Cost Based Nonlinear Pricing

Dirk Bergemann (Yale)       Tibor Heumann (UPC)
Stephen Morris (MIT)

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Introduction

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- optimal nonlinear pricing,
  - depends heavily on information about demand distribution
  - e.g., optimal mark-up is equal to reciprocal of demand elasticity
digital commerce on large (global) platforms comes with heterogeneous consumers and much demand variation across time and space
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  1. is independent of specific distribution of willingness-to-pay
  2. exhibits profit guarantee across all distributions without any restrictions on demand, such as moment restrictions, support restrictions, etc., profit guarantee must be relative (or proportional) rather than absolute.
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- Sellers have weak information about demand distribution.
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  2. Exhibits profit guarantee across all distributions.
- Without any restrictions on demand, such as moment restrictions, support restrictions, etc., profit guarantee must be relative (or proportional) rather than absolute.
Profit Guarantee as Competitive Ratio

- a mechanism will be evaluated by the ratio of the realized profit to feasible surplus (=complete information / first-degree price discrimination profit)
- the profit guarantee / "competitive ratio" is the infimum of this ratio across all demand distributions
- term originated in analysis of online vs. offline algorithms to express related informational constraints
Two Classes of Pricing Problems

1. quality differentiated pricing problems
   Mussa and Rosen (1978)
   - linear willingness-to-pay for quality
   - cost is increasing, convex function of quality

2. quantity differentiated pricing
   Maskin and Riley (1984)
   - concave willingness-to-pay for quantities
   - constant marginal cost of producing additional units
Today’s Talk

- quality differentiated pricing problems: Mussa and Rosen (1978)
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first pass: iso-elastic cost function
Today’s Talk

- quality differentiated pricing problems: Mussa and Rosen (1978)

  1. first pass: iso-elastic cost function
  2. then, general cost functions
First Result: Positive Profit Guarantee

- profit guarantee / competitive ratio is strictly positive and bounded away from zero
- profit guarantee / competitive ratio is a simple monotone function of cost elasticity
- profit guarantee / competitive ratio is sharp:
  - identify demand distribution under which robust policy coincides with Bayes optimal mechanism
Second Result: Constant Mark-Up

- derive indirect mechanism – quality tariff—that attains profit guarantee
- optimal tariff is a constant mark-up policy
- simple and transparent pricing policy that attains profit guarantee
- mark-up is determined by cost elasticity alone without reference to demand data, thus:
  - cost based nonlinear pricing
One More Result: Consumer Surplus

- profit guarantee is solution of profit optimization problem
- solution is agnostic about consumer surplus
- yet, robust pricing rule generates large consumer surplus
- how large? for every cost elasticity, find the largest share of consumer surplus across all distributions (and Bayes optimal mechanisms)
- the maximum is attained by robust pricing rule for all demand distributions
- thus robust pricing policy succeeds by creating consumer surplus
optimal monopoly pricing for single unit demand
Neeman (2003), Maglaras (2009), Hartline and Roughgarden (2014), etc...
with support $[1, h]$, competitive ratio: $1/(1 + \ln h)$
competitive ratio vanishes as support restriction weakens
single unit pricing requires randomized reserve price, i.e., many prices and assignment probabilities
menu already arises for efficient allocation, now finds second use to hedge against demand uncertainty
Model
Model

- buyer has value \( v \in \mathbb{R}_+ \) (willingness-to-pay) for quality \( q \in \mathbb{R}_+ \)
  \[
  u(v, q, t) = v \cdot q - t
  \]
- value \( v \) is private information
- seller offers quality differentiated products \( q \) at cost
  \[
  c(q) = q^\eta / \eta, \quad \eta \in (1, \infty)
  \]
- cost elasticity \( \eta \):
  \[
  \frac{dc(q)}{c(q)} \cdot \frac{dq}{q} = \frac{dc(q)}{dq} = \frac{c'(q)q}{c(q)} = \eta
  \]
Payoffs and Menu

