

THE FOLK THEOREM IN REPEATED GAMES WITH ANONYMOUS RANDOM MATCHING

JOYEE DEB

Yale School of Management, Yale University

TAKUO SUGAYA

Stanford Graduate School of Business, Stanford University

ALEXANDER WOLITZKY

Department of Economics, MIT

We prove the folk theorem for discounted repeated games with anonymous random matching. We allow non-uniform matching, include asymmetric payoffs, and place no restrictions on the stage game other than full dimensionality. No record-keeping or communication devices—including cheap talk communication and public randomization—are necessary.

KEYWORDS: Repeated games, random matching, folk theorem.

1. INTRODUCTION

IN A REPEATED GAME with anonymous random matching, a finite population of players repeatedly breaks into pairs to play 2-player games. Each period, a player observes only her partner’s action—not his identity, and not any other player’s action. We prove the folk theorem in this environment. In particular, when the players are sufficiently patient, they can attain the same payoffs as if everyone’s identity and actions were publicly observed at the end of each period.

Because players receive so little information under anonymous random matching, this environment has long been used as a benchmark against which to measure the value of various record-keeping devices and institutions, such as fiat money, merchant coalitions and guilds, credit bureaus, online rating systems, “standing” and “image scoring” in evolutionary biology, and monitoring within ethnic groups.¹ The main implication of our result is that, even in this information-poor benchmark environment, patient players can obtain any feasible and individually rational payoffs without any record-keeping devices or institutions beyond their individual memories and the ability to count periods. Thus, any role for such institutions must result from impatience of the players, or from the possi-

Joyee Deb: joyee.deb@yale.edu

Takuo Sugaya: tsugaya@stanford.edu

Alexander Wolitzky: wolitzky@mit.edu

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¹On money, see Kiyotaki and Wright (1989, 1993), Kocherlakota (1998), Wallace (2001), Araujo (2004), Aliprantis, Camera, and Puzzello (2007). On merchants, see Milgrom, North, and Weingast (1990), Greif (1993), Greif, Milgrom, and Weingast (1994). On credit bureaus, see Klein (1992), Padilla and Pagano (2000). On online rating systems, see Friedman and Resnick (2001). On standing and image scoring, see Sugden (1986), Nowak and Sigmund (1998). On ethnic conflict, see Fearon and Laitin (1996).

bility of constructing “simpler,” “more robust,” or “more realistic” equilibria when more information is available.²

Our folk theorem thus admits both positive and negative interpretations. The positive interpretation is that a wide range of cooperative behaviors are possible despite minimal information. The negative interpretation is that, in a finite population of patient long-run players, it is difficult to justify the value of information-sharing institutions on efficiency grounds alone. In particular, in these environments, the assumptions that monitoring is decentralized and players are anonymous—which might have been expected to restrict the set of attainable payoffs in some games—turn out to be completely payoff-irrelevant.³

Our approach is to view the repeated random matching game as a single N -player repeated game with imperfect private monitoring and apply techniques from the literature on the folk theorem with private monitoring. The main obstacle to this approach is that, when viewed as a single repeated game, the random matching game fails standard statistical identifiability conditions (e.g., Fudenberg, Levine, and Maskin’s (1994) pairwise identifiability) and full support conditions. To overcome this obstacle, we show that players can be given incentives to truthfully share information—despite communicating only via payoff-relevant actions—and that the aggregated information of a player’s opponents always identifies her action. Our paper thus connects three literatures: repeated games with random matching, repeated games with private monitoring, and secure communication in repeated games.

Random Matching. Kandori (1992), Ellison (1994), and Harrington (1995) showed that cooperation can be sustained in the repeated prisoner’s dilemma with anonymous random matching via “contagion strategies,” where a single defection triggers the breakdown of cooperation throughout the population. This approach does not generalize beyond the prisoner’s dilemma, because spreading contagion may not be incentive compatible when punishing is costly. Even within the prisoner’s dilemma, it cannot be used to support asymmetric equilibria, where, for example, a subset of players are allowed to defect while others must cooperate. In contrast, our theorem covers all games (subject to a mild full dimensionality condition) and all feasible and individually rational payoffs.

Deb (2020) proved the folk theorem for asymmetric games where players from distinct communities fill different player-roles, cheap talk communication between partners is allowed, and all players from the same community receive the same payoff. We instead consider random matching within a single population (though our approach generalizes to multiple communities), allow asymmetric payoffs, and—most importantly—disallow cheap talk.⁴ Deb and González-Díaz (2020) also disallowed cheap talk in the 2-community model, but they imposed some conditions on the stage game, restricted attention to symmetric payoffs that Pareto dominate a Nash equilibrium (obtaining a “Nash threat” folk theorem), and required the population to be sufficiently large. Their proof is completely

²Of course, our result first fixes the population size N and then takes $\delta \rightarrow 1$. If the population is very large, the required discount factor is very close to 1. For example, if one extended our model by introducing fiat money à la Kiyotaki and Wright (1989, 1993) or Wallace (2001), our theorem would immediately imply that, for any fixed N , money is inessential for sufficiently high δ ; however, for any fixed δ , for many stage games money is essential for sufficiently high N . This observation generalizes the conclusion of Araujo (2004) in the same way that our theorem generalizes the conclusions of Kandori (1992) and Ellison (1994).

³Another interpretation sometimes claimed by repeated games papers is that the constructed equilibrium is a positive description of behavior. We do not make such a claim here, and indeed think our construction is much too complicated to interpret this way. Our theorem is simply a benchmark possibility result.

⁴Ruling out cheap talk seems essential, as the point of our analysis is to see what outcomes are possible in the absence of record-keeping and communication devices.

different, as they generalized the contagion approach, while we build on the block belief-free approach introduced by Hörner and Olszewski (2006) to study repeated games with almost-perfect monitoring. We compare these two approaches below. Deb, González-Díaz, and Renault (2016) proved a general folk theorem for N -community games without discounting. Another difference from these papers is that our approach extends to non-uniform and even non-i.i.d. matching.

Other random matching models assume players directly observe some information about their partners' past play. Rosenthal (1979), Okuno-Fujiwara and Postlewaite (1995), and Dal Bó (2007) considered finite population models; notably, the latter paper allows asymmetric payoffs. Takahashi (2010), Dilmé (2016), Heller and Mohlin (2018), Bhaskar and Thomas (2020), and Clark, Fudenberg, and Wolitzky (2019a, 2019b) considered continuum models.

Private Monitoring. The literature on repeated games with imperfect private monitoring is too large to survey here. The folk theorem with public cheap talk communication was proved by Compte (1998) and Kandori and Matsushima (1998). Piccione (2002), Ely and Välimäki (2002), Matsushima (2004), Ely, Hörner, and Olszewski (2005), Hörner and Olszewski (2006), and Yamamoto (2012) developed belief-free techniques that we build on. Sugaya (2019) proved a general folk theorem under identifiability and full support conditions. These conditions are violated with anonymous random matching, but we use some ideas from Sugaya's proof.⁵ We explain the connection to this literature in Section 3.5.

Secure Communication. The most challenging part of our proof is providing incentives for secure communication with anonymous random matching, when communication can be executed only through payoff-relevant actions. As far as we know, ours is the first paper to address this problem. Incentives for secure communication have been studied in the related setting of repeated games played on fixed networks (Ben-Porath and Kahaneman (1996), Renault and Tomala (1998), Lippert and Spagnolo (2011), Laclau (2012, 2014), Nava and Piccione (2014), Wolitzky (2015)). While the technical overlap with this literature is slight, our non-uniform matching model can approximate a fixed network, as we allow the case where a player "almost always" interacts with the same partners.

2. MODEL AND FOLK THEOREM

There is a finite set of players $I = \{1, \dots, N\}$, with $N \geq 4$ even. In every period $t = 1, 2, \dots$, players match in pairs to play a finite, symmetric 2-player game with action set A and payoff function $u : A \times A \rightarrow \mathbb{R}$, with $|A| \geq 2$. Let $a^0, a^1 \in A$ denote two arbitrary, distinct actions.

Pairs are formed as follows: (i) a *matching* μ is a partition of the population into pairs, (ii) there is an exogenous distribution p over matchings, and (iii) the period- t matching μ_t is drawn from p i.i.d. across periods.⁶ We assume p has full support and let $\bar{\varepsilon} > 0$ denote the minimum of $p(\mu)$ over all matchings. Let $\mu(i)$ denote player i 's partner in matching μ . Let $p_{i,j} = \sum_{\mu: \mu(i)=j} p(\mu)$ denote the probability that players i and j are matched.

Players are anonymous—each player observes only the actions she faces and not her opponents' identities. Formally, letting $a_{i,t} \in A$ denote player i 's period- t action, player i 's observation in period t is the pair $(a_{i,t}, \omega_{i,t})$, where $\omega_{i,t} = a_{\mu_t(i),t}$. Say that a profile

⁵Fudenberg, Ishii, and Kominers (2014) also built on Hörner and Olszewski to prove a folk theorem in a setting where Sugaya's theorem does not apply, albeit a completely different one from ours.

⁶The extension to non-i.i.d. matching is discussed in Section 4.

of observations $(a_i, \omega_i)_{i \in I}$ is *feasible* if there exists an action profile $\mathbf{a} = (a_1, \dots, a_N) \in \prod_{i \in I} \mathcal{A} = \mathcal{A}^N$ and a matching μ such that $\omega_i = a_{\mu(i)}$ for all $i \in I$. Player i 's history at the beginning of period t is denoted $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1}$, with $h_i^0 = \emptyset$. Players maximize expected discounted payoffs with common discount factor $\delta < 1$. Let $E(\delta)$ denote the sequential equilibrium payoff set with discount factor δ .⁷

For any action profile $\mathbf{a} \in \mathcal{A}^N$, player i 's expected payoff at action profile \mathbf{a} is given by

$$\hat{u}_i(\mathbf{a}) = \sum_{j \neq i} p_{i,j} u(a_i, a_j).$$

Thus, the (convex hull of the) feasible payoff set in the N -player game is $F = \text{co}(\{\hat{\mathbf{u}}(\mathbf{a})\}_{\mathbf{a} \in \mathcal{A}^N})$, where $\hat{\mathbf{u}}(\mathbf{a}) = (\hat{u}_1(\mathbf{a}), \dots, \hat{u}_n(\mathbf{a}))$.⁸ Let $\bar{u} = \max_{(a,a') \in \mathcal{A}^2} |u(a, a')|$ be the greatest magnitude of any feasible payoff, and let $\underline{u} = \min_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$ be the minmax payoff. Let $\alpha^{\min} \in \arg\min_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$ be a minmax strategy in the 2-player game; to minmax player i in the N -player game, every player but i plays α^{\min} . Denote the set of feasible and individually rational payoffs by $F^* = \{\mathbf{v} \in F : v_i \geq \underline{u} \forall i \in I\}$. We assume F^* has dimension N . This condition is generic: letting

$$e^i = (u(a^0, a^1), ((1 - p_{j,i})u(a^1, a^1) + p_{j,i}u(a^1, a^0))_{j \neq i}) \in \mathbb{R}^N$$

be the payoff vector when player i plays a^0 and all other players play a^1 , the vectors $(e^i)_{i \in I}$ are linearly independent for generic values of $u(a^0, a^1)$, $u(a^1, a^0)$, and $u(a^1, a^1)$.⁹

In this setting, we establish the folk theorem:

THEOREM 1: *For all $\mathbf{v} \in \text{int}(F^*)$, there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

3. KEY IDEAS OF THE EQUILIBRIUM CONSTRUCTION

We provide a constructive proof of the folk theorem. The proof is deferred to the [Appendix](#). Here we describe the key ideas of the construction.

3.1. Overall Structure of the Construction

We view the repeated game as an infinite sequence of finite blocks of periods. Players follow automaton strategies. In each block, each player $i \in I$ has two possible states—denoted $x_i \in \{G, B\}$, for “good” and “bad.” A player's state in the current block, her history in the current block, and private randomization jointly determine her state in the next block. We specify each player i 's block strategy in state x_i —denoted $\sigma_i(x_i)$ —and the

⁷In defining sequential equilibrium, the choice of topology on the sets of beliefs and strategies does not matter for us—for concreteness, take it to be the product topology. This is another point of contrast with the approaches in [Deb \(2020\)](#) and [Deb and González-Díaz \(2020\)](#), where choosing the product topology is essential.

⁸The definition of the feasible payoff set accounts for anonymity. For example, if the stage game is the prisoner's dilemma, the payoff vector corresponding to everyone cooperating with player 1 and defecting against everyone else is not feasible.

⁹Full-dimensionality of F^* and full-dimensionality of the underlying 2-player game are logically independent. If the 2-player game is a pure coordination game (with payoff dimension 1), then F^* has full dimension.

Conversely, with $N = 4$ and uniform matching, the 2-player game $\begin{matrix} & a^0 & a^1 \\ a^0 & 4, 4 & 1, 3 \\ a^1 & 3, 1 & 0, 0 \end{matrix}$ has full dimension, but F^* has dimension 1.

state transition rules so that two properties hold. First, for every realization of the other players' states $x_{-i} \in \{G, B\}^{N-1}$, both $\sigma_i(G)$ and $\sigma_i(B)$ are optimal strategies for player i (i.e., the equilibrium is *block belief-free*, as in Hörner and Olszewski (2006)). Second, player i is the “arbiter” of the payoff of player $(i + 1) \pmod N$, in that i 's state determines $i + 1$'s equilibrium continuation payoff: whenever i 's state is G (B), $i + 1$'s expected continuation payoff is higher (lower) than the target equilibrium payoff, so that $i + 1$'s target payoff can be exactly attained by tuning i 's state transition rule. For example, if the state profile at the start of a block is $x = (B, B, G, G, \dots, G)$, so everyone except players 1 and 2 is in state G , then the strategy profile to be played in the block is one that guarantees that expected continuation payoffs are low for players 2 and 3 and high for everyone else. Full-dimensionality of the payoff set guarantees that such a strategy profile exists. For instance, in the prisoner's dilemma, such a profile might require players 2 and 3 to cooperate for 99% of the block, while everyone else cooperates for 95% of the block. Note that, while i is responsible for choosing $i + 1$'s continuation payoff via her state transition, she has no special role in delivering this payoff: once the state profile $x \in \{G, B\}^N$ is chosen, all players are equally responsible for following the prescribed equilibrium continuation.

