Parameters Used in the Construction

We fix some parameters that will be used in the construction. Let

$$K := \left\lceil \frac{\max\{G, L\}}{\eta/16} \right\rceil. \quad (12)$$

<sup>32</sup>For a vector  $x \in X^{N-1}$  and  $k \in \{1, \dots, N-1\}$ , we denote the  $k^{\text{th}}$  coordinate of  $x$  by  $x|_k$ .

<sup>33</sup>Note that it is possible that  $m_{-i}(j)|_n \neq m_{-i'}(j)|_n$  for some  $i \neq i' \neq n \neq i$ . Intuitively,  $m_{-i}(j) = \mathbf{error}$  means that  $j$  fails to infer a message through a chain of players excluding  $i$ , but not necessarily that the messages she infers through all chains are mutually consistent.

Fix  $Z$  sufficiently large such that  $Z \geq \bar{Z}$  (with  $c$  and  $\bar{Z}$  given in Lemma 11),

$$\exp(-cZ) \geq \frac{1}{2}, \text{ and} \quad (13)$$

$$(K+3) \left( \frac{1}{Z} + \exp(-cZ) \right) \bar{u} \leq \frac{\eta}{16}, \quad (14)$$

where  $\bar{u} = 2 \max\{L, 1+G\}$ . Next, let

$$\begin{aligned} T &= Z \log_2 N, \\ T^* &= (3 + (K-1)(Z+1) + Z)T, \text{ and} \\ T^{**} &= (3 + K(1+Z))T. \end{aligned} \quad (15)$$

Note that, by (12) and (15), we have

$$\frac{\eta}{16} T^* \geq ZT \times \max\{G, L\}. \quad (16)$$

Finally, take  $l$  sufficiently large such that

$$(1-\delta)(-L) + \delta\eta \sum_{\theta^*: |\theta^*| \geq \alpha N} \Pr(\theta^* | i \in \theta^*) \geq (1-\delta)(1+G), \quad (17)$$

$$\frac{1}{T^*} \left( \left| \frac{1-\delta^{T^{**}}}{1-\delta} - T^* \right| + 2 \sum_{t=1}^{T^*} (1-\delta^{t-1}) \right) \bar{u} \leq \frac{1}{8}\eta, \text{ and} \quad (18)$$

$$\frac{1-\delta}{\delta^{T^{**}}} \left( \left| \frac{1-\delta^{T^{**}}}{1-\delta} - T^* \right| + 2 \sum_{t=1}^{T^*} (1-\delta^{t-1}) + 4T^* \right) \bar{u} \leq \frac{1}{2}\eta. \quad (19)$$

### B.2.3 Period 1

The very first period of the repeated game plays a special role in our construction. We denote this period by  $1^*$  rather than 1, to clarify that this is the first period of the infinitely repeated game, rather than the first period of a block (see Section B.2.4 for the definition of the block). In period  $1^*$ , every normal player is supposed to play  $C$ . Given the outcome of period  $1^*$ , let  $\theta$  denote the set of players who took  $a_{i,1^*} = C$  as prescribed. (Note that  $\theta \subseteq \theta^*$ , as all committed players take  $D$ , and some rational players may also take  $D$  as the result of a deviation.) In our construction, only players in  $\theta$  will cooperate with each other. The strategies we construct will take  $\theta$  as a persistent “state variable,” and we denote the set of possible states  $\theta$  by  $\Theta = 2^I$ . Note that each player  $i$ ’s period- $1^*$  history,  $h_{i,1^*} = (\mu_{i,1^*}, a_{i,1^*}, a_{\mu_{i,1^*},1^*})$ , is directly informative of  $\theta$ ; for this reason, players’ period- $1^*$  histories will play a distinguished role in our construction.<sup>34</sup>

Fix  $(v^{\theta^*})_{\theta^*}$  such that  $v^{\theta^*} \in F^{\alpha,\eta}(\theta^*)$  for each  $\theta^*$ ; and let  $(\tilde{v}^{\theta,\theta^*})_{\theta,\theta^*}$  be such that, for each  $\theta$  and  $\theta^*$  with  $\theta \subseteq \theta^*$ , we have

$$\tilde{v}_i^{\theta,\theta^*} = \begin{cases} v_i^\theta & \text{if } i \in \theta, \\ 0 & \text{if } i \in \theta^* \setminus \{\theta\}. \end{cases}$$

Note that  $\tilde{v}^{\theta,\theta^*} \in \mathbb{R}^{|\theta^*|}$ . Since  $v^\theta \in F^{\alpha,\eta}(\theta)$ , the vector  $(\tilde{v}_i^{\theta,\theta^*})_{i \in \theta} \in \mathbb{R}^{|\theta|}$  satisfies the following

<sup>34</sup>We omit messages  $(m_{i,1^*}, m_{\mu_{i,1^*},1^*})$  in the description of  $h_{i,1^*}$ , as there is no communication in period  $1^*$  in our construction.

conditions:

1. For each  $\theta$  satisfying  $|\theta| \geq \alpha N$  and each  $i \in \theta$ , we have  $\prod_{i \in \theta} [\tilde{v}_i^{\theta, \theta^*} - \eta, \tilde{v}_i^{\theta, \theta^*} + \eta] \subseteq F^*(\theta)$ .

Since  $\mathbf{A}(\theta)$  specifies that players take  $D$  against opponents who are not in  $\theta$ , we have

$$\prod_{i \in \theta} [\tilde{v}_i^{\theta, \theta^*} - \eta, \tilde{v}_i^{\theta, \theta^*} + \eta] \times \prod_{i \in \theta^* \setminus \{\theta\}} \{0\} \subseteq F^*(\theta^*).$$

2. For each  $\theta$  satisfying  $|\theta| < \alpha N$  and each  $i \in \theta$ ,  $v_i^\theta = 0$ .

We will construct a sequential equilibrium where, for each  $\theta^*$ ,  $(h_{i,1^*})_{i \in I}$ ,  $\theta$ , and  $i \in \theta^*$ , player  $i$ 's continuation payoff starting from the second period of the repeated game equals  $\tilde{v}_i^{\theta, \theta^*}$ . To see why this suffices to prove the theorem, note that in state  $\theta^*$  a rational type's equilibrium payoff in period  $1^*$  is  $p^{\theta^*} - (1 - p^{\theta^*})L$ , and hence her repeated game payoff is  $(1 - \delta)(p^{\theta^*} - (1 - p^{\theta^*})L) + \delta v_i^{\theta^*}$ . Since  $v_i^{\theta^*} \geq \eta$  whenever  $i \in \theta^*$  and  $|\theta^*| \geq \alpha N$ , player  $i$ 's expected payoff from taking  $C$  in period  $1^*$  is

$$\begin{aligned} & (1 - \delta) \sum_{\theta^*} \Pr(\theta^* | i \in \theta^*) (p^{\theta^*} - (1 - p^{\theta^*})L) + \delta \sum_{\theta^*: |\theta^*| \geq \alpha N} \Pr(\theta^* | i \in \theta^*) v_i^{\theta^*} \\ & \geq (1 - \delta)(-L) + \delta \eta \sum_{\theta^*: |\theta^*| \geq \alpha N} \Pr(\theta^* | i \in \theta^*), \end{aligned}$$

while her expected payoff from taking  $D$  in period  $1^*$  is at most  $(1 - \delta)(1 + G)$ . By (17), taking  $C$  in period  $1^*$  is optimal.

## B.2.4 Block Belief-Free Structure

We now describe the general structure of our construction (following period  $1^*$ ) and present the corresponding equilibrium conditions.

**Block Strategies.** We view the repeated game from period 2 on as an infinite sequence of  $T^{**}$ -period blocks. At the beginning of every block, each player  $i$  selects a "strategy state"  $x_i^\theta \in \{G, B\}$  for each  $\theta \in \Theta$  from a full support probability distribution. Given the vector  $\mathbf{x}_i = (x_i^\theta)_{\theta \in \Theta}$  and player  $i$ 's period- $1^*$  history  $h_{i,1^*}$ , player  $i$  plays a behavioral strategy  $\sigma_i^*(\mathbf{x}_i, h_{i,1^*})$  (her *block strategy*) within the block. That is, in every period  $t = 1, \dots, T^{**}$  of the block,  $\sigma_i^*(\mathbf{x}_i, h_{i,1^*})$  specifies a probability distribution over cheap talk messages and actions as a function of player  $i$ 's *block history*  $h_i^t = \left( \left( \mu_{i,\tau}, m_{i,\tau}, m_{\mu_{i,\tau},\tau}, a_{i,\tau}, a_{\mu_{i,\tau},\tau} \right)_{\tau=1}^{t-1}, \mu_{i,t} \right)$ . Denote player  $i$ 's strategy set in the  $T^{**}$ -period game by  $\Sigma_i$ .

Players are prescribed to play  $C$  in period  $1^*$  and subsequently use the same strategy in each block. Thus, a player's entire repeated-game strategy can be summarized by a single block strategy, together with a policy for selecting the strategy state  $\mathbf{x}_i$  at the start of each block.