- seller chooses menu $M$ (or direct mechanism) with qualities $Q(v)$ at prices $T(v)$:

  $$M \triangleq \{(Q(v), T(v))\}_{v \in \mathbb{R}^+}$$

- incentive compatibility and participation constraints,

  $$vQ(v) - T(v) \geq vQ(v') - T(v');$$
  $$vQ(v) - T(v) \geq 0; \quad \forall v, v' \in \mathbb{R}^+$$

- profit and consumer surplus with menu $M$ and value $v$:

  $$\Pi_M (v) \triangleq T(v) - c(Q(v)),$$
  and

  $$U_M (v) \triangleq Q(v)v - T(v).$$
First Degree Price Discrimination

- profit with complete information is profit with perfect or first-degree price discrimination
  \[ \Pi(v) \triangleq \max_{q} \{ vq - c(q) \} = \]

- supported by socially efficient allocation:
  \[ Q(v) \triangleq \arg \max_{q} \{ vq - c(q) \} = v \frac{1}{n} \]

- first degree price discrimination captures social surplus
  \[ \Pi(v) \triangleq \max_{q} \{ vq - c(q) \} \triangleq S(v) \]
Second Degree Price Discrimination

- given distribution $F$:
  \[ F \in \Delta([v, \bar{v}]), \quad 0 \leq v < \bar{v} \leq \infty \]

- expected profit and surplus with $M$:
  \[ \Pi_{F,M} \triangleq \mathbb{E}[T(v) - c(Q(v))], \]
  and
  \[ U_{F,M} \triangleq \mathbb{E}[Q(v)v - T(v)]. \]

- Bayes optimal menu with distribution $F$:
  \[ M_F \triangleq \arg \max_M \Pi_{F,M}. \]

- with some abuse of notation
  \[ \Pi_F = \Pi_{F,M_F} \quad \text{and} \quad U_F = U_{F,M_F}. \]
Competitive Ratio

- we are interested in ratio of profit under unknown distribution to profit under known distribution
- competitive ratio of mechanism $M$:
  \[
  \inf_F \frac{\Pi_{F,M}}{\Pi_F}
  \]
- find optimal profit-guarantee menu $M^*$ defined as:
  \[
  M^* = \arg\max_M \inf_F \frac{\Pi_{F,M}}{\Pi_F}
  \]
- as by-product find distribution of values that minimizes seller’s normalized profit:
  \[
  \inf_F \max_M \frac{\Pi_{F,M}}{\Pi_F}
  \]
Analysis
Profit Guarantee

- consider a given mechanism \( M = \{Q(v), T(v)\} \)
- how well is mechanism \( M \) performing across different demand distributions \( F \)?
- how well is the mechanism \( M \) performing against a most challenging distribution \( F^* \)?
- referred to as competitive ratio of \( M \):

\[
\inf_F \frac{\Pi_{F,M}}{\Pi_F} < 1
\]

- profit-guarantee menu \( M^* \) maximizes competitive ratio

\[
M^* = \arg \max_M \inf_F \frac{\Pi_{F,M}}{\Pi_F}
\]
Competitive Ratio and Adversarial Nature

- profit-guarantee menu $M^*$ maximizes competitive ratio

$$M^* = \arg \max_M \inf_F \frac{\Pi_{F,M}}{\Pi_F}$$

- minmax theorem suggest distribution $F^*$ that minimizes

$$\max_M \frac{\Pi_{F,M}}{\Pi_F}$$

- and thus

$$F^* = \arg \min_F \max_M \frac{\Pi_{F,M}}{\Pi_F}$$

- and indeed there is a saddle-point:

$$\max_M \inf_F \frac{\Pi_{F,M}}{\Pi_F} = \min \sup_M \frac{\Pi_{F,M}}{\Pi_F}$$
First Step Toward Solution: Local