Play within a block proceeds as follows. First, there is an “initial talk sub-block,” where players communicate their states.¹⁰ This lets them coordinate on the block strategy profile based on the state profile $x \in \{G, B\}^N$. Then, players repeat the following “play-and-talk sub-block” multiple times: they play actions that attain the target payoffs at state profile x for many periods, and then communicate to see if anyone deviated. This is followed by a “final talk sub-block,” where players communicate a summary of the entire block history. Since all communication is executed via payoff-relevant actions, to attain the target payoffs the players must spend most of their time in the “play” phases: in particular, they cannot take the time to communicate about every play period. Instead, when players communicate to identify deviations, player i chooses one period at random from the preceding play phase and communicates this choice to the other players, who then share their information about that period only. This information is used to check if player $i + 1$ deviated in the chosen period. Since player $i + 1$ does not know in advance which period his arbiter i will choose, this scheme can provide incentives for the entire play phase.

If deviations are detected as a result of the communication among players (described in Section 3.2 below), then deviators are punished in two ways. First, if communication reveals that player $i + 1$ deviated, then everyone switches to mutual minmaxing for the rest of the current block (starting with the next play-and-talk sub-block). Second, at the beginning of the next block, player $i + 1$'s arbiter (player i) adjusts her state transition probability so as to reduce player $i + 1$'s expected continuation payoff; and for each other player $j \neq i + 1$, player j 's arbiter (player $j - 1$) adjusts player j 's expected continuation payoff to compensate her for any cost of punishing player i during the last block.¹¹

An implication of this block structure is that each player's continuation value is controlled separately across blocks. Therefore, the challenge is providing incentives within each block for correct on-path play and (especially) providing incentives for truthful communication. This is unlike contagion equilibria, where all players' payoffs are tied together, and so the key challenge is in providing incentives to carry out punishments.

¹⁰Recall that all communication is executed via actions.

¹¹This basic idea of “rewarding the punishers” dates back to Fudenberg and Maskin (1986). As in that paper, “rewards” compensate punishers for the cost of carrying out punishments, but a player may still be left worse-off than she was before an opponent's deviation.

3.2. How Communication Works

In our construction, players communicate by taking turns broadcasting information. Which player's turn it is to "talk" in each period is pre-determined.¹² We explain how a player sends a binary message $m \in \{0, 1\}$. Longer messages are sent by binary expansion.

To send message 1, the sender plays a^1 for T periods and then a^0 for another T periods, where T is a pre-determined large number. To send message 0, the order is reversed: first a^0 for T periods, then a^1 . The other players—the "receivers"—play only a^0 with high probability throughout the entire $2T$ -period interval. At the end of the interval, a receiver who observed a^1 during the first T periods only infers that the sender sent message 1. A receiver who observed a^1 during the last T periods only infers that the sender sent message 0. A receiver who observed any other pattern—that is, observed a^1 at least once in each half-interval, or never observed a^1 at all—receives a message of error.

This protocol has several desirable properties. First, if T is large, with high probability the sender matches with each receiver at least once in each T -period half-interval, and therefore the message transmits successfully when all players follow the protocol. Second, a key obstacle to communication is that, since players are anonymous, a receiver may be tempted to talk at the same time as the sender in an attempt to manipulate the message. Our protocol makes such a manipulation very unlikely to succeed: no matter what a given receiver does, every other receiver will either receive the correct message or receive error, so long as she meets the sender at least once in the half-interval where the sender plays a^1 —a very high-probability event. Hence, to deter this attempted manipulation, it suffices to punish all players whenever anyone receives error.

There are, however, two important challenges to implementing this simple scheme.

First, in the course of communication, a receiver might learn that a low-probability realization of the matching process has occurred, at which point her expected gain from manipulation can be much larger. For example, suppose a single receiver happens to see a^1 in *all* of the first T periods—this event is very unlikely, but it is not impossible. Since only one receiver at a time sees a^1 , this receiver can infer that she is the only one to have received the message successfully. This puts her at a large informational advantage over the other players, and it is difficult to predict how she may exploit this advantage in continuation play.

We address this receiver-learning problem by introducing *jamming*, a key innovation in our proof. Specifically, at the beginning of each block, with small probability each player is designated a *jamming player* for the block. (We defer the details of how this designation is determined.) Jamming players differ from regular players in that, when they are receivers, with small probability they continually play a^1 (which we refer to as *jamming communication*) rather than a^0 . Clearly, communication is very unlikely to succeed when a jamming player is present and jams communication—however, since jamming players are rarely present (and rarely jam communication when they are present), this has a negligible effect on equilibrium payoffs. Moreover, even a slight possibility that communication may be jammed is enough to solve the receiver-learning problem: now, if a receiver sees a^1 repeatedly, she infers that with high probability a jamming player is present and jammed communication, rather than inferring that a low-probability match realization occurred. In the former case, it is very likely that all players inferred that communication was jammed. Thus, the possibility of jamming greatly reduces the perceived informational advantage of a receiver who repeatedly observes a^1 . The resulting gain from manipulation

¹²Here we rely on the implicit assumption that the players share a common sense of calendar time.

is small enough that it can be offset by a small loss in continuation payoff at the start of the next block.

Second, when a receiver i receives `error`, the subsequent punishments must be incentive compatible. How this is ensured depends on where in the block the `error` occurs. If i receives `error` in the last communication phase in the block, she (costlessly) adjusts her transition probability for the next block (putting on more weight on $x_i = B$) so as to reduce player $i + 1$'s continuation payoff only. If instead i receives `error` at a time when there are still some play-and-talk sub-blocks remaining, mutual minmaxing commences at the start of the next sub-block, incentivized by the promise of compensation at the start of the next block.

Thus, on equilibrium path, if there is no jamming and no low-probability match realizations occur, then there are no punishments within a block, and all required continuation payoff adjustments are made across blocks. The structure of the different communication phases within a block is described in more detail in Section 3.4.

3.3. How Identification Works

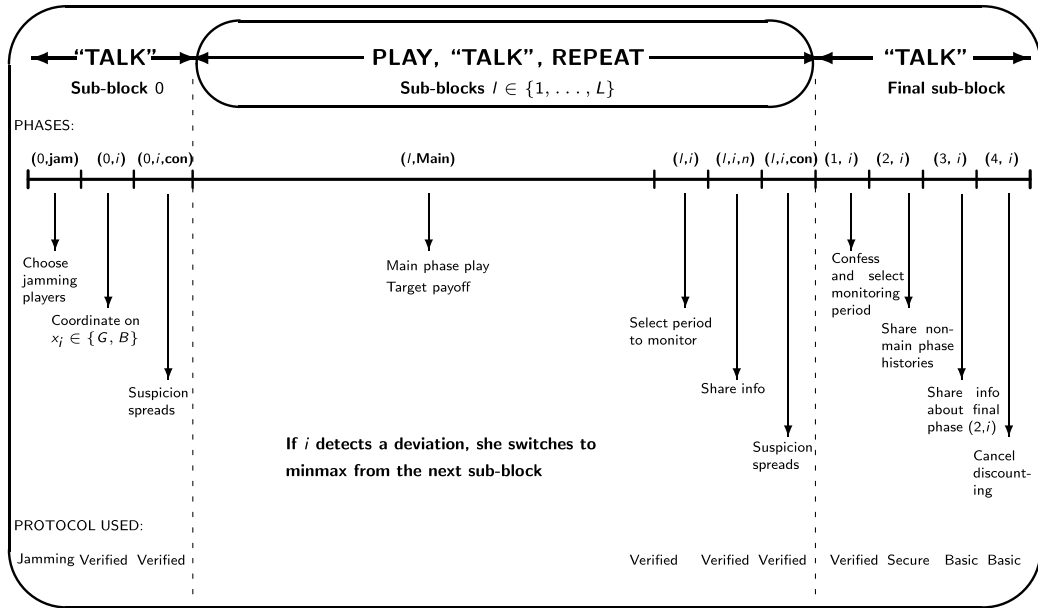
Another step in the proof is that, if player i 's opponents can successfully aggregate their information regarding a particular period of play, this information suffices to perfectly identify player i 's action and observation in that period. This step is straightforward. Since matching occurs in pairs, the total number of players who observe the same action they play (i.e., observe $\omega_n = a_n$) is always even. Therefore, if there exists $a \in A$ such that the number of i 's opponents for whom $\omega_n = a_n = a$ is odd, then $\omega_i = a_i = a$. If instead this number is even for every $a \in A$, then $a_i \neq \omega_i$. (Otherwise, the total number of players with $\omega_n = a_n = a_i$ would be odd.) In this case, there is one action a such that more of i 's opponents observe $\omega_n = a$ than play $a_n = a$, and there is another action ω such that more of i 's opponents play $a_n = \omega$ than observe $\omega_n = \omega$. This pair (a, ω) must then equal (a_i, ω_i) . Thus, if players $-i$ can aggregate their information, they can perfectly monitor player i .¹³

3.4. A Closer Look at the Communication Sub-Blocks

Next, we provide a little more detail on the “initial talk,” “play-and-talk,” and “final talk” sub-blocks noted above. “Talk” proceeds via *communication protocols*: finite repetitions of the stage game in which players communicate via actions. Our analysis consists of stringing together analyses of different communication protocols. Since we verify incentive compatibility essentially by backwards induction, we describe the protocols backwards from the end of a block.

Figure 1 provides a schematic of play within a block. The final talk sub-block comprises four phases. In the last phase, for each $i \in I$, player $i - 1$ chooses one period t at random from the previous periods in the block and communicates it to the other players, who then communicate their period- t information to player $i - 1$: intuitively, players $-i$ “talk about” player i 's play in period t . Player $i - 1$ then slightly adjusts her state transition probability such that the effect of discounting in player i 's payoff is cancelled

¹³This perfect monitoring property is not necessary for our approach: in the working-paper version (Deb, Sugaya, and Wolitzky (2018)), we extend our proof to almost-perfect monitoring within matches. Nonetheless, perfect monitoring simplifies the proof while letting us focus on its most novel element: incentivizing truthful communication.



The game is an infinite sequence of such blocks

FIGURE 1.—Schematic of play within a block.

out: when player $i - 1$ chooses period t , she increases player i 's continuation payoff by $(1 - \delta^{t-1})\hat{u}_i(\mathbf{a}_t) / \Pr(t \text{ is chosen})$, where \mathbf{a}_t is the period- t action profile identified from communication. This makes player i indifferent about the timing of her actions within a block. Hence, in all earlier phases, we may view the game as one without discounting, which is a substantial simplification.

Recall that player $i - 1$'s state affects player i 's payoff only. Thus, in the last communication phase, players $-i$ are indifferent to the outcome of communication, and are thus willing to report truthfully. Moreover, even player i has only a very small potential gain from manipulating communication when δ is large (once we fix the length of the block). Since it is always possible to provide small incentives without sacrificing much efficiency, we do not need to rely on jamming players in this phase, and a very simple communication protocol—the *basic communication protocol*, introduced in Section D.1 of the Appendix—is sufficient.

In the penultimate and third-to-last talk phases, players $-i$ aggregate their information from all previous talk phases in the block. Player $i - 1$ uses this information to adjust her state transition. As we will see, the impact of this adjustment on player i 's payoff can be large, so player i may have a strong incentive to manipulate the communication. Hence, for this phase we need a communication protocol where there is no history at which player i believes she can manipulate the outcome of communication to her benefit. This requires the *secure communication protocol*, introduced in Section D.2, which relies on jamming players.

In the first talk phase of the final talk sub-block, player $i - 1$ chooses one period t_l at random from each of the L main play sub-blocks and communicates it to the other players, who then communicate their period- t_l information to player $i - 1$. Players also confess

whether they have deviated in the current block so far.¹⁴ Similarly, in the talk phases of the play-and-talk sub-blocks, players communicate selected periods to monitor and share information about the monitoring periods with the sub-block. Finally, talk phases in the initial talk sub-block are used to determine jamming players for the block and to coordinate on the state $x \in \{G, B\}^N$. Communication in the initial sub-block and the play-and-talk sub-blocks is especially challenging. This is because these phases affect not only continuation payoffs at the end of the block but also continuation play within the block. Thus, all players (not only the one “about whom the others are talking”) may have a strong incentive to manipulate communication. We therefore need a protocol that no player can profitably manipulate. We construct the *verified communication protocol* (introduced in Section D.3) to have this property. The key additional feature of this protocol is that each receiver communicates the message she received back to the sender. This lets the players determine whether or not they all received the same message.¹⁵

3.5. Relation to the Private Monitoring Literature

Some readers may wish to understand how our construction relates to existing work on repeated games with private monitoring. Our goal is to construct a block belief-free equilibrium, as in Hörner and Olszewski (2006). To allow accurate communication under random matching, we have players repeat actions and messages and apply a concentration inequality (Lemma 3). In this sense, our construction joins the line of research combining belief-free equilibria and review strategies, following Matsushima (2004). The closest papers in this literature are Yamamoto (2012) and Sugaya (2019).

Yamamoto showed how to combine belief-free equilibria and review strategies in general repeated games. There are several important differences with our approach, but a crucial one is that Yamamoto assumed conditional independence: player i 's signal and player j 's signal are independent conditional on actions. Thus, i cannot learn j 's inference from her own signals. In contrast, with random matching signals are not conditionally independent. This is the “receiver-learning problem” noted above, which we address via the innovation of introducing jamming players.

Sugaya proved a general folk theorem by generalizing Yamamoto's construction to conditionally dependent monitoring. As in the current paper, mixed strategies are used to control incentives after erroneous histories that arise with small ex ante equilibrium probability. In particular, after observing such a history, a player believes this observation results from a rare realization of her opponents' mixed strategies. By specifying her continuation payoff to be constant after such erroneous realizations, the player is incentivized to adhere to the same continuation play as after non-erroneous histories. However, Sugaya's construction assumes pairwise identifiability (i.e., each player can unilaterally identify other players' deviations). This makes communication straightforward, as when player i “sends a message” to player j , player j can construct a statistic whose distribution depends on player i 's message but is independent of unilateral deviations by players $-i$. With anonymous random matching, pairwise identifiability is robustly violated.

¹⁴Confessions incentivize punishment during the main phases. Once a player observes an off-path history, she expects that the deviator (whoever he is) will confess in the final sub-block, and her own arbiter will adjust her continuation payoff accordingly. Meanwhile, the deviator is willing to confess because his confession is used only to adjust his opponents' continuation payoffs; in particular, his own punishment during the main phase is already sunk. This is as in Hörner and Olszewski (2006).