**Target Payoffs.** For each  $\theta$ ,  $\mathbf{x}$ , and  $i \in \theta^*$ , let

$$v_i^\theta(x_{i-1}^\theta) = \begin{cases} v_i^\theta - \text{sign}(x_{i-1}^\theta) \frac{\eta}{4} & \text{if } |\theta| \geq \alpha N \text{ and } i \in \theta, \\ 0 & \text{otherwise,} \end{cases} \quad (20)$$

where  $\text{sign}(x_{i-1}^\theta) := 1_{\{x_{i-1}^\theta=B\}} - 1_{\{x_{i-1}^\theta=G\}}$ . Note that

$$v_i^\theta \begin{cases} \in (v_i^\theta(B), v_i^\theta(G)) & \text{if } |\theta| \geq \alpha N \text{ and } i \in \theta, \\ = v_i^\theta(B) = v_i^\theta(G) & \text{otherwise.} \end{cases} \quad (21)$$

As will be seen, continuation payoffs at the beginning of a block conditional on state  $\theta$  and strategy state profile  $\mathbf{x}^\theta$  will equal  $(v_i^\theta(x_{i-1}^\theta))_{i \in I}$ .

**Continuation Payoffs.** Conditional on the persistent state being equal to  $\theta$ , player  $i$ 's equilibrium continuation payoff at the end of a block is a function only of player  $(i-1)$ 's state  $x_{i-1}^\theta$  and history  $h_{i-1}^{T^{**}}$  in the previous block. (Adopt here the convention that player-names are mod  $N$ , so player  $(1-1)$  is player  $N$ .) Denote this continuation payoff by  $w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}})$ .

Thus, player  $(i-1)$  is the "arbiter" of player  $i$ 's payoff, in that player  $(i-1)$ 's choice of her strategy state  $\mathbf{x}_{i-1}$  determines player  $i$ 's equilibrium continuation payoff in each state  $\theta$ . This feature is typical of block belief-free constructions, such as those in Hörner and Olszewski (2006), Deb, Sugaya, and Wolitzky (2019), and Sugaya and Yamamoto (2019).

**Beliefs.** Players' belief systems  $(\beta_i)_{i \in I}$  are specified as a function of the block strategy profile  $\sigma$ . Intuitively, players believe that trembles in the current block are much less likely than trembles in previous blocks, but that, within the current block, trembles in later periods are much more likely than trembles in earlier periods. This has two important implications. First, if a player reaches a history that can be explained by some past opponents' play that does not involve any deviations within the current block, she believes with probability 1 that no one deviated within the current block. Second, if a player reaches a history that cannot be explained without appealing to deviations within the current block, but can be explained by supposing that the only within-block deviation was made by her current opponent in the current period, then she believes with probability 1 that this is indeed what occurred.

To construct the belief system, first note that  $N$  and  $T^{**}$  determine the number of possible block history profiles  $(h_i^t)_{i \in I, t \leq T^{**}}$ .<sup>35</sup> Denote this number by  $\tilde{c}$ . Beliefs are derived from Bayes' rule along a sequence of completely mixed strategy profiles  $(\sigma^l)_{l \in \mathbb{N}}$ , in which each player  $i$  "trembles" uniformly over all messages and actions with probability  $(1/l)^{\tilde{c}(T^{**}b-t)}$  in period  $t \in \{1, \dots, T^{**}\}$  of block  $b$ . As  $l \rightarrow \infty$ , the resulting beliefs display the properties discussed above.

**Equilibrium Conditions.**

Fix a block strategy profile  $(\sigma_i^*(\mathbf{x}_i, h_i^{1*}))_{i \in I, \mathbf{x}_i \in \{G, B\}^{|\Theta|}, h_i^{1*} \in H_i^{1*}}$ , and continuation payoffs  $(w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}))_{i \in \theta, \theta \in \Theta, x_{i-1}^\theta \in \{G, B\}, h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}}$ . Since  $\theta \subseteq \theta^*$ , payoffs are well-defined.

We present a set of conditions which will guarantee the existence of a robust sequential equilibrium where payoffs from period 2 on are given by  $(\tilde{v}^{\theta, \theta^*})_{\theta, \theta^*}$ . In what follows,  $\Pr^\sigma(\cdot)$  and  $\mathbb{E}^\sigma[\cdot]$  denote conditional probability and expectation, respectively, under block strategy profile  $\sigma$ , with the corresponding belief system defined above given  $\sigma$ . We also write  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$  for a generic history in period  $t$  of block  $b$  of the infinitely repeated game, and write  $h_i^t \in H_i^t$  for a generic block history in period  $t$  of a block. (Thus,  $\tilde{h}_i^{b,t}$  records the outcomes of  $(b-1)T^{**} + t - 1$  periods of play, while  $h_i^t$  records the outcomes of  $t - 1$  periods.) Finally, we write  $\tilde{h}_i^{b,0} \in \tilde{H}_i^{b,0}$  for a generic repeated game history at the beginning of block  $b$ , before the determination of the first match in the block.

1. [*Sequential Rationality*] For each  $\theta \in \Theta$ , the following two conditions hold:

(a) For each  $i \notin \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^{**}\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i(h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t}) \Pr^{\sigma^*(\mathbf{x}, h^{1*})} \left( (a_{i,t}, a_{\mu_{i,t}t}) = (D, D) | \mathbf{x}, h^{1*}, \tilde{h}_i^{b,t} \right) = 1.$$

<sup>35</sup>The size of the message sets  $|M_{i,t}|$  used in the construction will be explicitly determined as a function of  $N$  and  $T^{**}$  in the course of the proof.

(That is, each player  $i \notin \theta$  takes  $D$  and expects her opponents to take  $D$  towards her.)

- (b) For each  $i \notin \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^{**}\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,  $\sigma_i^*(\mathbf{x}_i, h_i^{1*})$  is a maximizer (over  $\sigma_i \in \Sigma_i$ ) of

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i \left( h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right) \mathbb{E} \left( \sigma_i, \sigma_{-i}^* \left( \mathbf{x}_{-i}, h_{-i}^{1*} \right) \right) \left[ \begin{array}{l} (1 - \delta) \sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) \\ + \delta^{T^{**}} w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) \end{array} \middle| \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right].$$

(Here, the sum  $\sum_{\tau=1}^{T^{**}}$  is taken over all periods in the current block  $b$ , where the current period  $t \in \{(b-1)T^{**} + 2, \dots, bT^{**} + 1\}$  is some period in block  $b$ .)

Note that both sequential rationality conditions are imposed “ex post” over vectors  $\mathbf{x}_{-i} \in \{G, B\}^{(N-1)|\Theta|}$ . This is the defining feature of a block belief-free construction. However, optimality with respect to  $h_{-i}^{1*}$  is demanded only in expectation, not ex post.

2. [*Promise Keeping*] For each  $\theta \in \Theta$ ,  $i \in \theta$ ,  $x_{i-1}^\theta \in \{G, B\}^N$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}^{b,0} \in \tilde{H}^{b,0}$ ,

$$v_i^\theta(x_{i-1}^\theta) = \mathbb{E}^{\sigma^*}(\mathbf{x}, h^{1*}) \left[ \begin{array}{l} (1 - \delta) \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) \\ + \delta^{T^{**}} w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) \end{array} \middle| \tilde{h}^{b,0}, \theta \right].$$

(Note that player  $i$ 's continuation payoff  $v_i^\theta(x_{i-1}^\theta)$  is allowed to depend on  $\tilde{h}^b$  only through  $\theta$ .)

3. [*Self-Generation*] For each  $\theta \in \Theta$ ,  $i \in \theta$ ,

- (a) If  $\theta$  satisfies  $|\theta| \geq \alpha N$ , we have  $w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) \in (v_i^\theta(B), v_i^\theta(G))$  for each  $x_{i-1}^\theta \in \{G, B\}$  and  $h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}$ .
- (b) If  $\theta$  satisfies  $|\theta| < \alpha N$ , we have  $w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) = 0$  for each  $x_{i-1}^\theta \in \{G, B\}$  and  $h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}$ .