- competitive ratio is stated in terms of expectations:

\[
\inf_F \frac{\prod_{F,M}}{\prod_F} = \inf \left\{ \frac{\int (T(v) - c(Q(v))) \, dF(v)}{\int (\overline{T}(v) - c(\overline{Q}(v))) \, dF(v)} \right\}
\]

- given menu \( M = \{T(v), Q(v)\} \), nature chooses demand \( F \) that lowers the profit guarantee

- nature puts weight on values \( v \) where guarantee is weak:

\[
\inf_v \left\{ \frac{T(v) - c(Q(v))}{\overline{T}(v) - c(\overline{Q}(v))} \right\}
\]

- to defend against such attacks find menu \( M \) where pointwise (local) guarantee is as high as possible, uniformly across all \( v \):

\[
\frac{T(v) - c(Q(v))}{\overline{T}(v) - c(\overline{Q}(v))} = k, \quad \forall v.
\]
Second Step Toward Solution: Proportional

- social surplus is generated by efficient choice $\overline{Q}(v)$
- maintain profit guarantee by staying with a constant proportion $s$ of $\overline{Q}(v)$:
  \[
s \cdot \overline{Q}(v), \quad s \in (0, 1)
\]
- gross revenue grows at rate $s$, cost increases at rate $s^\eta$
- find optimal trade-off

\[
\max_s \{ s - s^\eta \} \iff s^* = \left( \frac{1}{\eta} \right)^{\frac{1}{\eta-1}}
\]
A Profit Guarantee Menu

- construct a menu with a profit guarantee

**Theorem (Profit Guarantee Mechanism)**

The menu $M^*$:

$$Q^*(v) = s^* \cdot \overline{Q}(v) = \left( \frac{1}{\eta} \right)^{\frac{1}{\eta-1}} \cdot \frac{1}{v^{\frac{1}{\eta-1}}},$$

generates a profit guarantee

$$\frac{\Pi^*(v)}{\Pi(v)} = \left( \frac{1}{\eta} \right)^{\frac{\eta}{\eta-1}},$$

for every value $v$ and a fortiori every distribution $F$.

- thus profit guarantee is share $s^*$ powered by elasticity $\eta$
Return to Minmax

- profit-guarantee menu $M^*$ must be Bayes-optimal
- given $F$, $M^*$ solves

$$\arg \max_{M} \frac{\Pi_{F,M}}{\Pi_{F}} \iff \arg \max_{M} \Pi_{F,M}$$

- candidate optimal quality $Q^*$ is constant share $s^*$ of socially efficient quality $\overline{Q}(v)$
- candidate optimal quality $Q^*$ is obtained by virtual value proportional to value
- Pareto distribution uniquely generates virtual value that is linear in value
Pareto Distribution

- Pareto distribution with shape parameter \( \alpha \in [1, \infty) \):

\[
F_\alpha(v) \triangleq \begin{cases} 
0, & \text{if } v < 1; \\
1 - \frac{1}{v^\alpha}, & \text{if } v \geq 1;
\end{cases}
\]

- virtual value with Pareto distribution

\[
\phi(v) \triangleq v - \frac{1 - F_\alpha(v)}{f_\alpha(v)} = v - \frac{v^\alpha}{\alpha v^{\alpha-1}} = \frac{\alpha - 1}{\alpha} v
\]

- \( \alpha = 1 \) is equal revenue distribution prominent in unit demand pricing analysis
Pareto Distribution and Virtual Values

- Pareto distribution and virtual values
Profit Guarantee Menu is Optimal

- profit guarantee gave us a specific lower bound, can we do better?