¹⁵As indicated in Figure 1, we also use the verified communication protocol in the first talk phase of the final talk sub-block.

4. EXTENSIONS

We have extended Theorem 1 to three more general environments: imperfect monitoring within matches, non-pairwise matching, and non-i.i.d. matching. We summarize these extensions here—formal statements and proofs may be found in the working-paper version of this article.

Almost-perfect within-match monitoring: It is not surprising that we can allow almost-perfect monitoring within a match, since we build on Hörner and Olszewski (2006), who proved the folk theorem with almost-perfect monitoring. The required modifications to our proof are relatively minor. First, we have jamming players mix over all actions, rather than just a^0 and a^1 . This makes players attribute unexpected observations to randomization by jamming players rather than monitoring errors. Second, reward functions must be adjusted to account for monitoring errors. Third, it is useful to introduce a small probability that the block is extended to include a final “long communication phase” on which the required reward adjustments can be based. Here we do allow public randomization, in contrast to both Theorem 1 and Hörner and Olszewski’s theorem. It is used to decide when to extend the block by including a long communication phase.

Non-pairwise matching and random player-roles: The assumption that matching is pairwise is restrictive. For example, this requires that all players “play the game” the same number of times, and thus rules out a distinction between frequent and infrequent participants. The assumption that each player has the same “role” in each match is also restrictive. It rules out games where, each period, one player in each match has an opportunity to do a favor for her partner, as in “monetary” models à la Kiyotaki and Wright (1989, 1993). Our approach can be extended to cover these settings, with some restrictions on the structure of the game and the target payoff set. The required modifications to the proof are again minor. For example, a player must now report her group size and player-role (if applicable) in addition to her action and observation. Notably, with this additional information, our identification argument generalizes to non-pairwise matching.

Non-i.i.d. matching: Our approach also extends to situations where (pairwise) matching is determined by a Markov process with a full-support transition kernel that depends on both the current match and the current action profile. This encompasses models with endogenous match separation, such as finite population versions of Shapiro and Stiglitz (1984), Datta (1996), Kranton (1996), Carmichael and MacLeod (1997), Eeckhout (2006), Fujiwara-Greve and Okuno-Fujiwara (2009), and Pęski and Szentes (2013). The proof now requires substantial modification. The basic idea is to use the fact that, for large enough T , any two matches separated by T periods are almost independent. This lets us preserve the block belief-free structure.

5. DISCUSSION

Multiple communities and player-roles: Our result can also be extended to allow multiple communities, where each community has a fixed role. For example, in a stage-game between a buyer and a seller, we can allow the case where each player is always either a buyer or a seller, and also that where each player can play different roles in different periods.

Cheap talk and public randomization: The folk theorem would be easy to prove if we allowed public cheap talk communication. This would make detecting deviations straightforward, and then cooperation could be sustained by punishing deviations through mutual minmaxing. Deb (2020) considered a setting with private (within-match) cheap talk

and showed that it is possible to partially detect deviations, and then applied the perfect monitoring version of Hörner and Olszewski. On the other hand, allowing public randomization would not simplify our construction much.¹⁶

Incomplete information: A concern with contagion equilibria is that they are not robust to incomplete information, for instance the possibility of a few “commitment types” who always defect. Our approach of considering a single N -player game and controlling each player’s continuation payoff separately should be more robust to these considerations. However, incomplete information can undermine our communication protocols. Nonetheless, we conjecture that our approach combined with that in Fudenberg and Yamamoto (2010) may yield a partial folk theorem for ex post equilibria in this setting.

Unknown population size: Another type of incomplete information is uncertainty about the number of players in the game. Suppose there is an underlying population of M players, any (even) number of whom may be selected by Nature to play the anonymous random matching game. We conjecture that our approach can be extended to this setting by, as in the ex post equilibrium approach of Fudenberg and Yamamoto (2010), having players keep track of a vector of continuation payoff profiles, one for each possible realization of the population playing the game; and augmenting our construction with a learning phase, where each player in the underlying population has a chance to report if she is “present” in the game. However, since one player can always pretend to be a different player (and will not be caught if the other player is not present), the extent to which payoff asymmetries among the players can be supported will be more limited than in the case with a known population.

Low discount factors: While block belief-free strategies let us establish a folk theorem, they have the disadvantage of requiring a very high discount factor as a function of the population size. In contrast, contagion strategies are remarkably effective (in the prisoner’s dilemma) even for fairly low δ .¹⁷ Nonetheless, following Hörner and Takahashi (2016), it can be shown that the asymptotic rate of convergence of our equilibrium set to F^* is at least $(1 - \delta)^{-1/2}$ for generic stage games. Formalizing and investigating performance criteria for low δ in general anonymous random matching games is an interesting future direction.

APPENDIX: PROOF OF THEOREM 1

APPENDIX A: OVERVIEW OF THE PROOF AND NOTATION

Section B presents the block belief-free equilibrium conditions, which reduce the infinitely repeated game to a finitely repeated game with final-period reward functions. **Section C** defines target payoffs and presents preliminary lemmas. **Section D** defines the communication protocols. **Section E** provides an overview of the equilibrium strategies. **Sections F and G** prove *reduction lemmas* to simplify the equilibrium conditions. We reduce the game to an undiscounted game with final-period reward functions, and show that reward functions can exhibit some dependence on other players’ histories. **Section H** constructs the *verified communication module*, which augments the verified communication

¹⁶In the final talk phase of our construction, each player i randomly chooses a set of periods to monitor and communicates this choice to her opponents. With public randomization, we could eliminate this phase by letting Nature select these random periods.

¹⁷See the calculations in Ellison (1994).

TABLE I
GLOSSARY OF TERMINOLOGY DESCRIBING TIMING

Terminology	Meaning
Block	T^{**} periods, structured as in Section E.
Sub-Block	There are $L + 2$ sub-blocks in each block: an initial talk sub-block, a final talk sub-block and L sub-blocks in between that comprise both play and talk. See Section E.
Phase	A major component of a sub-block: either a complete play of a communication protocol, or a set of periods where players take the targeted actions. See Section E.
Round	A major component of the verified protocol. See Section D.3.
Interval	$2T$ consecutive periods in the basic, secure, or verified protocol. See Section D.
Half-Interval	T consecutive periods in the basic, secure, or verified protocol.
Period	A single play of the game.

protocol defined in Section D with a reward function. **Section I** uses this module to further simplify the equilibrium conditions: we show that it suffices to establish optimality of a player’s strategy only at histories consistent with her opponents’ equilibrium strategies. **Section J** completes the description of the equilibrium strategies. **Section K** constructs the final reward function, which sums the rewards for main and non-main phases. **Sections L and M** verify the equilibrium conditions. The Supplemental Material (Deb, Sugaya, and Wolitzky (2020)) contains omitted proofs.

We use different terms to refer to sets of consecutive periods that are meaningful in the construction. We define these in Table I, from the longest (a block) to the shortest (a period).

We also collect in Tables II and III some additional notation that will be used repeatedly in the proof.

We use standard asymptotic notation: “ $f(T) = O(g(T))$ ” means “ $\exists C > 0, \exists \bar{T} > 0: \forall T > \bar{T}, |f(T)| \leq Cg(T)$.”

APPENDIX B: BLOCK BELIEF-FREE STRUCTURE

We view the repeated game as an infinite sequence of T^{**} -period blocks, with T^{**} to be specified. At the beginning of each block, each player i selects a state $x_i \in \{G, B\}$. Given x_i , player i plays a behavior strategy $\sigma_i^*(x_i)$ (her *block strategy*) within the block: in every period $t = 1, \dots, T^{**}$ of a block, $\sigma_i^*(x_i)$ specifies a mixed action as a function of player i ’s *extended block history* $(\mathbb{L}_i, h_i^{t-1})$, where \mathbb{L}_i encodes the result of a private

TABLE II
GLOSSARY OF NOTATION FOR PAYOFFS

Notation	Meaning
v_i	The target payoff.
$v_i(G)$	The lowest payoff when players coordinate on x with $x_{i-1} = G$ (see (5)).
$v_i(B)$	The highest payoff when players coordinate on x with $x_{i-1} = B$ (see (5)).
\underline{u}	The minmax payoff (see Section 2).
\bar{u}	The greatest magnitude of any feasible payoff (see Section 2).
u^G	The smallest feasible payoff (see (55)).
u^B	The largest feasible payoff (see (55)).

TABLE III
GLOSSARY OF NOTATION FOR REWARD FUNCTIONS

Notation	Meaning
$\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $v_i(x_{i-1})$ (see (7)).
$\pi_i^a(a_{-i}, \omega_{-i})$	Reward to give payoff 0 if $a_i = a$ and -1 otherwise (see (8)).
$-\mathbf{1}_{\{a_{j,t} \neq a_{j,i}^*(h_{-j})\}}$	Reward to give payoff 0 if player follows verified protocol in checking rounds, and give payoff -1 otherwise (see (42)).
$\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $u^{x_{i-1}}$, while satisfying self-generation (see (56)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $v_i(x_{i-1})$, while satisfying self-generation if all players play $a^k(x)$ (see (56)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} \alpha^{\min})$	Reward to make player indifferent over actions with payoff $v_i(x_{i-1})$ when opponents play α^{\min} (see (56)).

randomization conducted by player i at the beginning of the block (described below), and $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1} \in H_i^{t-1}$. Denote player i 's strategy set in the T^{**} -period game by Σ_i .

We require that player i 's state x_i is determined by a *transition probability* $\rho_i(\cdot | \tilde{x}_i, \tilde{h}_i^{T^{**}}) \in \Delta(\{G, B\})$ that depends only on player i 's state in the previous block, \tilde{x}_i , and her history in the previous block, $\tilde{h}_i^{T^{**}}$. Moreover, we require that player i 's payoff at the beginning of each block is determined solely by player $(i-1)$'s state, $x_{i-1} \in \{G, B\}$, and denote it by $v_i^*(x_{i-1}) \in \mathbb{R}$. Hence, player i 's continuation payoff at the end of a block is a function only of player $(i-1)$'s state and extended history. Denote this continuation payoff by $w_i^*(x_{i-1}, h_{i-1}^{T^{**}})$.

We present conditions under which a given payoff vector $\mathbf{v} \in \mathbb{R}^N$ is attainable in a block belief-free equilibrium. These are similar to the conditions in Hörner and Olszewski (2006), with one significant difference: Hörner and Olszewski assumed monitoring has full support, so in their model, Nash and sequential equilibrium coincide, and there is no need to keep track of players' beliefs. In contrast, our model does not have full support, so we must introduce beliefs, verify Kreps–Wilson consistency, and—most subtly—ensure that beliefs respect the block belief-free equilibrium structure, in that sequential rationality is satisfied conditional on each possible state vector $x_{-i} \in \{G, B\}^{N-1}$. To do this, we keep track of players' beliefs conditional on each vector $x_{-i} \in \{G, B\}^{N-1}$. This approach implicitly determines a complete, unconditional belief system, but since sequential rationality is always imposed conditional on x_{-i} , these unconditional beliefs do not enter into our analysis.

Formally, an *ex post belief system* $\beta = (\beta_i)_{i \in I}$ consists of, for each player $i \in I$, opposing state vector $x_{-i} \in \{G, B\}^{N-1}$, period $t \in \{1, \dots, T^{**}\}$, and block history $h_i^{t-1} \in H_i^{t-1}$, a probability distribution $\beta_i(\cdot | x_{-i}, h_i^{t-1}) \in \Delta(H_i^{t-1})$. Together with a block strategy profile $(\sigma_i(x_i))_{i \in I, x_i \in \{G, B\}}$, an ex post belief system is *consistent* if there exists a sequence of completely mixed block strategy profiles $((\sigma_i^k(x_i))_{i \in I, x_i \in \{G, B\}})_{k \in \mathbb{N}}$ converging pointwise to $(\sigma_i(x_i))_{i \in I, x_i \in \{G, B\}}$ such that, for each $i \in I$, $x_{-i} \in \{G, B\}^{N-1}$, $t \in \{1, \dots, T^{**}\}$, and $h_i^{t-1} \in H_i^{t-1}$, we have $\beta(h_i^{t-1} | x_{-i}, h_i^{t-1}) = \lim_{k \rightarrow \infty} \Pr^{(\sigma_j^k(x_j))_{j \neq i}}(h_i^{t-1} | x_{-i}, h_i^{t-1})$.¹⁸

¹⁸With this definition, it is clear that, whenever an ex post belief system is consistent, the corresponding unconditional belief system is consistent in the usual Kreps–Wilson sense.

We are now ready to present the equilibrium conditions. In what follows, $\mathbb{E}^\sigma[\cdot]$ denotes expectation with respect to strategy profile σ , and $\mathbb{E}^{(\sigma, \beta)}[\cdot]$ denotes conditional expectation with respect to assessment (strategy profile and beliefs) (σ, β) .

For all $\mathbf{v} \in \mathbb{R}^N$ and $\delta < 1$, if there exist $T^{**} \in \mathbb{N}$, strategies $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G, B\}}$, consistent ex post belief system β^* , values $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}}$, and continuation payoffs $(w_i^*(x_{i-1}, h_{i-1}^{T^{**}}))_{i \in I, x_{i-1} \in \{G, B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ such that the following conditions hold for all $i \in I$, then we have $\mathbf{v} \in E(\delta)$:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,¹⁹

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[(1 - \delta) \sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) | x_{-i}, h_i^{t-1} \right].$$

(Here, the sum $\sum_{\tau=1}^{T^{**}}$ could alternatively be written as $\sum_{t=1}^{T^{**}}$, since payoffs already incurred in h_i^{t-1} are sunk. In addition, sequential rationality is imposed for every vector $x_{-i} \in \{G, B\}^{N-1}$. This is the defining feature of a block belief-free construction.)

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)} \left[(1 - \delta) \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right].$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T^{**}}$, we have $w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \in [v_i^*(B), v_i^*(G)]$.

4. [Full Dimensionality] Player $i-1$ can randomize her initial state to deliver player i 's target payoff v_i : $v_i^*(B) < v_i < v_i^*(G)$.