Defining  $\pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) := \frac{\delta^{T^{**}}}{1-\delta} \left( w_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) - v_i^\theta(x_{i-1}^\theta) \right)$ , we can rewrite these conditions as follows:

1. [*Sequential Rationality*] For each  $\theta \in \Theta$ , the following two conditions hold:

- (a) For each  $i \notin \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^{**}\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i \left( h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right) \Pr^{\sigma^*}(\mathbf{x}, h^{1*}) \left( \left( a_{i,t}, a_{\mu_{i,t},t} \right) = (D, D) | \mathbf{x}, h^{1*}, \tilde{h}_i^{b,t} \right) = 1. \quad (22)$$

- (b) For each  $i \in \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^{**}\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,  $\sigma_i^*(\mathbf{x}_i, h_i^{1*})$  maximizes

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i \left( h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right) \mathbb{E} \left( \sigma_i, \sigma_{-i}^* \left( \mathbf{x}_{-i}, h_{-i}^{1*} \right) \right) \left[ \sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) \middle| \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right]. \quad (23)$$

2. [Promise Keeping] For each  $\theta \in \Theta$ ,  $i \in \theta$ ,  $x_{i-1}^\theta \in \{G, B\}^N$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}^{b,0} \in \tilde{H}^{b,0}$ ,

$$v_i^\theta(x_{i-1}^\theta) = \mathbb{E}^{\sigma^*(\mathbf{x})} \left[ \frac{1-\delta}{1-\delta^{T^{**}}} \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) | \tilde{h}^{b,0}, \theta \right]. \quad (24)$$

3. [Self-Generation] For each  $\theta \in \Theta$ ,  $i \in \theta$ ,  $x_{i-1}^\theta \in \{G, B\}$  and  $h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}$ ,

(a) If  $\theta$  satisfies  $|\theta| \geq \alpha N$ , we have

$$\text{sign} \left( x_{i-1}^\theta \right) \pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) > 0 \text{ and } \left| \frac{1-\delta}{\delta^{T^{**}}} \pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) \right| < \frac{\eta}{2}. \quad (25)$$

(b) If  $\theta$  satisfies  $|\theta| < \alpha N$ , we have

$$\pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) = 0. \quad (26)$$

**Lemma 12** For each  $l$ , suppose there exist  $(\sigma_i^*(\mathbf{x}_i, h_i^{1*}))_{i \in I, \mathbf{x}_i \in \{G, B\}^{|\Theta|}, h_i^{1*} \in H_i^{1*}}$  and  $(\pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}))_{i \in \theta, \theta \in \Theta, x_{i-1}^\theta \in \{G, B\}, h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}}$  such that Conditions (22)–(26) are satisfied. Then there exists a robust sequential equilibrium such that, for each  $\theta^*$ ,  $(h_{i,1^*})_{i \in I}$ , and  $\theta$ , payoffs from the second period of the repeated game equal  $(\tilde{v}_i^{\theta, \theta^*})_{i \in \theta^*}$ .

**Proof.** For  $i \in \theta$ , Conditions (24) and (25) imply that payoffs  $(v_i^\theta(x_{i-1}^\theta))_{i \in I, \theta \in \Theta, x_{i-1}^\theta \in \{G, B\}}$  can be delivered at the beginning of each block with full-support state transition probabilities. For  $i \in \theta^* \setminus \theta$ , Conditions (22) and (26) imply that payoff 0 can be delivered at the beginning of each block with full-support state transition probabilities. Condition (21) then implies that, by appropriately randomizing over  $(x_{i-1}^\theta)_{i \in I, \theta \in \Theta}$  before the first block (i.e., before period 2 of the repeated game), the target expected payoff vector  $(v_i^\theta)_{i \in \theta}$  can be delivered. This is as in, for example, Hörner and Olszewski (2006). In total, payoffs from the second period of the repeated game equal  $(\tilde{v}_i^{\theta, \theta^*})_{i \in \theta^*}$ .

For  $i \in \theta$ , Condition (23) is then a more stringent version of the resulting sequential rationality constraint, as it imposes sequential rationality for each realization of  $\mathbf{x}_{-i}$ , rather than only in expectation. For  $i \notin \theta$ , Condition (22) implies that the distribution of the action sequence that player  $i$  faces is independent of her strategy for each realization of  $\mathbf{x}_{-i}$ . It is therefore optimal for rational player  $i$  to take  $D$  in each period and send any messages, and this behavior is indeed what is prescribed for player  $i$  by (22). For bad player  $i$ , Condition (22) implies that her strategy does not affect the action sequence she faces, as required. ■

To prove Theorem 3, it thus suffices to show that there exist  $(\sigma_i^*(\mathbf{x}_i, h_i^{1*}))_{i \in I, \mathbf{x}_i \in \{G, B\}^{|\Theta|}, h_i^{1*} \in H_i^{1*}}$  and  $(\pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}))_{i \in I, \theta \in \Theta, x_{i-1}^\theta \in \{G, B\}, h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}}$  such that Conditions (22)–(26) are satisfied.

### B.2.5 Target Actions

We now define a target (opponent-identity-contingent) action profile  $\mathbf{a}^{x^\theta}$  for each state  $\theta \subseteq I$ .

For  $\theta$  satisfying  $|\theta| < \alpha N$ , we define  $\mathbf{a}_i^{x^\theta}(j) = D$  for all  $x^\theta \in \{G, B\}^N$  and  $i \neq j$ . That is, all players are prescribed defection. In this case, we define  $v_i^\theta(G) = v_i^\theta(B) = 0$  for each  $i \in \theta$ . Note that, to satisfy (25), this requires  $\pi_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}+1}) = 0$  for all  $x_{i-1}^\theta \in \{G, B\}$  and  $h_{i-1}^{T^{**}+1} \in H_{i-1}^{T^{**}+1}$ .

For  $\theta$  satisfying  $|\theta| \geq \alpha N$ , for each  $x^\theta \in \{G, B\}^N$ , there exists a distribution  $\hat{\alpha}^{x^\theta} \in \Delta(\mathbf{A}(\theta))$  such that, for each  $i \in \theta$ ,  $\hat{u}_i(\hat{\alpha}^{x^\theta}) > v_i^\theta + \frac{1}{2}\eta$  if  $x_{i-1}^\theta = G$ , and  $\hat{u}_i(\hat{\alpha}^{x^\theta}) < v_i^\theta - \frac{1}{2}\eta$  if  $x_{i-1}^\theta = B$ ; and for each  $i \notin \theta$ ,  $a_{\mu_i} = D$  with probability one for all  $x^\theta$ . Here existence follows from the fact that  $(v_i^\theta)_{i \in \theta}$  satisfies  $\prod_{i \in \theta} [v_i^\theta - \eta, v_i^\theta + \eta] \subseteq F^*(\theta)$  and (20).

Since  $\hat{\alpha}^{x^\theta} \in \Delta(\mathbf{A}(\theta))$ , we have

$$\hat{u}(\hat{\alpha}^{x^\theta}) = \sum_{\mathbf{a} \in \mathbf{A}(\theta)} \alpha^{x^\theta}(\mathbf{a}) \sum_{\mu} \frac{2}{N(N-1)} (u_i(\mathbf{a}_i(\mu_i), \mathbf{a}_{\mu_i}(i)))_{i \in I}.$$

(Here,  $\frac{2}{N(N-1)}$  is the probability of each match realization  $\mu = (\mu_i)_{i \in I}$ .)

Let  $A_{i,j}(\theta) = \{(a_i, a_j) \in \{C, D\}^2 : (\mathbf{a}_i(\mu_i), \mathbf{a}_{\mu_i}(i)) = (D, D) \text{ if } i \notin \theta \text{ or } j \notin \theta\}$ . Since  $\mathbf{A}(\theta)$  specifies that players take  $D$  towards opponents outside  $\theta$ , given  $\hat{\alpha}^{x^\theta}$ , when players  $i$  and  $j$  meet they take  $(a_i, a_j) \in A_{i,j}(\theta)$  with probability one. Hence, we have

$$u(\hat{\alpha}^{x^\theta}) = \left( \sum_{\mu_i \neq i} \frac{1}{N-1} \sum_{\mathbf{a}_i(\mu_i), \mathbf{a}_{\mu_i}(i) \in A_{i,\mu_i}(\theta)} \alpha_{i,\mu_i}^{x^\theta}(\mathbf{a}_i(\mu_i), \mathbf{a}_{\mu_i}(i)) u_i(\mathbf{a}_i(\mu_i), \mathbf{a}_{\mu_i}(i)) \right)_{i \in I},$$

where  $\alpha_{i,\mu_i}^{x^\theta}$  is the distribution of action pairs of player  $i$ 's and player  $\mu_i$ 's given  $\hat{\alpha}^{x^\theta}$ , conditional on the event that players  $i$  and  $\mu_i$  match.

Therefore, for  $\theta$  satisfying  $|\theta| \geq \alpha N$ , for each  $x^\theta \in \{G, B\}^N$ , there exists  $\alpha_{i,j}^{x^\theta} \in \Delta(A_{i,j}(\theta))$  for each  $i \neq j$  such that the following conditions hold: letting  $\alpha^{x^\theta} \in \Delta A$  be the distribution of action profiles when  $\mu$  is drawn uniformly and then  $(a_i, a_j)$  is drawn from  $\alpha_{i,j}^{x^\theta}$  given  $j = \mu_i$ , for each  $i \in \theta$ ,  $\hat{u}_i(\alpha^{x^\theta}) > v_i^\theta + \frac{3}{4}\eta$  if  $x_{i-1}^\theta = G$ , and  $\hat{u}_i(\alpha^{x^\theta}) < v_i^\theta - \frac{3}{4}\eta$  if  $x_{i-1}^\theta = B$ ; and for each  $i \notin \theta$ ,  $(a_i, a_{\mu_i}) = D$  with probability one for all  $x^\theta$ .