Theorem (Minmax Distribution)

Menu $M^*$ is Bayes optimal for Pareto distribution $\alpha$:

$$\alpha = \frac{\eta}{\eta - 1},$$

and attains infimum:

$$\inf_{F} \frac{\prod_{F}}{\prod_{\hat{F}}} = \left. \frac{\prod_{F\alpha}}{\prod_{\hat{F}\alpha}} \right|_{\alpha = \frac{\eta}{\eta - 1}} = \left( \frac{1}{\eta} \right)^{\frac{\eta}{\eta - 1}}.$$

- Pareto distribution $\alpha = \eta/(\eta - 1)$: least normalized profit
- profit guarantee is a sharp bound
Consumer Surplus

- minmax solution generates particular pair of surplus sharing among seller and buyers

**Corollary (Consumer Surplus with $M^*$)**

Menu $M^*$ generates constant consumer surplus:

$$\frac{U_{M^*}(v)}{\Pi(v)} = \frac{U_{M^*}(v)}{S(v)} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

for every value $v$ and a fortiori every distribution $F$.

- in profit-guarantee menu, each consumer receives the same share of the efficient social surplus.
- how does consumer surplus guarantee compare to consumer surplus attained across all Bayes optimal menus?
Maximum Consumer Surplus

- recall consumer surplus with known demand:

\[ U_F = U_{F,M_F} \]

Corollary (Maximum Consumer Surplus)

The consumer surplus is bounded above as follows,

\[ \sup_F \frac{U_F}{S_F} = \left( \frac{1}{\eta} \right)^{ \frac{1}{\eta - 1} } \]

and is attained by the Pareto distribution with shape parameter

\[ \alpha = \frac{\eta}{\eta - 1}. \]

- profit guarantee concedes consumer surplus to stay near efficient allocation
Profit Share and Cost Elasticity

- profit (share) guarantee

\[ \frac{n}{s} \]

\[ (1/\eta)^{\eta/(\eta-1)} \]

- limit \( \eta \to 1 \) corresponds to nearly constant marginal cost
- limit \( \eta \to \infty \) corresponds to selling an indivisible good.
Profit and Consumer Surplus

- Profit and consumer surplus share move in opposite direction as cost elasticity increases
Social Surplus

- profit and cs move in opposite direction as $\eta$ increases
- realized social surplus increases with cost elasticity $\eta$
- uniform lower bound $2/e$
Indirect Mechanism

- indirect mechanism (tariff) asks price $P(q)$ for quality $q$
- marginal price for quality, the price-per-quality increment:
  \[
  P'(q) \triangleq p(q)
  \]
- for quality $q$ the total payment is:
  \[
  P(q) = \int_0^q p(s)ds
  \]
- incentive compatibility will imply that
  \[
  p(q(v)) = Q^{-1}(q(v)) = v.
  \]
Mark-Up Pricing

Corollary (Constant Mark-Up)

The menu $M^*$ is implemented by offering quality increments $q \in \mathbb{R}$ at a price $p(q)$ satisfying:

$$\frac{p(q) - c'(q)}{c'(q)} = \eta - 1 \iff p(q) = \eta c'(q).$$

- constant mark-up of cost:

  $$\eta > 1$$

- price depend on cost information only, demand information is entirely absent

- alternatively, expressing pricing in terms of Lerner’s index:

  $$\frac{p(q) - c'(q)}{p(q)} = \frac{\eta - 1}{\eta}$$

- again, a constant measure of market power
Contrast to Bayesian Optimal Menu

- for a given prior distribution $F$ optimal quantity is:

$$q(v) \in \arg \max_q \left\{ \left( v - \frac{1 - F(v)}{f(v)} \right) q - c(q) \right\}.$$  

- first-order condition is given by:

$$v - \frac{1 - F(v)}{f(v)} - c'(q(v)) = 0,$$

- incentive compatible transfers:

$$T'(v) = q'(v)v.$$  

- price per marginal unit of quality is given by:

$$p(q(v)) = \frac{T'(v)}{q'(v)} = v.$$
Demand Elasticity

- demand for quality $q$ at incremental quality price $p(q(v))$:

$$D(p(q(v)))) = 1 - F(v).$$

- resulting markup:

$$\frac{p(q) - c'(q)}{p(q)} = \frac{v - (v - \frac{1-F(v)}{f(v)})}{v} = \frac{1 - F(v)}{f(v)v}.$$