Defining $\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) := \frac{\delta^{T^{**}}}{1-\delta} (w_i(x_{i-1}, h_{i-1}^{T^{**}}) - v_i^*(x_{i-1}))$, we rewrite the conditions below:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[\sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) | h_i^{t-1} \right]. \quad (1)$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)} \left[\frac{1-\delta}{1-\delta^{T^{**}}} \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right]. \quad (2)$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T^{**}}$,

$$\begin{aligned} \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(G, h_{i-1}^{T^{**}}) &\leq 0, & \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(B, h_{i-1}^{T^{**}}) &\geq 0, \\ \left| \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right| &\leq v_i^*(G) - v_i^*(B). \end{aligned} \quad (3)$$

¹⁹Throughout, when we write “for all $h_i^{t-1} \in H_i^{t-1}$,” this should be understood as applying for all $i \in I$ and all t .

4. [Full Dimensionality]

$$v_i^*(B) < v_i < v_i^*(G). \quad (4)$$

LEMMA 1—Hörner and Olszewski (2006): For all $\mathbf{v} \in \mathbb{R}^N$ and $\delta \in [0, 1)$, if there exist $T^{**} \in \mathbb{N}$, $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G, B\}}$, β^* , $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}}$, and $(\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}))_{i \in I, x_{i-1} \in \{G, B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ such that Conditions (1)–(4) are satisfied, then $\mathbf{v} \in E(\delta)$.

APPENDIX C: PRELIMINARIES

C.1. Target Payoff and Actions

Given $\mathbf{v} \in \text{int}(F^*)$, there exist payoff vectors $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}} \in \mathbb{R}^{2N}$ such that $(\bar{v}_i(x_{i-1}))_{i \in I} \in \text{int}(F^*) \forall (x_{i-1})_{i \in I} \in \{G, B\}^N$ and $\underline{u} < \bar{v}_i(B) < v_i < \bar{v}_i(G) \forall i \in I$. Define

$$\varepsilon^* := \frac{1}{10} \min_i \min \{ \bar{v}_i(G) - v_i, v_i - \bar{v}_i(B), \bar{v}_i(B) - \underline{u} \}.$$

We approximate $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}}$ by sequences of action profiles: for $\varepsilon^* > 0$ that we have just fixed, there exist $K_v \in \mathbb{N}$ and a sequence of action profiles $(\mathbf{a}^k(x))_{k=1}^{K_v} \in A^{N K_v} \forall x \in \{G, B\}^N$ such that, for all $i \in I$, we have $|\frac{1}{K_v} \sum_{k=1}^{K_v} \hat{u}_i(\mathbf{a}^k(x)) - \bar{v}_i(x_{i-1})| < \varepsilon^*$. Let $\hat{u}_i(x) = \frac{1}{K_v} \sum_{k=1}^{K_v} \hat{u}_i(\mathbf{a}^k(x))$. Next, fix $(v_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}} \in \mathbb{R}^{2N}$ and sequences of action profiles $((\mathbf{a}^k(x))_{k=1}^{K_v})_{x \in \{G, B\}^N} \in A^{2N K_v}$ such that, for all $i \in I$,

$$\begin{aligned} v_i(G) &= \min_{x: x_{i-1}=G} \hat{u}_i(x), & v_i(B) &= \max_{x: x_{i-1}=B} \hat{u}_i(x) > \underline{u} + 9\varepsilon^*, \quad \text{and} \\ v_i(B) + 9\varepsilon^* &< v_i < v_i(G) - 9\varepsilon^*. \end{aligned} \quad (5)$$

Players will repeat the target action sequence $(\mathbf{a}^k(x))_{k=1}^{K_v}$ over L “sub-blocks,” where

$$L := \left\lceil \frac{2\underline{u}}{\varepsilon^*} \right\rceil (K_v + 1). \quad (6)$$

(Throughout, $\lceil \cdot \rceil$ denotes the “round-up” function.) For $l > K_v$, let $a_i^l(x) = a_i^{l \pmod{K_v}}(x)$.²⁰

C.2. Identification

We record the observation made in Section 3.3 that the profile (a_{-i}, ω_{-i}) of i ’s opponents’ actions and observations perfectly identifies player i ’s action and observation, (a_i, ω_i) .

LEMMA 2: There exists a function $\varphi : A_{-i} \times A_{-i} \rightarrow A_i \times A_i$ such that, if $(a_i, \omega_i)_{i \in I}$ is feasible, then $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$.

²⁰Hörner and Olszewski (2006) and several subsequent papers presented their constructions assuming $K_v = 1$. With random matching, this assumption is usually with loss. For example, in the prisoner’s dilemma, to punish player 1 while keeping her opponents’ payoffs close to $u(C, C)$, we must cycle through action profiles where player 1 and most of her opponents cooperate, while different subsets of her opponents take turns defecting. We thus present our construction for arbitrary K_v .

By Lemma 2, for each x_{i-1} , there exists a function $\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) : A^{N-1} \times A^{N-1} \rightarrow [-2\bar{u}, 2\bar{u}]$ such that, for each $\mathbf{a} \in A^N$, we have

$$\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) = v_i(x_{i-1}). \quad (7)$$

Thus, the function $\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})$ cancels player i 's instantaneous utility. Similarly, for each $a \in A$, there exists $\pi_i^a(a_{-i}, \omega_{-i}) : A^{N-1} \times A^{N-1} \rightarrow \mathbb{R}$ such that, for each $\mathbf{a} \in A^N$, we have

$$\pi_i^a(a_{-i}, \omega_{-i}) = \begin{cases} 0 & \text{if } a_i = a, \\ -1 & \text{if } a_i \neq a. \end{cases} \quad (8)$$

Thus, the function $\pi_i^a(a_{-i}, \omega_{-i})$ punishes player i for deviating from a .

C.3. A Bound on the Probability of Matches

We repeatedly use the following exponential bound on the probability that a pair of players fails to match even once during a set of T periods:

LEMMA 3: For any set of T periods $\mathbb{T} \in \mathbb{N}^T$ and any pair of distinct players $i, j \in I$, we have $\Pr(\mu_t(i) \neq j \forall t \in \mathbb{T}) \leq \exp(-\bar{\varepsilon}T)$.

PROOF: $\Pr(\mu_t(i) \neq j \forall t \in \mathbb{T}) \leq (1 - \bar{\varepsilon})^T = \exp(T \log(1 - \bar{\varepsilon})) \leq \exp(-\bar{\varepsilon}T)$. *Q.E.D.*

Given a set of periods \mathbb{T} , we say the realized matching process is *erroneous over* \mathbb{T} if there exists a pair of players who do not match with each other during \mathbb{T} .

APPENDIX D: COMMUNICATION PROTOCOLS

A basic building block of the equilibrium strategy is a *communication protocol*: a strategy profile for players to communicate via actions in a finitely repeated game. The description of a communication protocol does not include payoff functions and thus entails no claims about incentive compatibility. After constructing the equilibrium strategy, we will construct a reward function and then verify sequential rationality.

We view each protocol as a distinct, finitely repeated game. If \mathbb{T} is the set of periods comprising a protocol, a *protocol history* for player i is a vector $h_i = (a_{i,t}, \omega_{i,t})_{t \in \mathbb{T}} \in H_i$. Denote the set of protocol history profiles by $H = \prod_{i \in I} H_i$.

D.1. Basic Communication Protocol

The basic protocol lets a player $i \in I$ broadcast a message m_i from a set $M_i = \{1, \dots, |M_i|\}$. We call player i the *sender* and call the other players *receivers*. The protocol takes $2Tb(M_i)$ periods, where $b(M_i) := \lceil \log_2 |M_i| \rceil$.²¹

Basic Communication Protocol for Player i to Send Message m_i with Repetition T :²²

- Divide the $2Tb(M_i)$ periods into $b(M_i)$ intervals of $2T$ periods each.

²¹We sometimes abusively write $b(|M_i|)$ for $b(M_i)$.

²²In what follows, instructions of the form “play action a in period t ” are to be read as unconditional on a player's past actions and observations. Thus, a communication protocol is formally a strategy profile, not just a description of on-path play.

- For $t \in \{1, \dots, b(M_i)\}$,
 - If the t th digit of the binary expansion of $m_i - 1$ is 0, player i plays a^0 for the first half of the t th interval (i.e., the first T periods in the interval) and plays a^1 for the second half of the t th interval (i.e., the last T periods in the interval).
 - If the t th digit of the binary expansion of $m_i - 1$ is 1, player i plays a^1 for the first half of the t th interval and plays a^0 for the second half of the t th interval.

We call a set of T periods where player i takes a constant action a *half-interval*.

- Each player $j \neq i$ plays a^0 throughout the protocol.
- At the end of the protocol, each player $j \neq i$ makes an *inference* $m_i(j) \in M_i \cup \{0\}$ as follows (based on history $(a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}$). If $m_i(j) = 0$, we say j *fails to infer a message*:
 - If, for some $t \in \{1, \dots, b(M_i)\}$, $\omega_{j,\tau} \notin \{a^0, a^1\}$ for some period τ in the t th interval, player j sets $m_i(j) = 0$.
 - If, for some $t \in \{1, \dots, b(M_i)\}$, $\omega_{j,\tau} \neq a^1$ for every period τ in the t th interval, player j sets $m_i(j) = 0$.
 - If, for some $t \in \{1, \dots, b(M_i)\}$, $\omega_{j,\tau} = \omega_{j,\tau'} = a^1$ for some period τ in the first half of the t th interval and some τ' in the second half of the t th interval, player j sets $m_i(j) = 0$.
 - Otherwise, player j constructs a number $\hat{m} \in \{0, \dots, b(M_i) - 1\}$ as follows:
 - * If $\omega_{j,\tau} = a^1$ for some period τ in the first half of the t th interval and $\omega_{j,\tau} = a^0$ for every period τ in the second half of the t th interval, player j sets the t th digit of the binary expansion of \hat{m} equal to 1.
 - * If $\omega_{j,\tau} = a^1$ for some period τ in the second half of the t th interval and $\omega_{j,\tau} = a^0$ for every period τ in the first half of the t th interval, player j sets the t th digit of the binary expansion of \hat{m} equal to 0.
 - If $\hat{m} \leq |M_i| - 1$, player j sets $m_i(j) = \hat{m} + 1$. If $\hat{m} \geq |M_i|$ (which is possible if $\log_2 |M_i|$ is not an integer), player j sets $m_i(j) = 0$.

When all players follow the protocol, $m_i(j) = m_i$ if and only if player j matches with player i at least once in every T -period half-interval where player i plays a^1 . Hence, by Lemma 3,

$$\Pr(m_i(j) = m_i) \geq 1 - b(M_i) \exp(-\bar{\epsilon}T) \quad \forall j \neq i. \quad (9)$$

Moreover, when all players follow the protocol, either j 's inference is correct or j fails to infer a message: if $m_i(j) \neq m_i$ then $m_i(j) = 0$.

D.2. Secure Communication Protocol

The secure protocol is a generalization of the basic protocol that lets player i send a message so that it is harder for any receiver to manipulate. In addition to the parameters $(i, m_i, \text{ and } T)$, the secure protocol takes as given a set of players $I_{\text{jam}} \subset I \setminus \{i\}$, called *jamming players*.

Secure Communication Protocol for Player i to Send Message m_i With Repetition T and Jamming Players I_{jam} :

- Divide the $2Tb(M_i)$ periods of the protocol into $b(M_i)$ intervals of $2T$ periods each.
- Player i behaves as in the basic communication protocol.
- Each player $j \notin I_{\text{jam}} \cup \{i\}$ behaves as in the basic communication protocol (i.e., plays a^0).
- For each player $j \in I_{\text{jam}}$, in the first period of each T -period half-interval (i.e., in periods $t = kT + 1$ for $k \in \{0, 1, \dots, 2b(M_i) - 1\}$), player j plays a^0 with probability $1 - \exp(-T^{\frac{1}{2}})$ and plays a^1 with probability $\exp(-T^{\frac{1}{2}})$. She then repeats the chosen action for the remainder of the half-interval (i.e., plays $a_{j,t} = a_{j,kT+1}$ for $t \in \{kT + 2, \dots, (k+1)T\}$).

• At the end of the protocol, each player $j \neq i$ infers a message $m_i(j) \in M_i \cup \{0\}$ as in the basic communication protocol.

For $j \in I_{\text{jam}}$ and $k \in \{0, 1, \dots, 2b(M_i) - 1\}$, if $a_{j,kT+1} = a^0$ we say player j *plays REG* (“regular”) in the k th half-interval, and if $a_{j,(k-1)T+1} \neq a^0$ we say player j *plays JAM* (“jamming”) in the k th half-interval. Thus, player j plays REG and JAM with probabilities $1 - \exp(-T^{\frac{1}{2}})$ and $\exp(-T^{\frac{1}{2}})$ in each half-interval, independently across each half-interval.

Denote the event that all jamming players play REG throughout the protocol by ALL-REG. Conditional on ALLREG, all players behave identically in the secure and basic protocols. In particular, conditional on ALLREG, inequality (9) holds and $m_i(j) \neq 0$ implies $m_i(j) = m_i \forall j \neq i$. Moreover,

$$\Pr^{o^{m_i}}(m_i(j) = m_i \forall j \neq i \cap \text{ALLREG}) \geq 1 - Nb(M_i)(\exp(-\bar{\varepsilon}T) + 2\exp(-T^{\frac{1}{2}})). \quad (10)$$

The key new property of the secure protocol is that, for each player $j \neq i$ with $I_{\text{jam}} \setminus \{j\} \neq \emptyset$ and every sequence of observations $(\omega_{j,t})_{t=1}^{2Tb(M_i)}$, either she believes with high probability that communication was jammed, or she believes with high probability that, conditional on the event that communication was not jammed, the message is likely to have transmitted successfully. Intuitively, the former case arises when player j observes a^1 frequently, and the latter case arises when she observes a^1 less frequently. To formalize this, let

$$\bar{\eta} := \max_{\gamma \in [0,1]} \min_{i,j,j'} \left\{ \gamma \log \frac{p_{i,j} + p_{j',j}}{p_{i,j}} + (1 - \gamma) \log \frac{1 - p_{i,j} - p_{j',j}}{1 - p_{i,j}}, \bar{\varepsilon}(1 - \gamma) \right\} > 0, \quad (11)$$

and let $\bar{\gamma}$ be the maximizer.