Since  $\hat{u}_i(\alpha^{x^\theta}) > v_i^\theta + \frac{3}{4}\eta$  and  $\hat{u}_i(\alpha^{x^\theta}) < v_i^\theta - \frac{3}{4}\eta$  are strict inequalities, we can assume that  $\alpha_{i,j}^{x^\theta}(a_i, a_j)$  is a rational number: there exists  $M^A \in \mathbb{N}$  such that, for all  $(a_i, a_j) \in \{C, D\}^2$ ,

$$\alpha_{i,j}^{x^\theta}(a_i, a_j) = \frac{m_{a_i, a_j}}{M^A} \text{ for some } m_{a_i, a_j} \in \mathbb{N}. \quad (27)$$

Arbitrarily label the four elements of  $\{C, D\}^2$  by  $(a_i^1, a_j^1)$ ,  $(a_i^2, a_j^2)$ ,  $(a_i^3, a_j^3)$ , and  $(a_i^4, a_j^4)$ . For each  $x^\theta$  and  $(i, j)$ , we can see  $\alpha_{i,j}^{x^\theta}$  as a uniform distribution over  $\{1, \dots, M^A\}$ . Given the realization  $m \in \{1, \dots, M^A\}$ , there exists a unique  $k \in \{1, 2, 3, 4\}$  such that

$$\sum_{0 \leq \bar{k} < k} m_{a_i^{\bar{k}}, a_j^{\bar{k}}} < m \leq \sum_{0 \leq \bar{k} \leq k} m_{a_i^{\bar{k}}, a_j^{\bar{k}}}. \quad (28)$$

For such  $k$ , we say that the realization of  $\alpha_{i,j}^{x^\theta}$  is  $(a_i^k, a_j^k)$ .

Note that, for each  $i \in \theta$ , we have

$$\left( \max_{x^\theta: x_{i-1}^\theta = B} \hat{u}_i(\alpha^{x^\theta}) \right)_+ < v_i^\theta(B) - \frac{\eta}{4} < v_i^\theta < v_i^\theta(G) + \frac{\eta}{4} < \min_{x^\theta: x_{i-1}^\theta = G} \hat{u}_i(\alpha^{x^\theta}). \quad (29)$$

### B.2.6 Structure of the Block

In what follows, recall that players always take action  $D$  while circulating information.

1. **1\*-communication sub-block** (the first  $T$  periods of the block): Players circulate information about  $h^{1*}$ .
2.  **$\mathbf{x}$ -communication sub-block** (the next  $T$  periods): Players circulate information about  $\mathbf{x}$ .
3. **Supplemental round 0** (the next  $T$  periods): Players circulate information about the first two sub-blocks.
4. **Main sub-block  $k$**  (there are  $K$  main sub-blocks, each lasting for  $(1 + Z)T$  periods, and each divided into the following two rounds):
  - (a) **Main round  $k$**  (the first  $ZT$  periods of the sub-block): Players  $(i, j)$  who are paired draw a joint controlled lottery according to the target action distribution  $\alpha_{i,j}^{x^\theta}$ . Given the realization, they take the pure action.  
(Formally, players use their inference of  $\theta$  and  $x^\theta$  based on 1\*-communication and  $\mathbf{x}$ -communication sub-blocks. Moreover, players draw the joint controlled lottery by cheap talk. These are the only cheap talk messages sent in the main round).
  - (b) **Supplemental round  $k$**  (the next  $T$  periods of the sub-block): Players circulate information about the history up to the end of main round  $k$ .

Given (15),  $T^{**}$  is in fact the length of the block, or equivalently the last period of supplemental round  $K$ ; and  $T^*$  is the last period of main round  $K$ .

### B.2.7 Reduction Lemma

We now show that, by communicating their histories during supplemental round  $K$  (the last such round in the block) and adjusting continuation payoffs appropriately, the players can effectively cancel the effects of discounting while letting continuation payoffs depend on  $(x_{-i}^\theta, h^{1*}, h^{T^*+1})$  rather than  $(x_{i-1}^\theta, h_{i-1}^{T^{**}+1})$  (when  $|\theta| \geq \alpha N$ ).<sup>36</sup>

Let  $\Sigma_i^{T^*}$  denote the set of  $i$ 's block strategies up to period  $T^*$ . We show that the following conditions guarantee the existence of a robust sequential equilibrium where payoffs from period 2 on equal  $(\tilde{v}_i^{\theta, \theta^*})_{i \in \theta^*}$ .

1. [*Sequential Rationality*] For each  $\theta \in \Theta$ , the following two conditions hold:

- (a) For each  $i \notin \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^*\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i \left( h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right) \Pr^{\sigma^*(\mathbf{x}, h^{1*})} \left( (a_i, a_{\mu_i, t}) = (D, D) | \mathbf{x}_{-i}, h^{1*}, \tilde{h}_i^{b,t} \right) = 1. \quad (30)$$

<sup>36</sup> A similar but more complicated argument appears in Deb, Sugaya, and Wolitzky (2020).

- (b) For each  $i \in \theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $h_i^{1*} \in H_i^{1*}$ ,  $t \in \{1, \dots, T^*\}$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ ,  $\sigma_i^{T^*}(\mathbf{x}_i, h_i^{1*})$  maximizes (over  $\sigma_i \in \Sigma_i^{T^*}$ )

$$\sum_{h_{-i}^{1*} \in H_{-i}^{1*}} \beta_i \left( h_{-i}^{1*} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \right) \mathbb{E}^{\left( \sigma_i, \sigma_{-i}^*(\mathbf{x}_{-i}, h_{-i}^{1*}) \right)} \left[ \begin{array}{l} 1_{\{\theta: |\theta| \geq \alpha N\}} \left( \sum_{\tau=1}^{T^*} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^\theta(x_{-i}^\theta, h_{-i}^{1*}, h^{T^*+1}) \right) \\ + 1_{\{\theta: |\theta| < \alpha N\}} \sum_{\tau=1}^{T^*} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) \\ | \mathbf{x}_{-i}, \tilde{h}_i^{b,t} \end{array} \right]. \quad (31)$$

2. [Promise Keeping] For each  $\theta \in \Theta$ ,  $\mathbf{x} \in \{G, B\}^{N|\Theta|}$ ,  $i \in \theta$ ,  $b \in \mathbb{N}$ , and  $\tilde{h}^{b,0} \in \tilde{H}^{b,0}$ ,

$$v_i^\theta(x_{i-1}^\theta) = \mathbb{E}^{\sigma^*(\mathbf{x}, h^{1*})} \left[ \begin{array}{l} 1_{\{\theta: |\theta| \geq \alpha N\}} \frac{1}{T^*} \left( \sum_{\tau=1}^{T^*} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}) \right) \\ + 1_{\{\theta: |\theta| < \alpha N\}} \frac{1-\delta}{1-\delta^{T^*}} \sum_{\tau=1}^{T^*} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) \end{array} \middle| \theta, \tilde{h}^{b,0} \right]. \quad (32)$$

3. [Self-Generation] For each  $\theta \in \Theta$ ,  $i \in \theta$ ,  $\mathbf{x}_{i-1} \in \{G, B\}^{|\Theta|}$ ,  $h^{1*} \in H^{1*}$ , and  $h^{T^*+1} \in H^{T^*+1}$ ,

$$\text{sign} \left( x_{i-1}^\theta \right) \pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}) \begin{cases} \geq \frac{1}{8} \eta T^* & \text{for } \theta \text{ satisfying } |\theta| \geq \alpha N, \\ = 0 & \text{for } \theta \text{ satisfying } |\theta| < \alpha N. \end{cases} \quad (33)$$

and

$$\left| \pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}) \right| \begin{cases} \leq 2\bar{u} T^* & \text{for } \theta \text{ satisfying } |\theta| \geq \alpha N, \\ = 0 & \text{for } \theta \text{ satisfying } |\theta| < \alpha N. \end{cases} \quad (34)$$

**Lemma 13** Suppose there exist  $(\sigma_i^{T^*}(\mathbf{x}_i, h_i^{1*}))_{i \in I, \mathbf{x}_i \in \{G, B\}^{|\Theta|}, h_i^{1*} \in H_i^{1*}}$  and  $(\pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}))_{i \in I, x_{-i}^\theta \in \{G, B\}^{N-1}, h^{1*}}$ , such that Conditions (31)–(34) are satisfied. Then there exists a robust sequential equilibrium such that, for each  $\theta^*$ ,  $(h_{i,1^*})_{i \in I}$ , and  $\theta$ , payoffs from the second period of the repeated game equal  $(\tilde{v}_i^{\theta, \theta^*})_{i \in \theta^*}$ .

**Proof.** We have fixed  $v_i^\theta(x_{i-1}^\theta)$ . Fix  $\sigma_i^{T^*}$  and  $\pi_i^\theta$  satisfying (31)–(34). We will construct  $\tilde{\sigma}_i^*$ , and  $\tilde{\pi}_i^\theta$  that satisfy (22)–(26).

We extend strategy  $\sigma_i^{T^*} \in \Sigma_i^{T^*}$  to a strategy  $\tilde{\sigma}_i^* \in \Sigma_i$  by specifying that players circulate message  $m = (m_i)_i = (\mathbf{x}_i, h_i^{1*}, h_i^{T^*+1})_i$  in supplemental round  $K$ .

Given player  $i-1$ 's history in supplemental round  $K$ , we define  $\tilde{\pi}_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^*+1})$  as follows.