- rhs is negative of reciprocal of demand elasticity:

$$\frac{1 - F(v)}{f(v)v} = -\frac{D(p(v))}{p(v)} = \frac{D'(p(v))}{D(p(v))}.$$

- classic formula for Lerner’s index
Lerner’s Index

- Classic formula for the Lerner’s index:

\[
\frac{p(q) - c'(q)}{p(q)} = 1 - F(v) = -\frac{D(p(v))}{p(v)}
\]

- Bayes-optimal mechanism determined by demand elasticity—expressed in terms of product of value \( v \) and hazard rate \( f(v) / (1 - F(v)) \)

- Profit-guarantee menu is determined only by cost elasticity

\[
\frac{p(q) - c'(q)}{p(q)} = \frac{\eta - 1}{\eta}
\]

- Profit-guarantee is accomplished across all possible distribution of values, no reference to specific distribution
Constant Mark-Up and Mirrlees

- we did not impose any restrictions on distribution of willingness-to-pay
  - no monotonicity or regularity restrictions on $F$
  - no support restrictions on $F$
- critical demand is Pareto with unbounded support
- thus "no distortion at the top" fails to hold, instead constant mark-up
- related insights in optimal taxation literature
Beyond Constant Elasticity
Non-Constant Cost Elasticity

- pointwise cost elasticity:

\[ \eta(q) = \frac{dc(q)}{dq} \frac{q}{c(q)}. \]

- pointwise marginal cost elasticity

\[ \gamma(q) = \frac{dc'(q)}{dq} \frac{q}{c'(q)}. \]

- cost with constant elasticity has simple relation:

\[ \gamma(q) = \eta(q) - 1, \]
Approximation

- use constant elasticity informed pricing
- obtain profit guarantees for non-constant elasticity
- weaker guarantees, transparent approximation
Proportional Mark Up Pricing

- tariff $P(q)$ and price $p(q)$ for quality increment:
  \[ p(q) = P'(q) \]

- price per quality proportional to marginal cost elasticity
  \[ \hat{p}(q) \triangleq (1 + \gamma(q))c'(q) \]

- tariff in terms of markup:
  \[ \frac{\hat{p}(q) - c'(q)}{c'(q)} = \gamma(q) \]
Lower Bound

- establish a relationship between profit and social surplus
- profit with $v$ is equal surplus with $w$, where
  \[ w(v) < v \]

Lemma (Profit as Downward Shifted Social Surplus)

The tariff attains profit as downward shifted social surplus:

\[ w(v) = \frac{v}{1 + \gamma(q(v))}, \]

and

\[ \Pi(v) = S(w(v)). \]

- monotone relationship between profit and social surplus
Bounded Cost Elasticity

- consider bounds on marginal cost elasticity:

\[ \gamma(q) \in [\gamma, \bar{\gamma}], \quad \forall q \]

**Proposition**

*Suppose the elasticity is bounded \( \gamma(q) \in [\gamma, \bar{\gamma}], \forall q, \) then:*

\[
\frac{\Pi(v)}{S'(v)} \geq \left( \frac{1}{\bar{\gamma} + 1} \right)^{\frac{\gamma+1}{\gamma}}.
\]

- relative to constant elasticity, bound is weaker as base and exponent are formed by lower and upper bound of marginal cost elasticity
- coincides with constant elasticity result if \( \gamma = \bar{\gamma} \)
Sharp Bound

- consider class of increasing cost elasticity:

\[ \gamma'(q) \geq 0 \quad \text{and} \quad \lim_{q \to \infty} \gamma(q) = \gamma < \infty. \]

Proposition

If marginal cost elasticity \( \gamma(q) \) is increasing with limit \( \gamma \), then proportional pricing generates decreasing ratio:

\[
\frac{\Pi(v)}{S(v)} \geq \left( \frac{1}{\gamma + 1} \right)^{\gamma+1}
\]

and the bound is attained in the limit \( v \to \infty \).