LEMMA 4: For any player $j \neq i$ with $I_{\text{jam}} \setminus \{j\} \neq \emptyset$ and any sequence of observations $(\omega_{j,t})_{t=1}^{2Tb(M_i)}$ that arises with positive probability when players $-j$ follow the secure protocol:

1. If $\omega_{j,t} = a^1$ for at least $\bar{\gamma}T$ periods in some half-interval, then, for all $(a_{j,t})_{t=1}^{2Tb(M_i)}$, we have

$$\Pr(\text{ALLREG} | (a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}) \leq \exp(-\bar{\eta}T + T^{\frac{1}{2}}). \quad (12)$$

2. If $\omega_{j,t} = a^1$ for at most $\bar{\gamma}T$ periods in each half-interval, then:

(a) For all $(a_{j,t})_{t=1}^{2Tb(M_i)}$, we have

$$\begin{aligned} \Pr(m_i(j') \in \{m_i, 0\} \forall j' \notin \{i, j\} | (a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}, \text{ALLREG}) \\ \geq 1 - Nb(M_i) \exp(-\bar{\eta}T). \end{aligned} \quad (13)$$

(b) If $a_{j,t} = a^0$ for all $t \in \{1, \dots, 2Tb(M_i)\}$, we have

$$\begin{aligned} \Pr(m_i(j') = m_i \forall j' \notin \{i, j\} | (a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}, \text{ALLREG}) \\ \geq 1 - Nb(M_i) \exp(-\bar{\eta}T). \end{aligned} \quad (14)$$

PROOF: Fix $j \neq i$ with $I_{\text{jam}} \setminus \{j\} \neq \emptyset$. Suppose there is a half-interval \mathbb{S} in which $\omega_{j,t} = a^1$ for γ periods, with $\gamma \geq \bar{\gamma}T$. Fix a player $j' \in I_{\text{jam}} \setminus \{j\}$. Let j' JAMS denote the event that, in half-interval \mathbb{S} , player j' plays JAM and all other jamming players play REG. Let $\mathbb{S}\text{REG}$

denote the event that all jamming players play REG in half-interval \mathbb{S} . Let $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}}$ denote the restriction of $(a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}$ to half-interval \mathbb{S} . Then

$$\begin{aligned} \frac{\Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | j' \text{JAMS})}{\Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | \mathbb{S} \text{REG})} &= \left(\frac{p_{i,j} + p_{j',j}}{p_{i,j}} \right)^\gamma \left(\frac{1 - p_{i,j} - p_{j',j}}{1 - p_{i,j}} \right)^{T-\gamma} \\ &\geq \exp \left(\left(\bar{\gamma} \log \frac{p_{i,j} + p_{j',j}}{p_{i,j}} + (1 - \bar{\gamma}) \log \frac{1 - p_{i,j} - p_{j',j}}{1 - p_{i,j}} \right) T \right), \end{aligned}$$

which is no less than $\exp(\bar{\eta}T)$. Hence, by Bayes's rule,

$$\begin{aligned} \Pr(\mathbb{S} \text{REG} | (a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}}) &\leq \left[1 + \frac{\Pr(j' \text{JAMS}) \Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | j' \text{JAMS})}{\Pr(\mathbb{S} \text{REG}) \Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | \text{ALLREG})} \right]^{-1} \\ &\leq \left[1 + \exp(-T^{\frac{1}{2}}) \frac{\Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | j' \text{JAMS})}{\Pr((a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}} | \text{ALLREG})} \right]^{-1} \\ &\leq [1 + \exp(\bar{\eta}T - T^{\frac{1}{2}})]^{-1} \leq \exp(-\bar{\eta}T + T^{\frac{1}{2}}). \end{aligned}$$

Since the event that a jamming player plays JAM is independent across half-intervals and the behavior of players $-j$ is independent of their past actions and observations, we have

$$\Pr(\text{ALLREG} | (a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}) \leq \Pr(\mathbb{S} \text{REG} | (a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}) = \Pr(\mathbb{S} \text{REG} | (a_{j,t}, \omega_{j,t})_{t \in \mathbb{S}}).$$

Combining the inequalities yields (12).

Next, suppose $\omega_{j,t} = a^1$ for at most $\bar{\gamma}T$ periods in every half-interval. Then, in each half-interval where player i plays a^1 , player i matches with a player other than j in at least $(1 - \bar{\gamma})T_0$ periods. Suppose player j plays a^0 throughout the protocol. For all $j' \notin \{i, j\}$, if player i matches with player j' at least once in each half-interval where player i plays a^1 , and ALLREG occurs, then $m_i(j') = m_i$. Hence, by Lemma 3,

$$\begin{aligned} \Pr(m_i(j') = m_i | (a^0, \omega_{j,t})_{t=1}^{2Tb(M_i)}, \text{ALLREG}) &\geq 1 - b(M_i) \exp(-\bar{\varepsilon}(1 - \bar{\gamma})T) \\ &\geq 1 - b(M_i) \exp(-\bar{\eta}T). \end{aligned}$$

Applying this bound repeatedly for each $j' \neq i, j$, we obtain

$$\Pr(m_i(j') = m_i \forall j' \notin \{i, j\} | (a^0, \omega_{j,t})_{t=1}^{2Tb(M_i)}, \text{ALLREG}) \geq 1 - Nb(M_i) \exp(-\bar{\eta}T).$$

This establishes (14). Similarly—regardless of player j 's behavior—if player i matches with player $j' \neq i, j$ in some period in each half-interval where player i plays a^1 , then $m_i(j') \in \{m_i, 0\}$. (In particular, $m_i(j') = 0$ if j ever matches with j' while playing $a_j \notin \{a^0, a^1\}$, or if i and j match with j' while playing a^1 in different halves of the same interval, and $m_i(j') = m_i$ otherwise.) Hence, (13) also holds. Q.E.D.

D.3. Verified Communication Protocol

In the verified communication protocol, player i first broadcasts a message $m_i \in M_i$ in $2b(M_i)$ periods using the basic communication protocol (with $T = 1$). Then, each player (including player i herself) sequentially broadcasts her actions and observations from

these $2b(M_i)$ periods using the secure communication protocol with repetition T (with T to be specified). The verified protocol thus takes a total of $\mathcal{T}(M_i, T)$ periods, where

$$\mathcal{T}(M_i, T) := 2b(M_i) + 2b(A^{4b(M_i)})NT. \quad (15)$$

Verified Communication Protocol for Player i to Send Message m_i With Repetition T :

At the beginning of the verified protocol, each player j has two possible types, denoted $\zeta_j \in \{\text{reg}, \text{jam}\}$. A strategy in the protocol is thus a mapping from $\{\text{reg}, \text{jam}\}$ and protocol histories to actions. Let $\mathcal{I}_{\text{jam}} = \{j : \zeta_j = \text{jam}\}$. The protocol consists of $N + 1$ rounds.

- *Message round*

- Player i sends message $m_i \in M_i$ as in the basic communication protocol with $T = 1$.²³
- Each player $j \neq i$ plays a^0 throughout the round.

Let $\mathbb{T}(\text{msg})$ denote the set of $2b(M_i)$ periods comprising the message round.

- *j -checking round, for each $j \in I$.* Each checking round consists of $b(A^{4b(M_i)})$ intervals. Each interval consists of $2T$ periods. Let $\mathbb{T}(j)$ denote the set of $2Tb(A^{4b(M_i)})$ periods comprising the j -checking round.

- Player j sends message $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})} \in A^{4b(M_i)}$ as in the basic protocol.
- Each player $n \notin \mathcal{I}_{\text{jam}} \cup \{j\}$ plays a^0 throughout the round.
- In each half-interval, each player $n \in \mathcal{I}_{\text{jam}} \setminus \{j\}$ mixes between REG and JAM with probabilities $1 - \exp(-T^{\frac{1}{2}})$ and $\exp(-T^{\frac{1}{2}})$, as in the secure protocol.
- Each player $n \neq j$ infers message $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} \in A^{4b(M_i)} \cup \{0\}$ as in the basic protocol.

- At the end of the protocol, each player $n \in I$ creates a *final inference* $m_i(n) \in M_i \cup \{0\}$ as follows:

- If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ for some $j \neq n$, then $m_i(n) = 0$.
- Otherwise, if the vector $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is not feasible—that is, for some $j' \in I$ and $t \in \mathbb{T}(\text{msg})$, $(a_{j',t}(n), \omega_{j',t}(n)) \neq \varphi((a_{j,t}(n), \omega_{j,t}(n))_{j \neq j'})$ (see Lemma 2 for the definition of φ)—then $m_i(n) = 0$.
- If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is feasible and $(a_{i,t}(n))_{t \in \mathbb{T}(\text{msg})}$ corresponds to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n) = \hat{m}_i$.
- If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is feasible but $(a_{i,t}(n))_{t \in \mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n)$ is set equal to an arbitrary, pre-determined element of M_i —for concreteness, let $m_i(n) = 1$.

In the verified protocol, we call player i the *initial sender*, and we say player $j \in I$ is a *sender in period t* if $t \in \mathbb{T}(j)$ or $[j = i \text{ and } t \in \mathbb{T}(\text{msg})]$. We say *players coordinate on m_i* if $m_i(n) = m_i$ for all $n \in I$.

For each $j \in I$, say that player j is *suspicious* at protocol history h_j , denoted $\text{susp}(h_j) = 1$, if $m_i(j) = 0$. Otherwise, $\text{susp}(h_j) = 0$. Note that $\text{susp}(h_j) = 1$ only if some player deviates, some jamming player plays JAM, or the realized matching process is erroneous over some half-interval. We will derive some key properties of the function $\text{susp}(\cdot)$ in Section H.

D.4. Jamming Coordination Protocol

Finally, we describe how players coordinate on the identities of the jamming players $\mathcal{I}_{\text{jam}} \subset I$.

²³To make following the verified communication protocol sequentially rational, we will subsequently slightly modify player i 's prescribed behavior after she herself deviates from the protocol. See Section H.

Jamming Coordination Protocol With Parameter T :

• In each of the two periods, each player i plays a^1 with probability $\exp(-T^{\frac{1}{3}})$ and plays each $a \neq a^1$ with probability $(1 - \exp(-T^{\frac{1}{3}}))/(|A| - 1)$, independently across periods.

Given a protocol history h_i , we define $\zeta_i(h_i) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$. That is, a player becomes a jamming player if she observes a^1 in either period.

Let $P_i(h_i) = \Pr(\zeta_j(h_j) = \text{jam} \forall j \neq i | h_i)$. For every protocol history h_i , the probability that all players in $I \setminus \{i, \mu_t(i)\}$ play a^1 in both periods t and $\mu_1(i) \neq \mu_2(i)$ is at least $\bar{\varepsilon} \exp(-(N-2)T^{\frac{1}{3}})$. Conditional on this event, the probability that $\zeta_j(h_j) = \text{jam} \forall j \neq i$ is 1. Hence,

$$P_i(h_i) \geq \bar{\varepsilon} \exp(-(N-2)T^{\frac{1}{3}}). \quad (16)$$

APPENDIX E: EQUILIBRIUM STRATEGIES: OVERVIEW

We now define the equilibrium block strategies, deferring some details to Section J. The length of a block is parameterized by a number $T_0 \in \mathbb{N}$. We fix T_0 sufficiently large such that the following three inequalities hold:

$$\begin{cases} \frac{2 + \bar{u}}{\bar{\varepsilon} \min\{\varepsilon^*, 1\}} 300L^2N^4|A| \log_2 T_0 \leq (T_0)^{\frac{1}{10}}, \\ (T_0)^4 (\exp(-(T_0)^{\frac{1}{6}}) + \exp(-\bar{\varepsilon}T_0 + 2(T_0)^{\frac{5}{6}})) \leq 1, \\ (T_0)^4 \exp(-(T_0)^{\frac{1}{3}}) \leq \frac{\varepsilon^*}{2}. \end{cases} \quad (17)$$

Below, we give a precise description of how play proceeds within a block (and an intuitive description in parentheses).

1. *Sub-block 0*: This sub-block consists of the following $2 + 2N$ phases:

(a) *Jamming coordination phase* (0, jam): Players play the jamming coordination protocol for 2 periods. (“The players coordinate on who will be jamming players.”)

(b) *Coordination phase* (0, i) (repeat for each $i = 1, \dots, N$): Player i sends $x_i \in \{G, B\}$ using the verified communication protocol with repetition T_0 . Since the message set $M_i = \{G, B\}$ has cardinality 2, this phase takes $\mathcal{T}(M_i, T) = 2b(2) + 2b(A^{4b(2)})NT_0 \approx 4 + 16NT_0 \log_2 |A|$ periods.²⁴ (“The players coordinate on x .”)

(c) *Contagion phase* (0, i , con) (repeat for $i = 1, \dots, N$): Player i sends $\text{susp}(h_i) \in \{0, 1\}$ using the verified protocol with repetition T_0 . This phase also takes $\approx 4 + 16NT_0 \log_2 |A|$ periods. (“If any player is suspicious, her suspicion spreads.”)

2. *Sub-block $l = 1, \dots, L$* : This sub-block consists of the following $1 + 3N$ phases:

(a) *Main phase* (l , main): This phase takes $(T_0)^3$ periods, and is described in Section J. Roughly, if player i is not suspicious, she plays $a_i^l(x(i))$ in every period; otherwise, she plays α^{\min} in every period.

Let $\mathbb{T}(l, \text{main})$ denote the set of $(T_0)^3$ periods in main phase (l , main). At the end of the phase, each player i selects a period $t_i(l) \in \mathbb{T}(l, \text{main})$, uniformly at random. (“Each player selects a random period to monitor.”)

²⁴Throughout this section, we use \approx to indicate equality up to rounding up all \log_2 terms: formally, we write $f(x) \approx g(\log_2 y_1, \dots, \log_2 y_m)$ if $g(\log_2 y_1, \dots, \log_2 y_m) \leq f(x) \leq g(\lceil \log_2 y_1 \rceil, \dots, \lceil \log_2 y_m \rceil)$.

(b) *Communication phase* (l, i) (repeat for $i = 1, \dots, N$): Player i sends $t_i(l) \in \mathbb{T}(l, \text{main})$ using the verified protocol with repetition T_0 . Since the message set has cardinality $|\mathbb{T}(l, \text{main})| = (T_0)^3$, this phase takes $2b((T_0)^3) + 2b(A^{4b((T_0)^3)})NT_0 \approx 6\log_2 T_0 + 24NT_0 \log_2 T_0 \log_2 |A|$ periods. (“Players communicate selected monitoring periods.”)