- (i) If  $m_{-i}(i-1) = \text{error}$ , then  $\tilde{\pi}_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^*+1}) = 0$  for each  $x_{i-1}^\theta, h_{i-1}^{T^*+1}$ . (ii) Otherwise, player  $i-1$  infers  $(h_{-i}^{1*}(i-1), h_{-i}^{T^*+1}(i-1))$ . Since matching is pairwise, there exists a unique  $(h^{1*}(i-1), h^{T^*+1}(i-1))$  that is consistent with  $(h_{-i}^{1*}(i-1), h_{-i}^{T^*+1}(i-1))$ . Given  $h^{T^*+1}(i-1)$ , let  $\mathbf{a}_t(i-1)$  be the action in period  $t$ . We define

$$\begin{aligned} & \tilde{\pi}_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^*+1}) \\ = & \begin{cases} \left( \frac{1-\delta^{T^*}}{1-\delta} - T^* \right) v_i^\theta(x_{i-1}^\theta) + \frac{\sum_{t=1}^{T^*} (1-\delta^{t-1}) \hat{u}_i(\mathbf{a}_t(i-1)) + \pi_i^\theta(\mathbf{x}_{-i}(i-1), h^{1*}(i-1), h^{T^*+1}(i-1))}{\Pr(m_{-i}(i-1) \neq \text{error})} & \text{if } |\theta| \geq \alpha N, \\ 0 & \text{if } |\theta| < \alpha N. \end{cases} \end{aligned}$$

It remains to show that  $\tilde{\sigma}_i^*(\mathbf{x}_i, h_i^{1*})$ ,  $\tilde{\beta}^*$ ,  $\tilde{\pi}_i^\theta$ , and  $v_i^\theta(x_{i-1}^\theta)$  satisfy (22)–(26).

For  $i \notin \theta$ , since players take  $D$  in supplemental round  $K$ , (30) implies (22). For  $i \in \theta$ , note that player  $i$ 's payoff depends on the outcome of play in supplemental round  $K$  only through her stage game payoffs (rational types maximize them by taking  $D$ ) and  $m_{-i}(i-1)$ . Since player  $i$  cannot

affect the distribution of  $m_{-i}(i-1)$ , following  $\sigma_i^*(\mathbf{x}_i, h_i^{1*})|_{h_i^{T^*+1}}$  is optimal. Given this, by the law of iterated expectation, in period  $t \leq T^*$ , the expected value of

$$\sum_{\tau=t}^{T^*} \delta^{\tau-t} \hat{u}_i(\mathbf{a}_\tau) + \tilde{\pi}_i^\theta(x_{i-1}^\theta, h_{i-1}^{T^{**}})$$

given  $\theta$ ,  $\mathbf{x}$ ,  $h^{1*}$ , and  $\tilde{h}^t$  is equal to

$$\begin{aligned} & \mathbf{1}_{\{|\theta| \geq \alpha N\}} \left( \left( \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right) v_i^\theta(x_{i-1}^\theta) + \sum_{\tau=t}^{T^*} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}) \right) \\ & + \mathbf{1}_{\{|\theta| < \alpha N\}} \sum_{\tau=t}^{T^*} \delta^{\tau-t} \hat{u}_i(\mathbf{a}_\tau). \end{aligned}$$

Ignoring the constant  $\left( \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right) v_i^\theta(x_{i-1}^\theta)$ , (31) implies (23).

For  $|\theta| < \alpha N$ , (24)–(26) hold since all the payoffs and rewards are zero regardless of  $x_{i-1}^\theta$  and  $h_{i-1}^{T^{**}+1}$ . For  $|\theta| \geq \alpha N$ , (24) follows from (32). In addition, (25) holds since

$$\Pr(m_{-i}(i-1) \neq \mathbf{error}) \geq \frac{1}{2} \text{ by Lemma 11 and (13),}$$

$$\begin{aligned} & \text{sign}(x_{i-1}^\theta) \left( \left( \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right) v_i^\theta(x_{i-1}^\theta) + \frac{\sum_{t=1}^{T^*} (1 - \delta^{t-1}) \hat{u}_i(\mathbf{a}_t(i-1)) + \pi_i^\theta(\mathbf{x}_{-i}(i-1), h^{1*}(i-1), h^{T^*+1}(i-1))}{\Pr(m_{-i}(i-1) \neq \mathbf{error})} \right) \\ & \geq - \left( \left| \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right| \bar{u} + \frac{\sum_{t=1}^{T^*} (1 - \delta^{t-1}) \bar{u}}{\frac{1}{2}} \right) + \frac{1}{8} \eta T^* \text{ by (33)} \\ & > 0 \text{ by (18),} \end{aligned}$$

and

$$\begin{aligned} & \frac{1 - \delta}{\delta^{T^{**}}} \left| \left( \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right) v_i^\theta(x_{i-1}^\theta) + \frac{\sum_{t=1}^{T^*} (1 - \delta^{t-1}) \hat{u}_i(\mathbf{a}_t(i-1)) + \pi_i^\theta(\mathbf{x}_{-i}(i-1), h^{1*}(i-1), h^{T^*+1}(i-1))}{\Pr(m_{-i}(i-1) \neq \mathbf{error})} \right| \\ & \leq \frac{1 - \delta}{\delta^{T^{**}}} \left( \left| \frac{1 - \delta^{T^{**}}}{1 - \delta} - T^* \right| \bar{u} + \frac{\sum_{t=1}^{T^*} (1 - \delta^{t-1}) \bar{u}}{\frac{1}{2}} \right) + \frac{1 - \delta}{\delta^{T^{**}}} \frac{2\bar{u}T^*}{\frac{1}{2}} \text{ by (34)} \\ & < \frac{\eta}{2} \text{ by (19).} \end{aligned}$$

■

## B.2.8 Equilibrium Strategies

We now complete the description of the equilibrium strategies.

It will be useful to define the notion of a “detectable deviation” by player  $i$ . As we will see, given player  $i$ ’s period  $1^*$  history  $h_i^{1*}$  and her strategy state  $\mathbf{x}_i$ , her block strategy is pure along the equilibrium path of play except for the joint controlled lottery drawn at the beginning of each period of the main round. Given  $h_i^{1*}$  and an on-path period  $t$  block history  $h_i^t$ , we say that a period  $t$  message  $m_{i,t}$  is a *detectable deviation* if there does not exist a strategy state  $\hat{\mathbf{x}}_i$  such that  $(h_i^t, m_{i,t})$

occurs with positive probability given  $(\hat{\mathbf{x}}_i, h_i^{1*})$ ; similarly, given a triple  $(h_i^t, m_{i,t}, m_{\mu_{i,t},t})$ , an action  $a_{i,t}$  is a *detectable deviation* if there does not exist a strategy state  $\hat{\mathbf{x}}_i$  such that  $(h_i^t, m_{i,t}, m_{\mu_{i,t},t}, a_{i,t})$  occurs with positive probability given  $(\hat{\mathbf{x}}_i, h_i^{1*})$ . We say a player *detectably deviates* if she plays a detectable deviation.

**1\*-Communication Sub-Block** Players circulate message  $m = (m_i)_i$ , where  $m_i$  is the set of players whom player  $i$  knows to have taken  $C$  in period 1\*: that is,  $m_i = \{i, \mu_{i,1^*}\} \cap \theta$ .

Let  $h_i^{T+1}$  be player  $i$ 's history at the end of the sub-block. We define  $\theta(h_i^{T+1}) = \emptyset$  if, for some  $j \neq i$ , either  $\zeta_{i,T}^{T,-j} \neq -j$  (i.e.,  $i$  does not receive each player's message through a path excluding  $j$ ) or  $m_{-j}(i) = \mathbf{error}$  (i.e.,  $i$  receives inconsistent messages through a path excluding  $j$ ). We also define  $\theta(h_i^{T+1}) = \emptyset$  if there exist  $j \neq j' \neq k \neq j$  such that  $m_{-j}(i)|_k \neq m_{-j'}(i)|_k$  (i.e.,  $i$  receives inconsistent messages through a path excluding  $j$  and through a path excluding  $j'$ ). Otherwise, we define  $\theta(h_i^{T+1}) = \bigcup_{j \neq i} \bigcup_{k \neq j} m_{-j}(i)|_k$  (i.e.,  $\theta(h_i^{T+1})$  is the set of players who  $i$  has been told took  $C$  in period 1\*).

Lemma 11 immediately implies the following result.

**Lemma 14** *Suppose all players follow the protocol. There exist  $c > 0$  and  $\bar{Z} > 0$  such that, for all  $Z > \bar{Z}$  and all  $l$ , we have*

$$\Pr(\theta(h_i^{T+1}) = \theta \forall i) \geq 1 - \exp(-cZ).$$

We record two key properties of player  $i$ 's beliefs about  $\theta$ . Suppose the current block is block  $b$ .

First, for each  $t \geq T + 1$ ,  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ , and  $\mathbf{x}_{-i} \in \{G, B\}^{|\theta|(N-1)}$ , player  $i$  believes that  $\theta \supseteq \theta(h_i^{T+1})$ :

$$\sum_{\theta \supseteq \theta(h_i^{T+1})} \beta_i(\theta | \mathbf{x}_{-i}, \tilde{h}_i^{b,t}) = 1. \quad (35)$$

This is trivial if  $\theta_i(h_i^{T+1}) = \emptyset$ . Otherwise, since trembles in earlier blocks are more likely,  $\beta_i(\theta = \theta(h_i^{T+1}) | \mathbf{x}_{-i}, \tilde{h}_i^{b,T+1}) = 1$  (i.e., player  $i$  believes that  $\theta = \theta(h_i^{T+1})$  at the end of the 1\*-communication sub-block). Moreover, since trembles are more likely in later periods within the block, player  $i$  continues to believe that  $\theta = \theta(h_i^{T+1})$  for the duration of the block.