- a generalization of Pareto distribution with variable shape parameter delivers a Bayesian optimal mechanism
Additional Demand Information
Additional Demand Information

- we have worked without any information about demand
- with additional information about demand, we may increase the profit guarantee
- suppose we know lower and upper bounds on the support of the value distribution, thus
  \[ 0 \leq v < \bar{v} < \infty. \]
New Results

- main insights of Theorem 1 and 2 remain in the presence of additional support information:
  1. there exists a minmax solution
  2. competitive ratio between realized profit and social surplus are constant at every point in support of demand

- some changes with finite support:
  1. optimal menu does not display constant mark-up anymore
  2. "no-distortion at the top" result re-emerges
Minmax Solution

- find allocation $q(v)$ and distribution $F(v)$ such that:

$$\max_{\{q: \underline{v}, \bar{v} \rightarrow \mathbb{R}\}} \inf_{F \in \Delta [\underline{v}, \bar{v}]} \frac{\int \Pi_q(v) dF(v)}{\int S(v) dF(v)},$$

- denote a solution by $(q^*, F^*)$.

Proposition (Existence)

There exists $(q^*, F^*)$ such that:

$$\max_{\{q: \underline{v}, \bar{v} \rightarrow \mathbb{R}\}} \inf_{F \in \Delta [\underline{v}, \bar{v}]} \frac{\int \Pi_q(v) dF(v)}{\int S(v) dF(v)} = \min_{F \in \Delta [\underline{v}, \bar{v}]} \sup_{\{q: \underline{v}, \bar{v} \rightarrow \mathbb{R}\}} \frac{\int \Pi_q(v) dF(v)}{\int S(v) dF(v)}.$$

And $q^*$ is the optimal Bayesian mechanism when the distribution is $F^*$. 
Construction of Constant Competitive Ratio

- define a family of allocations, parameterized by $\beta \in [0, 1]$, denoted by $q_{\beta} : [\underline{v}, v_{\beta}] \rightarrow \mathbb{R}$
- defined implicitly by:

$$\frac{\Pi_{q_{\beta}}(v)}{S(v)} = \beta$$

- upper bound of domain $v_{\beta}$ is upper bound on which condition can be maintained

Proposition (Constant Profit-Surplus Ratio)

1. The profit-guarantee mechanism $q^*$, is given by $q^*(v) = q_{\beta}(v)$, with $\beta$ such that $v_{\beta} = \bar{v}$.

2. The allocation rule $q_{\beta}$ is increasing in $\beta$ and $v_{\beta}$ is decreasing in $\beta$. 
Upper Bound

- with finite support there is no explicit solution for competitive ratio even with constant elasticity
- provide an upper bound by means of a Bayes optimal mechanism
- converges to exact solution as upper bound of support diverges to $\infty$.

Proposition (Bounded Support)

There exists a distribution $F$ with support in $[v, \bar{v}]$ such that the Bayesian optimal mechanism generates normalized profits:

$$\frac{\Pi}{S} = \frac{1}{\eta \frac{n}{\eta-1}} + \left(1 - \frac{1}{\eta \frac{n}{\eta-1}}\right) \frac{1}{1 + \frac{n}{\eta-1} \log(\bar{v})}.$$ 

- constitutes an upper bound on profit guarantee
Approximation and Finite Support

- how does competitive ratio degrade with size of support?