(c) *Communication phase* (l, i, n) (repeat for $i = 1, \dots, N$ and $n = 1, \dots, N$): Player n sends $(a_{n,t}, \omega_{n,t})$ using the verified protocol with repetition T_0 , where t equals player n 's inference of $t_i(l)$ in phase (l, i) . Since the message set has cardinality $|A|^2$, this phase takes $2b(|A|^2) + 2b(A^{4b(|A|^2)})NT_0 \approx 4\log_2 |A| + 16NT_0(\log_2 |A|)^2$ periods. (“Players share information about the monitoring periods.”)

(d) *Contagion phase* (l, i, con) (repeat for $i = 1, \dots, N$): A repetition of phase $(0, i, \text{con})$, but for the current histories h_i . Again, this phase takes $\approx 4 + 16NT_0 \log_2 |A|$ periods. (“Suspicion spreads.”)

Let $\mathbb{L}_i = (t_i(l))_{l=1}^L$ be the collection of random monitoring periods selected by player i .

Let T^* be the final period of the last contagion phase, phase (L, N, con) . Let $\mathbb{T}^* = \{1, \dots, T^*\}$ be the set of periods up to period T^* . Let \mathbb{T}' be the set of non-main phase periods up to period T^* :

$$\mathbb{T}' = \mathbb{T}^* \setminus \bigcup_{l=1}^L \mathbb{T}(l, \text{main}). \quad (18)$$

Given that T_0 satisfies (17), it can be checked that $|\mathbb{T}'| \leq (T_0)^{1.1}$.²⁵ (In what follows, all comparisons of numbers of periods involving T_0 assume (17).)

Let $\chi_n \in \{0, 1\}$ be a function of $(x_n, h_n^{T^*})$, where $\chi_n = 1$ if and only if there exists $t \in \{1, \dots, T^*\}$ such that $a_{n,t} \notin \text{supp}(\sigma_n^*(x_n)|_{h_n^{t-1}})$ (i.e., player n deviated from $\sigma_n^*(x_n)$ in the first T^* periods).

3. *Final Talk Sub-block*: This sub-block consists of the following $4N$ phases:

(a) *Phase* $(\text{final}, 1, i)$ (repeat for $i = 1, \dots, N$): Player $i - 1$ sends the list of periods $\mathbb{L}_{i-1} \in \{1, \dots, (T_0)^3\}^L$ using the verified protocol with repetition T_0 . Next, sequentially, each player $n \neq i, i - 1$ sends the following two messages using the secure protocol with repetition T_0 : (i) $\chi_n \in \{0, 1\}$ (i.e., player n “confesses” if she deviated in the first T^* periods). (ii) $(a_{n,t}, \omega_{n,t})_{t \in \mathbb{L}_i(n)}$, where $\mathbb{L}_i(n)$ is player n 's inference of \mathbb{L}_i . (If $\mathbb{L}_i(n) = 0$, then player n sends $(a_{n,t}, \omega_{n,t}) = (a^0, a^0)$.) (“Players confess any deviations and re-send their information about the monitoring periods.”²⁶) Player i 's message set has cardinality $(T_0)^{3L}$ and the message set of each player $n \neq i, i - 1$ has cardinality $2A^{2L}$. Hence, the length of

²⁵In particular,

$$\begin{aligned} \mathbb{T}'(T_0) &= 2 + 2N(2b(2) + 2b(A^{4b(2)})NT_0) + L(N(2b((T_0)^3) + 2b(A^{4b((T_0)^3)})NT_0) \\ &\quad + N^2(2b(|A|^2) + 2b(A^{4b(|A|^2)})NT_0) + N(2b(2) + 2b(A^{4b(2)})NT_0)) \\ &= 2 + 8N + 64\lceil \log_2 |A| \rceil N^2 T_0 + 12LN\lceil \log_2 T_0 \rceil + 96\lceil \log_2 |A| \rceil LN^2\lceil \log_2 T_0 \rceil T_0 \\ &\quad + 8LN^2\lceil \log_2 |A| \rceil + 64\lceil \log_2 |A| \rceil^2 LN^3 T_0 + 4NL + 32\lceil \log_2 |A| \rceil LN^2 T_0 \\ &\leq (T_0)^{1.1} \quad (\text{by (17)}). \end{aligned}$$

Elsewhere in the proof, similar calculations show that (17) guarantees a sufficiently high value of T_0 . We omit such calculations going forward.

²⁶Confessing deviations and re-sending past messages play a similar role here as in Hörner and Olszewski (2006) and Yamamoto (2012).

this phase is

$$\begin{aligned} T(\text{final}, 1, i) &= 2b((T_0)^{3L}) + 2b(A^{4b((T_0)^{3L})})NT_0 + (N-2)2b(2|A|^{2L})T_0 \\ &\approx 6L \log_2 T_0 + 24NT_0L \log_2 T_0 \log_2 |A| + 2(N-2)L T_0(1 + 2 \log_2 |A|). \end{aligned}$$

Let T_1 be the final period of phase (final, 1, N). Let $\mathbb{T}_1 = \{1, \dots, T_1\}$. Let

$$\mathbb{T}'' = \mathbb{T}_1 \setminus \bigcup_{l=1}^L \mathbb{T}(l, \text{main}). \quad (19)$$

It can be checked that $|\mathbb{T}''| \leq (T_0)^{1.1}$. Let $\mathbb{T}(\text{final}, 1, i)$ be the set of periods in phase (final, 1, i).

(b) *Phase* (final, 2, i) (repeat for $i = 1, \dots, N$): Sequentially, each player $n \neq i, i-1$ sends x_n and $(a_{n,t}, \omega_{n,t})_{t \in \mathbb{T}''}$ using the secure protocol with repetition T_0 . (“Players share their non-main phase histories.”) The length of this phase is $T(\text{final}, 2) = (N-2)2b(2A^{2\mathbb{T}''})T_0 \approx 2(N-2)T_0 \log_2(2A^{2\mathbb{T}''})$. Let T_2 be the final period of phase (final, 2, N). Let $\mathbb{T}_2 = \{1, \dots, T_2\}$. It can be checked that $T_2 \leq L(T_0)^3 + (T_0)^{2.1}$. Let $\mathbb{T}(\text{final}, 2, i)$ be the set of periods in phase (final, 2, i).

(c) *Phase* (final, 3, i) (repeat for $i = 1, \dots, N$): Sequentially, each player $n \neq i, i-1$ sends $(a_{n,t}, \omega_{n,t})_{t \in \bigcup_{j \in I} \mathbb{T}(\text{final}, 2, j)}$ using the basic protocol with repetition T_0 . (“Players share their information about each other’s non-main phase histories.”) The length of this phase is $T(\text{final}, 3) = (N-2)2b(N \times T(\text{final}, 2))T_0 \approx 2(N-2)T_0 \log_2(N \times T(\text{final}, 2))$. Let T_3 be the final period of phase (final, 3, N). It can be checked that $T_3 \leq L(T_0)^3 + (T_0)^{2.1}$.

(d) *Phase* (final, 4, i) (repeat for $i = 1, \dots, N$): Player $i-1$ selects a period $t_{i-1} \in \{1, \dots, T_3\}$, uniformly at random. Player $i-1$ sends the realization of t_{i-1} using the basic protocol with repetition T_0 . Next, sequentially, each player $n \neq i-1, i$ sends her inference $t_{i-1}(n) \in \{0, 1, \dots, T_3\}$ and $(a_{n,t_{i-1}(n)}, \omega_{n,t_{i-1}(n)})$ using the basic protocol with repetition T_0 . (If $t_{i-1}(n) = 0$, then n sends $(a_{n,t_{i-1}(n)}, \omega_{n,t_{i-1}(n)}) = (a^0, a^0)$.) (“Each player monitors one extra period to cancel the effects of discounting.”) The length of the phase is

$$\begin{aligned} T(\text{final}, 4) &= 2b(T_3)T_0 + (N-2)2b((T_3+1) \times A^2)T_0 \\ &\approx 2T_0 \log_2(T_3) + (N-2)2T_0(\log_2(T_3+1) + 2 \log_2 |A|). \end{aligned}$$

Finally, we have $T^{**} = T_3 + T(\text{final}, 4)$. It can be checked that $T^{**} \leq L(T_0)^3 + (T_0)^{2.1}$.

APPENDIX F: REDUCTION LEMMAS: PHASES (final, 3, i) AND (final, 4, i)

F.1. Basic Communication Module

We analyze the equilibrium block strategies by backwards induction. Since the basic communication protocol is used in the last phases (phases (final, 3, i) and (final, 4, i)), we start by considering payoffs and reward functions for this protocol. We call the resulting finitely repeated game the *basic communication module*.

For each player $n \in I$, payoff functions in the module take the form

$$\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \pi_n(x_{n-1}, h_{n-1}) + w_n(h), \quad (20)$$

where \hat{u}_n is the stage-game payoff function; π_n is a *reward function* that depends only on player $n - 1$'s state and module history (where the state vector $(x_n)_{n \in I}$ is taken as fixed and commonly known); and w_n is a *continuation payoff function* that depends on the entire module history. We wish to construct a reward function such that, when viewed as a strategy profile in this finitely repeated game, the basic protocol is a belief-free equilibrium.

DEFINITION 1: A strategy profile σ is a belief-free equilibrium (BFE) if, for each player i and history h_i , the continuation strategy $\sigma_i|_{h_i}$ is a best response against $\sigma_{-i}|_{h_{-i}}$ for every opposing history profile h_{-i} .

We say that *the premise for basic communication with magnitude K is satisfied* if the following conditions hold:

1. Player i is indifferent about the result of communication: $w_i(h) = 0$ for all h .
2. For all $n \neq i$, the range of $w_n(h)$ is bounded by K : $\max_{h, \tilde{h}} |w_n(h) - w_n(\tilde{h})| \leq K$.

LEMMA 5: For each $i \in I$, x_{i-1} , M_i , T , $\delta \in [0, 1]$, w , and $K \geq 2\bar{u}/\bar{\varepsilon}$ satisfying the premise for basic communication with magnitude K , there exists a family of functions $(\pi_n(x_{i-1}, \cdot) : H_{n-1}^T \rightarrow \mathbb{R})_{n \in I}$ such that the following hold:

1. With payoff functions (20), the basic protocol is a BFE.
2. For each $n \in I$ and $m_i \in M_i$, $\mathbb{E}[\sum_{t \in T} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \pi_n(x_{n-1}, h_{n-1})] = T v_n(x_{n-1})$.
3. For each $n \in I$ and $t \in T$,

$$\max_{x_{n-1}, h_{n-1}, \tilde{h}_{n-1}} |\pi_n(x_{n-1}, h_{n-1}) - \pi_n(x_{n-1}, \tilde{h}_{n-1})| \leq \left(\bar{u} + 2 \frac{\bar{u} + K}{\bar{\varepsilon}} \right) T. \quad (21)$$

The proof is relegated to the Supplemental Material (as are all other omitted proofs). Here is a sketch: For each receiver $n \neq i$, player $n - 1$ rewards player n every time she observes a^0 , which incentivizes player n to play a^0 throughout the module. Although whether player i (the sender) plays a^0 or a^1 also affects the probability that player $n - 1$ observes a^0 in a given period (since i and $n - 1$ may match), the expected number of rewards is independent of m_i because player i plays a^0 and a^1 with the same frequency for every m_i . In addition, whether player i plays a^0 in the first or second half-interval affects player n 's instantaneous utility through discounting, so we must adjust the rewards to cancel this effect.

For player i , player $i - 1$ makes her indifferent between playing a^0 and a^1 in every period. This is straightforward since player $i - 1$'s observations statistically identify player i 's actions.

Note that Lemma 5 concerns the complete information game where the states and continuation payoff functions $(x_n, w_n)_{n \in I}$ are known. However, as the statement of the lemma holds for each realization of $(x_n, w_n)_{n \in I}$, the same argument applies for the incomplete information game where $(x_n, w_n)_{n \in I}$ is unknown but the premise for communication is satisfied for each $(x_n, w_n)_{n \in I}$. The same remark applies for Lemmas 8, 13, and 17 introduced later.

F.2. Reduction Lemma 6: Undiscounted, Finitely Repeated Game

We show that the equilibrium conditions of Lemma 1 can be replaced by corresponding undiscounted conditions:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[\sum_{\tau=1}^{T_3} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) | x_{-i}, h_i^{t-1} \right]. \quad (22)$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) = \frac{1}{T_3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{\tau=1}^{T_3} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) \right]. \quad (23)$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T_3} \in H_{i-1}^{T_3}$,

$$\operatorname{sign}(x_{i-1}) \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) \geq -7\epsilon^* T_3, \quad (24)$$

where, for $x_{i-1} \in \{G, B\}$, define

$$\operatorname{sign}(x_{i-1}) := \begin{cases} -1 & \text{if } x_{i-1} = G, \\ 1 & \text{if } x_{i-1} = B. \end{cases}$$

Note that Condition (4) is omitted, as $v_i(x_{i-1})$ is fixed to satisfy it by (5). The third inequality in (3) (which here would be $|\frac{1-\delta}{\delta T_3} \pi_i^*(x_{i-1}, h_{i-1}^{T_3})| \leq v_i(G) - v_i(B)$) is also omitted, as we have fixed T_3 , $\pi_i^*(x_{i-1}, h_{i-1}^{T_3})$, and $v_i(G) > v_i(B)$ (by (5)) and will take $\delta \rightarrow 1$.

LEMMA 6: Suppose that, in the T_3 -period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$, consistent ex post belief system β^* , and reward functions $(\pi_i^*(x_{i-1}, h_{i-1}^{T_3}))_{i,x_{i-1},h_{i-1}^{T_3}}$ such that Conditions (22)–(24) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

The proof shows that, for any strategies $(\sigma_i^*(x_i))_{i,x_i}$ in the T_3 -period game satisfying the conditions of the lemma, the T^{**} -period game that results from concatenating these strategies with the phase (final, 4, i) $_{i \in I}$ strategies described in Section E (in which players share information about a random past period) satisfies the equilibrium conditions of Lemma 1. To prove this, we augment the reward functions from the T_3 -period game by giving each player a small reward if the newly monitored period reveals that she took an action yielding a higher payoff later in the block, so as to leave her indifferent to the timing of her actions within the first T_3 periods. Condition (22) then ensures sequential rationality for the first T_3 periods. Moreover, as $\delta \rightarrow 1$, the size of the new reward goes to 0. Hence, Lemma 5 guarantees the existence of a reward function that incentivizes players to follow the basic communication protocol in the last $T^{**} - T_3$ periods. Finally, since $(T^{**} - T_3)/T_3$ is small, communication takes a short enough time that Conditions (23) and (24) imply Conditions (2) and (3), given the slack in (5).