Second, for each  $t \geq T + 1$ ,  $\tilde{h}_i^{b,t} \in \tilde{H}_i^{b,t}$ , and  $\mathbf{x}_{-i} \in \{G, B\}^{|\theta|(N-1)}$ , player  $i$  believes that  $\theta \supseteq \theta(h_j^{T+1})$  for each  $j \neq i$ :

$$\sum_{\theta \supseteq \theta(h_j^{T+1})} \beta_{i,t}(\theta | \mathbf{x}_{-i}, \tilde{h}_i^{b,t}) = 1. \quad (36)$$

This holds by similar reasoning. Note that, if player  $i$  deviates in the 1\*-communication sub-block, this can only switch  $\theta(h_j^{T+1})$  from  $\theta$  to  $\emptyset$  (this is because, to have  $\theta(h_j^{T+1}) \neq \emptyset$ , player  $j$  needs to receive messages from every player except for  $i$  through the path excluding  $i$ ; hence, by telling a lie, player  $i$  can only create an inconsistency in the messages that player  $j$  receives), and hence cannot affect the probability that  $\theta \supseteq \theta(h_j^{T+1})$ .

**x-Communication Sub-Block** Players circulate message  $m = (m_i)_i = (\mathbf{x}_i)_i$ . Slightly abusing notation, let  $h_i^{\leq 0}$  denote player  $i$ 's history at the end of the  $\mathbf{x}$ -communication sub-block.

If player  $i$  infers  $m_{-j}(i) = \mathbf{error}$  for some  $j \neq i$ , we define  $\mathbf{x}(i) = \mathbf{B}$ , where  $\mathbf{B} \in \{G, B\}^{N|\Theta|}$  denotes the vector with  $B$  in every component. If instead  $i$  infers some  $m_{-j}(i) \in \times_{n \neq i} M_n$  for each  $j \neq i$ , then:

1. If there exists  $\hat{\mathbf{x}}_{-i} \in \{G, B\}^{(N-1)|\Theta|}$  such that  $m_{-j}(i)|_n = \hat{\mathbf{x}}_{-i}|_n$  for all  $j \neq i \neq n \neq j$ , we define  $\mathbf{x}(i) = (\mathbf{x}_i, \hat{\mathbf{x}}_{-i})$ .
2. Otherwise, we define  $\mathbf{x}(i) = \mathbf{B}$ .<sup>37</sup>

Finally, we define  $x(i) = \mathbf{x}(i)^{\theta(h_i^{T+1})}$  with  $\mathbf{x}(i)^\emptyset = \mathbf{B}$ .

**Supplemental Round 0** Players circulate message  $m = (m_i)_i = (h_i^{\leq 0})_i$ . Let  $h_i^{\leq 1}$  denote player  $i$ 's history at the end of supplemental round 0.

We define  $I^D(h_i^{\leq 1}) = 1$  if any of the following hold:

1.  $|\theta(h_i^{T+1})| < \alpha N$ .
2. Player  $i$  detectably deviates in either the  $\mathbf{x}$ -communication sub-block or supplemental round 0.
3.  $m_{-j}(i) = \mathbf{error}$  for some  $j \neq i$  in either the  $\mathbf{x}$ -communication sub-block or supplemental round 0.

Otherwise, for each  $j \neq i$ ,  $m_{-j}(i) = \times_{n \neq i} h_n^{\leq 0}$  for some  $\times_{n \neq i} h_n^{\leq 0} \in \times_{n \neq i} H_n^{\leq 0}$ . If there exists a player  $j \neq i$  such that, according to history  $(h_i^{\leq 0}, m_{-j}(i))$ , player  $j$  detectably deviated in the 1\*-communication sub-block or the  $\mathbf{x}$ -communication sub-block, then we define  $I^D(h_i^{\leq 1}) = 1$ . Otherwise, we define  $I^D(h_i^{\leq 1}) = 0$ .

**Main Sub-Block  $k$** ,  $k \in \{1, \dots, K\}$  For each  $k \in \{1, \dots, K\}$ , each player  $i$  enters sub-block  $k$  with state variables  $x(i) \in \{G, B\}^N$  and  $I^D(h_i^{\leq k}) \in \{0, 1\}$ . The state variable  $x(i)$  was determined at the end of the  $\mathbf{x}$ -communication sub-block, and remains constant throughout the main sub-blocks. The state variable  $I^D(h_i^{\leq 1})$  was determined at the end of supplemental round 0; the state variable  $I^D(h_i^{\leq k})$  may switch from 0 to 1 during some main sub-block, in which case it remains equal to 1 for the duration of the block.

We now define player  $i$ 's strategy in main sub-block  $k$  as a function of  $x(i)$  and  $I^D(h_i^{\leq k})$ , and then specify how  $I^D(h_i^{\leq k+1})$  evolves.

*Main round actions as a function of  $x(i)$  and  $I^D(h_i^{\leq k})$ :* At the beginning of each period, suppose  $j = \mu_i$ . Players  $(i, j)$  draw a joint controlled lottery, using the cheap talk. Specifically, each player draws  $m_{i,t} \in \{1, \dots, M^A\}$  uniformly at random. Define  $m_{(i,j),t} = m_{i,t} + m_{j,t} \pmod{M^A}$ . Note that, regardless of player  $i$ 's strategy,  $m_{(i,j),t}$  is distributed uniformly at random over  $\{1, \dots, M^A\}$ . As seen in (28), given  $x(i)$ , we can see that  $m_{(i,j),t}$  uniquely determines the realization of  $\alpha_{i,j}^{x(i)}$ . Let  $a_i^{x(i)}(m_{i,t}, m_{j,t})$  be the realization.

If  $I^D(h_i^{\leq k}) = 1$ , then player  $i$  takes  $D$  throughout the round. If  $I^D(h_i^{\leq k}) = 0$ , then if player  $i$  has not detectably deviated during the current main round, player  $i$  takes  $a_i^{x(i)}(m_{i,t}, m_{j,t})$ . Otherwise, she takes  $D$ .

<sup>37</sup>Note that this can occur even if  $m_{-j}(i) \neq \mathbf{error}$ , as in the situation noted in footnote 33.

Let  $h_i^{\leq k}$  denote player  $i$ 's history at the end of main round  $k$ . Note that we can recursively define that player  $i$  detectably deviates

1. if  $I^D(h_i^{\leq k}) = 0$ , she has not detectably deviated during the current main round, but she takes  $a_i \neq a_i^{x(i)}(m_{i,t}, m_{j,t})$ , or
2. if either  $I^D(h_i^{\leq k}) = 1$  or she has detectably deviated during the current main round, but she takes  $a_i \neq D$ .

*Supplemental round communication as a function of  $h_i^{\leq k}$* : Players circulate message  $m = (m_i)_i = (h_i^{\leq k})_i$ . For each  $j \neq i$ , we define  $m_{-j}(i)$  as in supplemental round 0.

*Determination of  $I^D(h_i^{\leq k+1})$* : Set  $I^D(h_i^{\leq k+1}) = 1$  if any of the following hold:

1.  $I^D(h_i^{\leq k}) = 1$ .
2. Player  $i$  detectably deviated during main sub-block  $k$ .
3.  $m_{-j}(i) = \mathbf{error}$  for some  $j \neq i$  during supplemental round  $k$ .
4. For each  $j \neq i$ ,  $m_{-j}(i) = \times_{n \neq i} h_n^{\leq k}$  for some  $\times_{n \neq i} h_n^{\leq k} \in \times_{n \neq i} H_n^{\leq k}$ , and there exists a player  $j \neq i$  such that, according to history  $(h_i^{\leq k}, m_{-j}(i))$ , player  $j$  detectably deviated during main round  $k$ .

Otherwise, set  $I^D(h_i^{\leq k+1}) = 0$ .

### B.2.9 Reward Function

Given the above block strategy profile, we now define the reward function  $\pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1})$ . For  $\theta$  satisfying  $i \notin \theta$ ,  $\pi_i^\theta$  is not defined (see in Lemma 13). For  $\theta$  satisfying  $|\theta| < \alpha N \wedge i \in \theta$ , define  $\pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) = 0$  for all  $(x_{-i}^\theta, h^{1^*}, h^{T^*+1})$ . This satisfies Conditions (32)–(34); we verify Condition (31) (sequential rationality) in the next subsection. For the remainder of this section, assume  $|\theta| \geq \alpha N \wedge i \in \theta$ .

Given  $(h^{1^*}, h^{T^*+1})$ , we define  $\chi_i(h^{1^*}, h^{T^*+1}) = 1$  if there exists a player  $j \neq i$  who detectably deviated from the prescribed block strategy (according to  $h^{T^*+1}$ ) or if the match realization was erroneous in any round in the current block (again, according to  $h^{T^*+1}$ ). We define  $\chi_i(h^{1^*}, h^{T^*+1}) = 0$  otherwise. Lemma 11 immediately implies the following result.