Here $\eta = 2, v = 1$; result are invariant for $\overline{v}/\underline{v}$
Boundaries of Surplus Sharing
Constrained Efficient Surplus Sharing

- profit guarantee is attained as Bayes optimal outcome for specific Pareto distribution
- upper frontier of the feasible consumer surplus and profit share across all distributions and Bayes optimal solutions:

$$\sup_F \left\{ \frac{U_F}{S_F} : \frac{\Pi_F}{S_F} = \beta \right\}$$

- identify maximum consumer surplus given profit is greater than or equal to some fraction $\beta \in [0, 1]$ of the social surplus
Proposition (Surplus Frontier)

The surplus frontier is given by:

\[
\sup_{F} \left\{ \frac{U_F}{S_F} : \frac{\Pi_F}{S_F} = \beta \right\} = \frac{\eta}{\eta - 1} \left( \beta^\frac{1}{\eta} - \beta \right).
\]

The constraint is feasible if and only if \( \beta \in \left[ 1/\eta^{\frac{\eta}{\eta-1}}, 1 \right] \).

lower bound is given by profit guarantee
Surplus Frontier

- surplus frontier and elasticity $\eta$

- all equilibrium points on surplus frontier by Pareto distributions with different shape parameters $\alpha \geq \frac{\eta}{(\eta - 1)}$
we note that:

$$\left. \frac{U_{P\alpha}}{S_{P\alpha}} \right|_{\alpha=1} = 0 \quad \text{and} \quad \left. \frac{\Pi_{P\alpha}}{S_{P\alpha}} \right|_{\alpha=1} = \frac{1}{\eta}$$

when distribution of values is the Pareto distribution with shape parameter $\alpha = 1$ the consumer’s surplus is 0

**Proposition (Lower Bound on Social Surplus)**

*When $\eta \geq 2$, social surplus is bounded below by:*

$$\inf_F \frac{U_F + \Pi_F}{S_F} = \left. \frac{U_{P\alpha} + \Pi_{P\alpha}}{S_{P\alpha}} \right|_{\alpha=1} = \frac{1}{\eta}.$$
Entire Surplus Set I

Figure: Equilibrium feasible normalized profits and consumer surplus for quadratic cost, $\eta = 2$
Figure: Illustration of Consumer Surplus and Profits for Different Distributions with Quadratic Costs
Conclusion

- cost-based rather than demand-based pricing can attain positive profit guarantee, and even higher social surplus guarantee
- menu in nonlinear pricing acts as a hedge against demand uncertainty
- menu provides stronger profit guarantee than could be anticipated from single-unit analysis
- robust menu attains guarantee through simple, transparent mark-up pricing
Variations
Quantity Discrimination

- provide a profit guarantee for the case of multiplicatively separable utility functions:

\[ u(v, q) = v \frac{\eta}{\eta + 1} q^{\frac{n+1}{n}}, \]

for some

\[ \eta \in (-\infty, -1) \]

- utility function is increasing and concave
- cost of production is linear \( c(q) = cq \), wlog \( c = 1 \)
- demand is inverse of marginal utility:

\[ D(v, p) \triangleq u_q^{-1}(v, p), \]

- demand elasticity is

\[ \frac{\partial D(v, p)}{\partial p} \frac{p}{D(v, p)} \triangleq \eta \]
Profit Guarantee with Quantity Discrimination

- Pareto distribution with shape parameter $\alpha \in (1, \infty)$

**Theorem (Profit Guarantee with Quantities)**

The uniform-price menu $t = p^* q$ with

$$p^* = \eta / (\eta + 1) > 1,$$

guarantees profits:

$$\Pi^* (v) = (\eta / (\eta + 1))^\eta S(v),$$

for every $v$ and every $F$.