F.3. Lemma 7: Letting Rewards Depend on h_{-i}

Next, consider phase (final, 3, i), during which players $n \neq i, i-1$ send messages $(a_{n,t}, \omega_{n,t})_{t \in \bigcup_j \mathbb{T}(\text{final}, 2, j)}$ using the basic communication protocol. Player $i-1$ then uses her

history in phase (final, 3, i) to compute player i 's reward for phase (final, 2, j) $_{j \in I}$ so that, at the end of phase (final, 2, N), player i 's expected reward is equal to

$$\begin{aligned} & \sum_{j \neq i} \sum_{t \in \mathbb{T}(\text{final}, 2, j)} \pi_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ & + \sum_{t \in \mathbb{T}(\text{final}, 2, i)} (\pi_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) + \pi_i^{a^0}(a_{-i,t}, \omega_{-i,t})). \end{aligned} \quad (25)$$

Given Conditions (7) and (8), player i 's expected payoff in phases ((final, 2, j)) $_{j \in I}$ equals

$$\sum_{t \in \mathbb{T}(\text{final}, 2, i)} v_i(x_{i-1}) - \sum_{t \in \mathbb{T}(\text{final}, 2, i)} \mathbf{1}_{a_{i,t} \neq a^0}. \quad (26)$$

Note that player i has a strict incentive to play a^0 during phase (final, 2, i). Based on this construction, we further reduce the conditions for Lemma 6:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\begin{aligned} \sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} & \left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \sum_{t \in \mathbb{T}(\text{final}, 2, i)} v_i(x_{i-1}) - \sum_{t \in \mathbb{T}(\text{final}, 2, i)} \mathbf{1}_{a_{i,t} \neq a^0} \right. \\ & \left. + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) | x_{-i}, h_i^{t-1} \right]. \end{aligned} \quad (27)$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$\begin{aligned} v_i(x_{i-1}) = \frac{1}{T_2} \mathbb{E}^{\sigma^*(x)} & \left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \sum_{t \in \mathbb{T}(\text{final}, 2, i)} v_i(x_{i-1}) \right. \\ & \left. - \sum_{t \in \mathbb{T}(\text{final}, 2, i)} \mathbf{1}_{a_{i,t} \neq a^0} + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) \right]. \end{aligned} \quad (28)$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T_2} \in H_{i-1}^{T_2}$,

$$\operatorname{sign}(x_{i-1}) \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) \geq -6\epsilon^* T_2. \quad (29)$$

Note that the slack in the self-generation constraint has been reduced to $6\epsilon^* T_2$, compared to $7\epsilon^* T_3$ in Condition (24). This is because some slack is “used up” when replacing $\pi_i^*(x_{i-1}, h_{i-1}^{T_1})$ with (25) and $\pi_i^*(x_{-i}, h_{i-1}^{T_2})$.

LEMMA 7: Suppose that, in the T_2 -period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i, x_i}$, consistent ex post belief system β^* , and reward functions $(\pi_i^*(x_{i-1}, h_{i-1}^{T_2}))_{i, x_{i-1}, h_{i-1}^{T_2}}$ such that Conditions (27)–(29) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

The proof shows that, for any strategies $(\sigma_i^*(x_i))_{i, x_i}$ in the T_2 -period game satisfying the conditions of the lemma, the T_3 -period game that results from concatenating these strategies with the phase (final, 3, i) $_{i \in I}$ strategies described in Section E satisfies the equilibrium

conditions of Lemma 6. Since the phase (final, 3, i) $_{i \in I}$ strategies are used only to compute the rewards π_i^{cancel} and $\pi_i^{a^0}$, and these rewards are of order \bar{u} , Lemma 5 with K of order \bar{u} guarantees the existence of a reward function that incentivizes players to follow the basic communication protocol in the last $T_3 - T_2$ periods.

APPENDIX G: REDUCTION LEMMA: PHASE (final, 2, i)

G.1. Secure Communication Module

In phase (final, 2, i), the secure protocol is used. We consider payoffs and reward functions for this protocol. The resulting finitely repeated game is the *secure communication module*.

We need only consider the case where I_{jam} is a singleton. Fix the sender i and another player i^* with $i \neq i^*$, $i^* - 1$. Let $I_{\text{jam}} = \{i^* - 1\}$. Intuitively, we consider a situation where player i must communicate a message m_i to player $i^* - 1$, but player i^* may gain if player $i^* - 1$ infers some $m'_i \neq m_i$, while other players are indifferent.

For each $n \in I$, payoff functions in the secure communication module are given by

$$-\mathbf{1}_{\{n=i^*\}} \sum_{t \in \mathbb{T}} \mathbf{1}_{\{a_{n,t} \neq a_0\}} + w_n(h), \quad (30)$$

for some function $w_n : H^{\mathbb{T}} \rightarrow \mathbb{R}$. Let $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ denote the strategy profile in the secure protocol. Note that only the sender's strategy depends on m_i . We will give conditions on $(w_n)_{n \in I}$ under which $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ is an " i^* -quasi-belief-free equilibrium" of the resulting finitely repeated game. Intuitively, this means that the strategy of each player $n \neq i^*$ is sequentially rational for every opposing history profile, and player i^* 's strategy is sequentially rational for some consistent belief system. In addition, sequential rationality for player i^* is imposed ex post with respect to m_i . This ensures that the module remains incentive compatible when viewed as one part of the infinitely repeated game.

DEFINITION 2: A family of strategy profiles $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ is an i^* -quasi-belief-free equilibrium (i^* -QBFE) if (i) for each player $n \neq i^*$ and history h_n , the continuation strategy $\sigma_n|_{h_n}$ is a best response against $\sigma_{-n}|_{h_{-n}}$ for every opposing history profile h_{-n} and every possible message m_i , and (ii) for player i^* , there exists a sequence of families of completely mixed strategy profiles $((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i})_{k=1}^{\infty}$ and a corresponding family of belief systems $\beta(h_{-i^*}|m_i, h_{i^*})$ (where $\beta(h_{-i^*}|m_i, h_{i^*})$ is the limit of conditional probabilities derived from $((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{k=1}^{\infty})$ such that, for each m_i and $h_{i^*}^{t-1}$,

$$\sigma_{i^*} \in \operatorname{argmax}_{\tilde{\sigma}_{i^*} \in \Sigma_{i^*}} - \sum_{t \in \mathbb{T}} \mathbf{1}_{\{a_{i^*,t} \neq a^0\}} + \mathbb{E}^{(\tilde{\sigma}_{i^*}, \sigma_{-i^*}^{m_i})} [w_{i^*}(h)|m_i, h_{i^*}^{t-1}],$$

where the expectation is taken with respect to $\beta(h_{-i^*}^{t-1}|m_i, h_{i^*}^{t-1})$.

We say that *the premise for secure communication for player i^* with magnitude K is satisfied* if the following conditions hold:

1. All players but player i^* are indifferent about the result of communication: $w_n(h) = 0$ for all h and $n \neq i^*$.

2. If player $i^* - 1$ deviates from σ_{i^*-1} or ALLREG does not occur,²⁷ then $w_{i^*}(h) = 0$ for all h .

3. If player $i^* - 1$ follows σ_{i^*-1} and ALLREG occurs, then the following conditions hold:

- (a) If $m_i(i^* - 1) \in M_i \cup \{0\}$ is the same at protocol histories h and \tilde{h} , then $w_{i^*}(h) = w_{i^*}(\tilde{h})$. Under this condition, we abuse notation and write $w_{i^*}(h) = w_{i^*}(m_i(i^* - 1))$.
- (b) The range of $w_{i^*}(m_i(i^* - 1))$ is bounded by K :

$$\max_{m_i, \tilde{m}_i \in M_i \cup \{0\}} |w_{i^*}(m_i) - w_{i^*}(\tilde{m}_i)| \leq K. \quad (31)$$

- (c) $w_{i^*}(0) \leq w_{i^*}(m_i(i^* - 1))$ for all $m_i(i^* - 1) \in M_i$.

We now specify player i^* 's beliefs. In particular, we specify that, after any off-path observation, she assigns probability 1 to the event that player $i^* - 1$ deviated (and hence, if the above premise holds, $w_{i^*}(h) = 0$). This belief is clearly consistent: for concreteness, define $((\sigma_i^{m_i, k}, \sigma_{-i}^k)_{m_i \in M_i})_{k=1}^\infty$ by letting player $i^* - 1$ tremble uniformly over all actions with probability k^{-1} at each history, and letting every other player tremble uniformly over all actions with probability k^{-k} at each history.

LEMMA 8: For each $i^* \in I$, $i \in I \setminus \{i^* - 1, i^*\}$, M_i , w , and K satisfying the premise for secure communication for player i^* with magnitude K , if

$$b(M_i)K \exp(-\bar{\eta}T + T^{\frac{1}{2}}) \leq 1, \quad (32)$$

then with payoff functions (30) the secure communication protocol, together with the above belief system for player i^* , is an i^* -QBFE.

PROOF: By construction, players other than i^* are indifferent over all actions throughout the module. For player i^* , fix a period $t \in \mathbb{T}$ and history $(a_{i^*, \tau}, \omega_{i^*, \tau})_{\tau \in \mathbb{T}, \tau \leq t-1}$. Suppose $\omega_{i^*, \tau} \in \{a^0, a^1\}$ for each $\tau \leq t-1$. By the same argument as for Lemma 4, for every possible continuation history $(a_{i^*, \tau}, \omega_{i^*, \tau})_{\tau \in \mathbb{T}, \tau \geq t}$, with probability at least

$$1 - b(M_i) \exp(-\bar{\eta}T + T^{\frac{1}{2}}) \quad (33)$$

conditional on $(a_{i^*, \tau}, \omega_{i^*, \tau})_{\tau \in \mathbb{T}}$, either ALLREG does not occur or $[m_i(i^* - 1) \in \{m_i, 0\}]$, and $m_i(i^* - 1) = m_i$ if $a_{i^*, \tau} = a^0$ for all $\tau \in \mathbb{T}$. Moreover, if $(\omega_{i^*, \tau})_{\tau \in \mathbb{T}}$ is such that $[m_i(i^* - 1) \in \{m_i, 0\}]$, and $m_i(i^* - 1) = m_i$ if $a_{i^*, \tau} = a^0$ for all $\tau \in \mathbb{T}$, then by definition of $m_i(i^* - 1)$, we have $m_i(i^* - 1) = m_i$ if and only if player i^* takes a^0 whenever she meets player $i^* - 1$ in a half-interval where player i takes a^0 . Hence, since $w_{i^*}(0) \leq w_{i^*}(m_i(i^* - 1))$ for all $m_i(i^* - 1) \in M_i$, taking $a_{i^*, \tau} = a^0$ for each $\tau \geq t$ maximizes $w_{i^*}(h)$ with probability at least (33). Given this, conditions (31) and (32) imply that the reward term $-\mathbf{1}_{\{a_{i^*, t} \neq a^0\}}$ in payoff (30) outweighs any possible benefit to player i^* from playing $a \neq a^0$ in an attempt to manipulate $m_i(i^* - 1)$. If instead $\omega_{i^*, \tau} \notin \{a^0, a^1\}$ for some $\tau \leq t-1$, then by construction of the belief system player i^* believes $w_{i^*}(h) = 0$ with probability 1. Hence, player i^* maximizes the reward term $-\mathbf{1}_{\{a_{i^*, \tau} \neq a^0\}}$ in payoff (30), so playing a^0 as prescribed is optimal. Q.E.D.

²⁷Player $i^* - 1$ follows σ_{i^*-1} if, for each τ , her action $a_{i^*-1, \tau}$ is in the support of σ_{i^*-1} given $(a_{i^*-1, t}, \omega_{i^*-1, t})_{t \leq \tau-1}$. Since $i^* - 1 \neq i$, the support is independent of m_i . Player $i^* - 1$ deviates from σ_{i^*-1} if she does not follow σ_{i^*-1} .

G.2. Reduction Lemma 9: Letting Rewards Depend on Other Players' Non-Main Phase Histories

We now use phases $((\text{final}, 2, n))_{n \in I}$ to further simplify equilibrium conditions. Player $i - 1$ uses the result of this communication to construct the reward function so that the expected reward at the end of phase $(\text{final}, 1, N)$ is the same as if player $i - 1$ knew the histories of players $-(i - 1, i)$ for all non-main phase periods. We write the reward function as $\pi_i(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''})$, where \mathbb{T}'' is the set of non-main phase periods, from (19). We wish to replace $\pi_i^*(x_{i-1}, h_{i-1}^{T_2})$ with $\pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''})$ in Conditions (27)–(29), yielding the following:

1. [Range Restriction] The range of the reward function is bounded by $8\bar{u}T_1$:

$$\sup_{x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}} |\pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''})| \leq 8\bar{u}T_1. \quad (34)$$

2. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}) | h_i^{t-1} \right]. \quad (35)$$

3. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) = \frac{1}{T_1} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t=1}^{T_1} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}) \right]. \quad (36)$$

4. [Self-Generation] For all $x_{-i}, h_{i-1}^{T^*}$, and $h_{-i}^{\mathbb{T}''}$,

$$\operatorname{sign}(x_{i-1}) \pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}) \geq -5\epsilon^* T_1. \quad (37)$$

LEMMA 9: Suppose that, in the T_1 -period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i, x_i}$, consistent ex post belief system β^* , and reward functions $(\pi_i^*(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}))_{i, x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}}$ such that Conditions (34)–(37) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

APPENDIX H: VERIFIED COMMUNICATION MODULE

In phase $(\text{final}, 1, i)$ and earlier communication phases, the verified communication protocol is used. We now establish some key properties of this protocol, and then augment it with payoffs and reward functions. The resulting *verified communication module* is the most complicated of our modules.