**Lemma 15** *Suppose player  $i$ 's opponents follow the prescribed strategy. For all  $l$  and all  $\theta$  (and regardless of player  $i$ 's own strategy), we have*

$$\Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 1 | \theta) \leq (3 + K) \exp(-cZ).$$

Next, define  $I_i^D(h^{1^*}, h^{T^*+1}) = 1$  if player  $i$  detectably deviated from the prescribed strategy, and define  $I_i^D(h^{1^*}, h^{T^*+1}) = 0$  otherwise. Finally, given  $(h^{1^*}, h^{T^*+1})$  satisfying  $I_i^D(h^{1^*}, h^{T^*+1}) = 0$ , define  $\hat{\mathbf{x}}_i(h^{1^*}, h^{T^*+1})$  to be that value of  $\hat{\mathbf{x}}_i$  for which the history  $(h_i^{1^*}, h_i^{T^*+1})$  is consistent with player  $i$  taking strategy  $\sigma_i^{T^*}(\hat{\mathbf{x}}_i, h_i^{1^*})$ . Such  $\hat{\mathbf{x}}_i$  is uniquely determined since player  $i$  communicates  $\hat{\mathbf{x}}_i$  in the  $\mathbf{x}$ -communication sub-block.

Define the function  $\pi_i^{\text{cancel}}(x_{i-1}^\theta, \mathbf{a}) : \{G, B\} \times A^N \rightarrow [-\bar{u}, \bar{u}]$  such that, for each  $\mathbf{a} \in A^N$ , we have

$$\begin{cases} \hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}^\theta, \mathbf{a}) = \text{sign}(x_{i-1}^\theta) \frac{1}{2} \bar{u} \\ \text{sign}(x_{i-1}^\theta) \pi_i^{\text{cancel}}(x_{i-1}^\theta, \mathbf{a}) \geq 0 \end{cases} \quad (37)$$

Thus, the function  $\pi_i^{\text{cancel}}(x_{i-1}^\theta, \mathbf{a})$  cancels player  $i$ 's instantaneous utility and leaves player  $i$  a negative (resp., positive) payoff when  $x_{i-1}^\theta = G$  (resp.,  $B$ )

If  $\chi_i(h^{1^*}, h^{T^*+1}) = 1$ , define

$$\hat{\pi}_i^\theta(x_{-i}, h^{T^*+1}) = \sum_{t=1}^{T^*} \pi_i^{\text{cancel}}(x_{i-1}^\theta, \mathbf{a}_t). \quad (38)$$

If  $\chi_i(h^{1^*}, h^{T^*+1}) = 0$ , define

$$\hat{\pi}_i^\theta(x_{-i}, h^{T^*+1}) = \begin{cases} \frac{1}{16} \{I_i^D(h^{1^*}, h^{T^*+1})=0\} \frac{\eta}{16} T^* & \text{if } x_{i-1}^\theta = B, \\ -1 \{I_i^D(h^{1^*}, h^{T^*+1})=1\} \bar{u} T^* & \text{if } x_{i-1}^\theta = G. \end{cases}$$

That is, if  $x_{i-1}^\theta = B$  then player  $i$  is rewarded if she follows the prescribed strategy; and if  $x_{i-1}^\theta = G$  then she is punished if she detectably deviates. Note that, for each  $(h^{1^*}, h^{T^*+1})$ , we have

$$\text{sign}(x_{i-1}^\theta) \hat{\pi}_i^\theta(x_{-i}, h^{T^*+1}) \geq 0. \quad (39)$$

Let

$$u_i(x^\theta, h^{1^*}) = \frac{1}{T^*} \mathbb{E}^{\sigma(\mathbf{x})} \left[ \sum_{\tau=1}^{T^*} \hat{u}_i(\mathbf{a}_\tau) + \hat{\pi}_i^\theta(x_{-i}, h^{T^*+1}) | h^{1^*}, \theta \right]. \quad (40)$$

Note that the right hand side of (40) depends only on  $x^\theta$  and  $h^{1^*}$  since (i) when  $\chi_i(h^{1^*}, h^{T^*+1}) = 0$ , the action pair  $(a_i, a_{\mu_i})$  is drawn from  $\alpha_{i, \mu_i}^{x^\theta}$  given  $\mu$  in main rounds, (ii) when  $\chi_i(h^{1^*}, h^{T^*+1}) = 1$ , (37) implies that player  $i$ 's payoff from  $\hat{u}_i$  and  $\pi_i^{\text{cancel}}$  depends only on  $x_{i-1}^\theta$  in main rounds, (iii) the distribution of  $\chi_i(h^{1^*}, h^{T^*+1})$  is determined by the match realization, and (iv) in non-main rounds, players take defection for sure.

Note that

$$\begin{aligned} & \left| u_i(x^\theta, h^{1^*}) - \hat{u}_i(\alpha^{x^\theta}) \right| \\ &= \Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 0 | \theta) \left| \frac{1}{T^*} \mathbb{E}^{\sigma(\mathbf{x})} \left[ \left( \frac{\sum_{\tau=1}^{T^*} \hat{u}_i(\mathbf{a}_\tau)}{+\hat{\pi}_i^\theta(x_{-i}, h^{T^*+1})} \right) | h^{1^*}, \theta, \chi_i(h^{1^*}, h^{T^*+1}) = 0 \right] - \hat{u}_i(\alpha^{x^\theta}) \right| \\ & \quad + \Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 1 | \theta) \left| \frac{1}{T^*} \mathbb{E}^{\sigma(\mathbf{x})} \left[ \left( \frac{\sum_{\tau=1}^{T^*} \hat{u}_i(\mathbf{a}_\tau)}{+\hat{\pi}_i^\theta(x_{-i}, h^{T^*+1})} \right) | h^{1^*}, \theta, \chi_i(h^{1^*}, h^{T^*+1}) = 1 \right] - \hat{u}_i(\alpha^{x^\theta}) \right| \\ &\leq \Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 0 | \theta) \left( \frac{(3+K)T}{T^*} \bar{u} + \frac{\eta}{16} \right) + \Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 1 | \theta) \left( \bar{u} + \frac{\eta}{16} \right) \\ &\leq \frac{\eta}{16} + \left( \frac{3+K}{Z} + \Pr(\chi_i(h^{1^*}, h^{T^*+1}) = 1 | \theta) \right) \bar{u} \\ &\leq \frac{\eta}{16} + (3+K) \left( \frac{1}{Z} + \exp(-cZ) \right) \bar{u} \quad (\text{by Lemma 15}) \\ &\leq \frac{\eta}{8} \quad (\text{by (14)}). \end{aligned} \quad (41)$$

Here the first inequality follows because (i) when  $\chi_i(h^{1^*}, h^{T^*+1}) = 0$ , the action pair  $(a_i, a_{\mu_i})$  is drawn from  $\alpha_{i, \mu_i}^{\theta}$  given  $\mu$  in main rounds, (i.e., in all but  $(3 + K)T$  periods), (ii) the magnitude of  $\hat{u}_i(\mathbf{a}_\tau)$  is bounded by  $\bar{u}/2$ , and (iii) on-path (i.e., when  $I_i^D(h^{1^*}, h^{T^*+1}) = 0$ ), the magnitude of  $\hat{\pi}_i^\theta(x_{-i}^\theta, h^{T^*+1})$  is bounded by  $\frac{\eta}{16}T^*$ .

We now define the reward function

$$\pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) = \left( v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1^*} \right) \right) T^* + \hat{\pi}_i^\theta(x_{-i}^\theta, h^{T^*+1}). \quad (42)$$

We verify that, with this reward function, Conditions (31)–(34) are satisfied. This will complete the proof. We first establish Conditions (32)–(34), deferring Condition (31) (sequential rationality) to the next subsection.

Since  $I_i^D(h^{1^*}, h^{T^*+1}) = 0$  on path, (40) implies that expected per-period block payoffs given  $|\theta| \geq \alpha N \wedge i \in \theta$  equal  $v_i^\theta(x_{i-1}^\theta)$ . Hence, (32) holds.

By (41) and (29), we have

$$\text{sign} \left( x_{i-1}^\theta \right) \left( v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1^*} \right) \right) \geq \frac{1}{8}\eta.$$

Together with (39), this implies

$$\text{sign} \left( x_{i-1}^\theta \right) \pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) \geq \frac{1}{8}\eta T^*, \quad (43)$$

and hence (33).

Moreover, if  $I_i^D(h^{1^*}, h^{T^*+1}) = 0$  then

$$\left| \pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) \right| \leq \left| v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1^*} \right) \right| T^* + \frac{\eta}{16}T^* \leq 2\bar{u}T^*;$$

and if  $I_i^D(h^{1^*}, h^{T^*+1}) = 1$  then

$$\left| \pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) \right| \leq \left| v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1^*} \right) \right| T^* + \bar{u}T^* \leq 2\bar{u}T^*.$$

Hence, (34) holds.