- profit-guarantee menu is Bayes optimal with Pareto distribution and $\alpha = |\eta|$:  

$$\lim_{\alpha \to |\eta|} \frac{\Pi_{P^\alpha}}{S_{P^\alpha}} = \left( \frac{\eta}{\eta + 1} \right)^\eta.$$
Nonlinear Utility

- nonlinearity in utility function:
  
  \[ u(v, q, t) = h(v, q) - t, \]

  where \( h \) is concave in \( q \) given \( v \) distributed with \( F \)

- cost of production remains linear \( c(q) = cq \) wlog \( c = 1 \)

- demand function is inverse of marginal utility:
  
  \[ D(v, p) \triangleq h_q^{-1}(v, p), \]

- demand elasticity
  
  \[ \eta(v, p) \triangleq \frac{\partial D(v, p)}{\partial p} \frac{p}{D(v, p)}, \quad \eta(v, p) < 0, \forall v, p \]

- all \( p \in [1, \infty] \),
  
  \[ \eta(v, p) \text{ is non-increasing in } p \text{ and } \eta(v, p) \in [\bar{\eta} - 1, \bar{\eta}], \]

  for some \( \bar{\eta} \in (-\infty, -1) \)
Robust Profit Guarantee

- for given $D(v, p)$, optimal uniform price $\hat{p}$:

  $$\hat{p} = \arg \max_p D(v, p)(p - c).$$

- first-order condition:

  $$\hat{p} = c \frac{\eta(v, \hat{p})}{\eta(v, \hat{p}) + 1}.$$ 

- and thus

  $$\hat{p} \leq c \frac{\bar{\eta}}{\bar{\eta} + 1}.$$
Robust Profit Guarantee

Theorem (Robust Profit-Guarantee Mechanism)

The uniform-price menu $t = p^* q$, where

$$p^* = \frac{\bar{\eta}}{\bar{\eta} + 1}$$

guarantees a profit share of the social surplus:

$$\Pi^* \geq \left( \frac{\bar{\eta}}{\bar{\eta} + 1} \right)^{\bar{\eta}} S.$$
Procurement
Procurement

- single buyer procures from sellers with private information about their cost
- robust procurement policies by competitive ratio
- seller has cost $\theta \cdot c(q)$ to provide a good of quality $q$
- $\theta$ is private information for seller with distribution $F$
- costs have constant elasticity

$$\theta c(q) = \theta \frac{q^n}{\eta}, \quad \eta > 1$$
effcient social surplus:

\[ S(\theta) = \max_q \{q - \theta c(q)\} . \]

effcient quality is inversely related to cost parameter \( \theta \) :

\[ q^* = \left(\frac{1}{\theta}\right)^{\frac{1}{\eta-1}} \]

generates a social surplus of:

\[ S(\theta) = \frac{\eta - 1}{\eta} \left(\frac{1}{\theta}\right)^{\frac{1}{\eta-1}} . \]
Surplus Guarantee

- constant $p$ for every marginal unit of quality:

$$\theta c'(q) = p \iff q = (\frac{p}{\theta})^{\frac{1}{\eta-1}}.$$ 

Corollary (Surplus Guarantee Mechanism)

The surplus guarantee menu has constant unit price

$$p = \frac{1}{\eta}$$

for incremental quality and the buyer is guaranteed a share:

$$\left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

of the efficient social surplus.
Procurement with Concave Cost

- buyer has a utility function $u(q)$
  \[ u(q) = \eta q^{(\eta+1)/\eta} / (\eta + 1) \]
  with a demand elasticity:
  \[ \eta \in (-\infty, -1) \]

- seller has cost
  \[ c(q) = \theta \cdot q, \]

- marginal cost $\theta$ is private information for the seller and given by a common prior distribution
first-best surplus is:

\[ S(\theta) = \max_q \{ u(q) - \theta q \} . \]

efficient quantity is:

\[ q^* = \theta^\eta \]

social surplus is

\[ S(\theta) = -\frac{\theta^{\eta+1}}{\eta + 1}. \]
Corollary (Surplus Guarantee Mechanism)

The surplus guarantee menu has constant mark-up

\[ p(q) = \frac{\eta + 1}{\eta} q^{1/\eta} \]

for quantity and the buyer is guaranteed a share:

\[ \left( \frac{\eta}{\eta + 1} \right)^{\eta + 1} \]

of the optimal social surplus.