### B.2.10 Verifying Sequential Rationality (Conditions (30) and (31))

Given (38) and  $|\theta| \geq \alpha N \wedge i \in \theta$ , if  $\chi_i(h_{1^*}, h^{T^*+1}) = 1$ , then any action is optimal for player  $i$ . Since  $\Pr(\chi_i(h_{1^*}, h^{T^*+1}) = 1 | \sigma_i, \theta)$  is independent of  $\sigma_i$ , it is without loss to verify sequential rationality conditional on the event  $\{\chi_i(h^{1^*}, h^{T^*+1}) = 0 \vee |\theta| < \alpha N \vee i \notin \theta\}$ . We thus restrict attention to pairs  $(\mathbf{x}_{-i}, \tilde{h}_i^{b,t})$  such that  $\Pr(\{\chi_i(h^{1^*}, h^{T^*+1}) = 0 \vee |\theta| < \alpha N \vee i \notin \theta\} | \mathbf{x}_{-i}, \tilde{h}_i^{b,t}) > 0$ . Note this implies that (35) and (36) hold conditional on the triple  $(\{\chi_i(h^{1^*}, h^{T^*+1}) = 0 \vee |\theta| < \alpha N \vee i \notin \theta\}, \mathbf{x}_{-i}, \tilde{h}_i^{b,t})$ .

We consider separately the cases  $|\theta| < \alpha N \vee i \notin \theta$  and  $\{\chi_i(h^{1^*}, h^{T^*+1}) = 0 \wedge |\theta| \geq \alpha N \wedge i \in \theta\}$ .

Conditional on  $|\theta| < \alpha N \vee i \notin \theta$ , by (36), player  $i$  believes that  $|\theta(h_n^{T+1})| < \alpha N \vee i \notin \theta(h_n^{T+1})$  for each  $n \neq i$ . Hence, player  $i \in \theta$  believes that players  $-i$  take  $D$  throughout the block regardless of her own strategy. In addition, player  $i \in \theta$  believes that  $\pi_i^\theta(x_{-i}^\theta, h^{1^*}, h^{T^*+1}) = 0$  regardless of her own strategy (recall that  $\pi_i^\theta$  is defined only for  $i \in \theta$ ). It is therefore optimal for rational player  $i$  to take  $D$  in each period and send any messages, and this behavior is indeed what is prescribed for player  $i$ , since (35) implies that  $|\theta(h_i^{T+1})| < \alpha N \vee i \notin \theta(h_i^{T+1})$ . For bad player  $i$ , her strategy does

not affect the action sequence that she faces.

It remains to verify sequential rationality conditional on  $\{\chi_i(h^{1*}, h^{T^*+1}) = 0 \wedge |\theta| \geq \alpha N \wedge i \in \theta\}$ . We proceed in three steps.

*It is optimal to take D and send any message after player i detectably deviates after the 1\*-communication sub-block.*

Let  $\tau$  be the first period in which player  $i$  detectably deviated. First, suppose that  $\tau$  is before supplemental round 0. Then, regardless of player  $i$ 's behavior after period  $\tau$ , the fact that  $\chi_i(h^{1*}, h^{T^*+1}) = 0$  (and hence matching is regular) implies that players  $-i$  will become aware of player  $i$ 's deviation at the end of supplemental round 0 and will then take  $D$  for the rest of the block. Moreover, the reward function is constant:

$$\begin{aligned} \pi_i^\theta(x_{-i}^\theta, h^{1*}, h^{T^*+1}) &= \left( v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1*} \right) \right) T^* + \hat{\pi}_i^\theta(x_{-i}^\theta, h^{T^*+1}) \\ &= \left( v_i^\theta(x_{i-1}^\theta) - u_i \left( \hat{x}_i^\theta(h^{T^*+1}), x_{-i}^\theta, h^{1*} \right) \right) T^* + \begin{cases} 0 & \text{if } x_{i-1}^\theta = B, \\ -\bar{u}T^* & \text{if } x_{i-1}^\theta = G. \end{cases} \end{aligned}$$

Hence, taking  $D$  and sending any messages is optimal for player  $i$ .

Second, suppose  $\tau$  is in or after supplemental round 0. Then, regardless of player  $i$ 's behavior after period  $\tau$ , players  $-i$  take  $\hat{\mathbf{a}}^{x^\theta}$  in the main sub-block and take  $D$  in other rounds until next supplemental round; and subsequently (since matching is regular) they will switch to  $D$  for the rest of the block. Again, the reward is constant. Hence, taking  $D$  and sending any messages is optimal.

*It is optimal not to detectably deviate from the equilibrium strategy at on-path histories.*

We compare the maximum gain in within-block payoffs from a detectable deviation to the minimum loss in the reward function. Since matching is regular, players  $-i$  switch to  $D$  starting in the next main round. Hence, the maximum gain in within-block payoffs is at most  $ZT \times \max\{G, L\}$ . In contrast, if  $x_{i-1}^\theta = B$ , the loss in the reward function from switching  $I_i^D(h^{1*}, h^{T^*+1})$  from 0 to 1 is at least  $\frac{\eta}{16}T^*$ ; this comes from the  $\hat{\pi}_i^\theta(x_{-i}^\theta, h^{T^*+1})$  term in the reward function. By (16),  $\frac{\eta}{16}T^* \geq ZT \times \max\{G, L\}$ , so deviating is unprofitable when  $x_{i-1}^\theta = B$ . If instead  $x_{i-1}^\theta = G$ , the loss in the reward function from switching  $I_i^D(h^{1*}, h^{T^*+1})$  from 0 to 1 is at least  $\bar{u}T^* \geq ZT \times \max\{G, L\}$ . In total, for any  $x_{i-1}^\theta$ , the net deviation gain is negative.

*It is optimal to send message  $\hat{\mathbf{x}}_i = \mathbf{x}_i$  in the  $\mathbf{x}$ -communication sub-block.*

We show that, for any  $\mathbf{x}_{-i}$ , player  $i$  is indifferent among the block strategies  $(\sigma_i(\mathbf{x}_i))_{\mathbf{x}_i}$ . By (40) and (42), player  $i$ 's expected payoff conditional on  $|\theta| \geq \alpha N \wedge i \in \theta$  equals

$$\mathbb{E} \left[ \frac{1}{T^*} \mathbb{E}^{\sigma(\mathbf{x})} \left[ \sum_{\tau=3T+1}^{T^*} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^\theta(x_{-i}^\theta, h^{T^*+1}) | h^{1*}, \theta \right] | \mathbf{x}_{-i} \right] = v_i^\theta(x_{i-1}^\theta).$$

Since these payoffs depend on  $\mathbf{x}$  only through  $x_{i-1}^\theta$ , and additionally  $\Pr(\chi_i(h^{1*}, h^{T^*+1}) = 1)$  is independent of  $\mathbf{x}$ , it follows that player  $i$ 's expected payoff conditional on  $\{\chi_i(h^{1*}, h^{T^*+1}) = 0 \wedge |\theta| \geq \alpha N \wedge i \in \theta\}$  also depends on  $\mathbf{x}$  only through  $x_{i-1}^\theta$ . This completes the proof of Theorem 3.

### B.3 De-Coupling Interaction and Community Frequency in Theorem 3

We explain how the proof of Theorem 3 must be modified when interaction and communication meetings are de-coupled.

In the proof of Theorem 3 (for our main model with coupled interaction and communication), the length of each communication round (i.e., the 1\*-communication sub-block, the  $\mathbf{x}$ -communication sub-block, and the supplemental rounds) must be much greater than  $\log N$ , so that messages spread

throughout the population with high probability, while the number of communication rounds is independent on  $N$ . The proof requires that discounting in each communication round is negligible, which holds if  $(1 - \delta) \log N \rightarrow 0$ . In addition, since players take  $D$  during communication rounds, the length of the main rounds must also increase with  $N$ , so that the vast majority of periods occur in main rounds rather than communication rounds (and thus players' overall repeated game payoffs are determined by their payoffs in the main rounds). The proof also requires that discounting in each main round is negligible; since the length of a main round can be taken to be of the same order as the length of a communication round, this also holds if  $(1 - \delta) \log N \rightarrow 0$ .

If interaction and communication are de-coupled, so that players communicate every  $\Delta_M$  periods and interact every  $\Delta_A$  periods (with a constant real-time discount rate), the length (i.e., the number of meetings) in each communication round must again be much greater than  $\log N$ , and discounting in each communication round must be negligible. This now holds if  $\Delta_M \log N \rightarrow 0$ . Moreover, since now players do not accrue payoffs during communication rounds, the length of the main rounds may be fixed independently of  $N$ . This proof again requires that discounting in each main round is negligible, but since the length of the main rounds is now independent of  $N$ , this requires only that  $\Delta_A \rightarrow 0$ .

Given these observations, the proof of the de-coupled version of Theorem 3 under the assumptions that  $\Delta_M \log N \rightarrow 0$  and  $\Delta_A \rightarrow 0$  is very similar to the proof for the coupled version, but is somewhat simpler because we no longer need to account for the payoffs that players accrue during the communication rounds.

(One subtlety is that the proof of Theorem 3 relies on cheap talk within the main rounds to coordinate matched partners on a correlated action distribution  $\alpha_{i,j}^{x(\theta)}$ . This main round communication can be dispensed with at the cost of complicating the messages players exchange during the communication rounds. It can also be dispensed with without changing the proof if the target payoff  $v_i^\theta$  lies in  $[0, 1]$  for each  $i \in \theta$  and  $|\theta| \geq \alpha N$ , because there then exists a degenerate distribution  $\alpha^{x^\theta}$  that satisfies the relevant condition, inequality (29).)