Let $\sigma^{*, m_i} = (\sigma_i^{*, m_i}, \sigma_{-i}^*)$ denote the prescribed protocol strategy profile when player i sends message m_i . For each $j, j' \in I$, player j 's equilibrium strategy in the j' -checking round is determined by $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})}$ and $\zeta_j \in \{\text{reg}, \text{jam}\}$ (independently of m_i). We say player j follows σ_j^* in the j' -checking round if, for each $\tau \in \mathbb{T}(j')$, her action $a_{j,\tau}$ is in the support of σ_j^* given $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})}$, $\zeta_j \in \{\text{reg}, \text{jam}\}$, and $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(j'), t \leq \tau-1}$. Let $H^{< j'}$ denote the set of protocol history profiles at the beginning of $\mathbb{T}(j')$ that arise with positive probability under some strategy profile σ . Given $h^{< j'} \in H^{< j'}$, let $H_j^{\mathbb{T}(j')}|_{h^{< j'}}$ denote

the set of protocol history profiles during $\mathbb{T}(j')$ that are reached from $h^{<j'}$ with positive probability under some strategy profile $(\sigma_j, \sigma_{-j}^*)$ with $\sigma_j \in \Sigma_j$ (i.e., when players $-j$ follow the protocol).

H.1. Regular and Erroneous Opponents' Histories

We classify each of player j 's opponents' history profiles as *regular* or *erroneous*, $\theta_j(h_{-j}, \zeta) \in \{R, E\}$. Roughly, a profile of player j 's opponents' histories h_{-j} is “erroneous” if it arises whenever some jamming player plays JAM or the realized matching process is erroneous.

This classification—which will affect player j 's reward function—depends on players $-j$'s protocol history h_{-j} and the type profile $\zeta = (\zeta_n)_{n \in I}$. By Lemma 9, player j 's reward function can depend on her opponents' non-main phase histories. As verified communication protocol histories and jamming coordination protocol histories (which will determine ζ) are non-main phase histories, player j 's reward function can depend on h_{-j} and ζ .

For $j, j' \in I$, we first define $\theta_j(h_{-j}, \zeta, j') = E$ (“ j 's opponents' histories in the j' -checking round are erroneous”) if and only if one or more of the following four conditions holds:

1. $\zeta_j = \text{jam}$.
2. There exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}(j')$.
3. [Condition FAIL] $j \neq j'$ and there exists a half-interval \mathbb{S} in $\mathbb{T}(j')$ and a player $n \neq j'$ such that player j' plays a^1 throughout \mathbb{S} but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Whether this event occurs is determined by h_{-j} , as Lemma 2 implies that h_j is uniquely determined by h_{-j} .)
4. [Condition FAIL j'] $j = j'$, player j' follows $\sigma_{j'}^*$ in the j' -checking round, and there exist a half-interval \mathbb{S} in $\mathbb{T}(j')$ and a player $n \neq j'$ such that player j' plays a^1 throughout \mathbb{S} but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Again, this event is determined by h_{-j} , by Lemma 2.)

(Note that $\theta_j(h_{-j}, \zeta, j')$ depends on h_{-j} only through $h_{-j}^{\mathbb{T}(j')}$ and $h_{-j}^{\mathbb{T}(\text{msg})}$, the latter because whether player j' follows $\sigma_{j'}^*$ in the j' -checking round (in [Condition FAIL j']) depends on $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$.)

We define $\theta_j(h_{-j}, \zeta) = E$ if and only if either $\theta_j(h_{-j}, \zeta, j') = E$ for some $j' \in I$ or some player $j' \neq j$ deviates from $\sigma_{j'}^*$ in any checking round. Otherwise, define $\theta_j(h_{-j}, \zeta) = R$. In addition, for each $j' \in I$, let $\text{JAM}_{j',-j}$ denote the event that there exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}(j')$. Let $\text{REG}_{j',-j}$ denote the complementary event.

LEMMA 10: For each player $j \in I$, each type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$, and each history profile $h^{<j'} \in H^{<j'}$:

1. If all players follow σ^* in the j' -checking round, then $\Pr(\theta_j(h_{-j}, \zeta, j') = E | h^{<j'}, \zeta)$ is the same for every $h^{<j'} \in H^{<j'}$.
2. $\sigma_{j'}^* \in \arg\max_{\sigma_{j'} \in \Sigma_{j'}} \Pr^{(\sigma_{j'}, \sigma_{-j'}^*)}(\theta_j(h_{-j}, \zeta, j') = E | \zeta, h^{<j'})$.
3. If all players follow σ^* in the j' -checking round and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for some $n \in I$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ and $\theta_j(h_{-j}, \zeta, j') = E$.
4. If player j' follows $\sigma_{j'}^*$ in the j' -checking round, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for some $n \in I$, and $\theta_j(h_{-j}, \zeta, j') = R$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$.
5. If $j \neq j'$, players $-j$ follow σ_{-j}^* in the j' -checking round, and $(a_{j',t}(j), \omega_{j',t}(j))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$, then $\theta_j(h_{-j}, \zeta, j') = E$.

PROOF: 1. For any message $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$, player j' plays a^1 the same number of times in each interval. Hence, the probability that FAIL (or FAIL j') holds is independent of $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$.

2. If player j' deviates from $\sigma_{j'}^*$ then FAIL j' does not hold. Moreover, Conditions 1 and 2 for $\theta_j(h_{-j}, \zeta, j') = E$ are independent of σ_j , and FAIL only applies when $j \neq j'$. Hence, the conclusion holds.

3. If $j \in \mathcal{I}_{\text{jam}}$ or a player in $\mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ plays JAM in some half-interval, then $\theta_j(h_{-j}, \zeta, j') = E$ by construction. If $j \notin \mathcal{I}_{\text{jam}}$ and all players $\mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ play REG in every half-interval, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ only if player n does not observe a^1 in some half-interval where player j' plays a^1 . Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ and FAIL or FAIL j' holds.

4. If $\theta_j(h_{-j}, \zeta, j') = R$, then each $n \neq j'$ observes a^1 in each half-interval where player j' plays a^1 . So, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ implies $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$.

5. When players $-j$ follow σ_{-j}^* , $(a_{j',t}(j), \omega_{j',t}(j))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ only if player j does not observe a^1 in some half-interval where player j' plays a^1 . Hence, FAIL holds. Q.E.D.

H.2. Statistical Properties of the Verified Protocol

LEMMA 11: Suppose that

$$2N(N-1)b(A^{4b(M_i)})\exp(-T^{\frac{1}{2}}) + N(N-1)b(A^{4b(M_i)})\exp(-\bar{\varepsilon}T) \leq \exp(-T^{\frac{1}{3}}). \quad (38)$$

Then the following claims hold for every $m_i \in M_i$ and every type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$:

1. For any $j \neq i$ and any $\sigma_j \in \Sigma_j^T$, given strategy profile $(\sigma_j, \sigma_{-j}^{*,m_i})$, either (i) $m_i(n) = m_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq j$, or (iii) $\theta_j(h_{-j}, \zeta) = E$. Moreover, $\text{susp}(h_j) = 1$ implies $\theta_j(h_{-j}, \zeta) = E$.

2. For any $\sigma_i \in \Sigma_i^T$, given $(\sigma_i, \sigma_{-i}^*)$, either (i) there exists $\hat{m}_i \in M_i$ with $m_i(n) = \hat{m}_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq i$, or (iii) $\theta_i(h_{-i}, \zeta) = E$. Moreover, $\text{susp}(h_i) = 1$ implies $\theta_i(h_{-i}, \zeta) = E$.

3. Given σ^{*,m_i} , for any $j \in I$, either (i) $m_i(n) = m_i$ and $\text{susp}(h_n) = 0$ for all $n \in I$, or (ii) $\theta_j(h_{-j}, \zeta) = E$.

4. Given σ^{*,m_i} , with probability at least $1 - \exp(-T^{\frac{1}{3}})$, all the following events occur: (i) $m_i(n) = m_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 0$ for all $n \in I$, and (iii) $\theta_n(h_{-n}, \zeta) = R$ for all $n \notin \mathcal{I}_{\text{jam}}$.

5. For any $m_i, m'_i \in M_i$ and $j \in I$, $\Pr^{\sigma^{*,m_i}}(\theta_j(h_{-j}, \zeta) = R | \zeta) = \Pr^{\sigma^{*,m'_i}}(\theta_j(h_{-j}, \zeta) = R | \zeta)$.

The intuition is that $\theta_j(h_{-j}, \zeta) = E$ only if some player plays JAM or matching is erroneous, which is unlikely. Moreover, since the sender plays a^1 with the same frequency for all m_i , the probability of this event is independent of m_i .

The next lemma is analogous to Lemma 4. Unlike Lemmas 10 and 11, this lemma involves conditions on players' beliefs about the type profile $(\zeta_n)_{n \in I} \in \{\text{reg}, \text{jam}\}^N$. To express these conditions, we assume each player n has a prior probability distribution over $(\zeta_n)_{n \in I}$ at the beginning of the protocol. Let $\Pr_n(\cdot)$ denote conditional probability under player n 's prior.

LEMMA 12: Fix any $j \in I$, $j' \neq j$, and $h^{<j'} \in H^{<j'}$. Suppose that, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, we have $\Pr_j(\zeta_{j'} = \text{jam} \forall j' \neq j | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}) \geq \exp(-T^{\frac{1}{2}})$. Then, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, at least one of the following two conditions holds:

1. We have

$$\Pr_j(\text{JAM}_{j',-j}|m_i, h^{<j'}, h_j^{\mathbb{T}(j')}) \geq 1 - \exp(-\bar{\eta}T + 2T^{\frac{1}{2}}). \quad (39)$$

2. The following two conditions hold:

(a) For all $(a_{j,t})_{t \in \mathbb{T}(j')}$,

$$\begin{aligned} \Pr_j((a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \in \{0, (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}\} \forall n \neq j | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, \text{REG}_{j',-j}) \\ \geq 1 - Nb(|A|^{4b(M_i)}) \exp(-\bar{\eta}T + 2T^{\frac{1}{2}}). \end{aligned} \quad (40)$$

(b) If $a_{j,t} = a^0$ for all $t \in \mathbb{T}(j')$, then

$$\begin{aligned} \Pr_j((a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})} \forall n \neq j | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, \text{REG}_{j',-j}) \\ \geq 1 - Nb(|A|^{4b(M_i)}) \exp(-\bar{\eta}T + 2T^{\frac{1}{2}}). \end{aligned} \quad (41)$$

PROOF: The same as Lemma 4, except that $2T^{\frac{1}{2}}$ replaces $T^{\frac{1}{2}}$ in the inequality (12), as now $\mathcal{I}_{\text{jam}} \setminus \{j\}$ is non-empty with probability at least $\exp(-T^{\frac{1}{2}})$ rather than 1. *Q.E.D.*

H.3. Payoffs and Incentives

Throughout this subsubsection, fix $m_i^* \in M_i$ and let $\sigma^* = \sigma^{*,m_i}$.

For each $j \in I$ and $t \in \mathbb{T}(j)$, given $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})}$ identified from h_{-j} by Lemma 2, calculate the equilibrium action $a_{j,t}^*(h_{-j})$. Suppose each player j 's payoff equals

$$-\mathbf{1}_{\{\zeta_j = \text{reg}\}} \sum_{t \in \mathbb{T} \setminus \mathbb{T}(j)} \mathbf{1}_{\{a_{j,t} \neq a_{j,t}^*(h_{-j})\}} - \sum_{t \in \mathbb{T}(j)} \mathbf{1}_{\{a_{j,t} \neq a_{j,t}^*(h_{-j})\}} + w_j(h, \zeta). \quad (42)$$

(This is similar to (30), but now player j is rewarded for following the equilibrium strategy $a_{j,t}^*(h_{-j})$ in round $\mathbb{T}(j)$.)

We say that *the premise for verified communication to send message $m_i^* \in M_i$ with magnitude K is satisfied* if there exist $(v_j^E)_{j \in I} \in \mathbb{R}^N$, and $(v_j^{m_i})_{j \in I, m_i \in M_i \cup \{0\}} \in \mathbb{R}^N$ such that, for all $j \in I$ and $h \in H$, the following conditions hold:

1. If $\theta_j(h_{-j}, \zeta) = E$, then $w_j(h, \zeta) = v_j^E$.
2. If $\theta_j(h_{-j}, \zeta) = R$ and $\text{susp}(h_n) = 1$ for some $n \neq j$, then $w_j(h, \zeta) = v_j^0$.
3. If $\theta_j(h_{-j}, \zeta) = R$, $\text{susp}(h_n) = 0$ for all $n \neq j$, and $\exists \hat{m}_i \in M_i$ such that $m_i(n) = \hat{m}_i$ for all $n \in I$, then $w_j(h, \zeta) = v_j^{\hat{m}_i}$.
4. $v_j^0 \leq \min\{\min_{m_i \in M_i} v_j^{m_i}, v_j^E\}$.
5. $v_j^{m_i^*} \geq v_j^{\hat{m}_i}$ for all $\hat{m}_i \in M_i \cup \{0\}$.
6. The range of $w_j(h, \zeta)$ is bounded by K : $K \geq \max_{j \in I} \{\max\{v_j^E, (v_j^{m_i})_{m_i \in M_i}\} - v_j^0\}$.

The interpretation is that v_j^E is player j 's continuation payoff after erroneous opposing histories; v_j^0 is player j 's punishment payoff (which results if $\theta_j(h_{-j}, \zeta) = R$ and $\text{susp}(h_n) = 1$ for some $n \neq j$); and $v_j^{m_i}$ is j 's continuation payoff after players coordinate on message m_i .

We modify player i 's strategy in the message round after she herself deviates as follows: Recall that we define $m_i(n) = 1$ if player n infers some $(a_{i,t})_{t \in \mathbb{T}(\text{msg})}$ not corresponding to the binary expansion of any message. We can thus view the play of such $(a_{i,t})_{t \in \mathbb{T}(\text{msg})}$ as

sending message $m_i = 1$. With this interpretation, for each h_i^{t-1} , let $M_i(h_i^{t-1}) \subset M_i$ be the (non-empty) set of messages \tilde{m}_i such that $(a_{i,\tau})_{\tau=1}^{t-1}$ is consistent with the binary expansion of \tilde{m}_i ; and let $M_i^*(h_i^{t-1}) = \operatorname{argmax}_{m_i \in M_i(h_i^{t-1})} v_i^{m_i}$ be the elements that maximize $v_i^{m_i}$. Given h_i^{t-1} , if $m_i^* \in M_i^*(h_i^{t-1})$, player i plays $a_{i,t}$ corresponding to the binary expansion of m_i^* ; otherwise, she plays $a_{i,t}$ corresponding to the binary expansion of some $m_i \in M_i^*(h_i^{t-1})$.

Call a history σ -consistent if it is reached with positive probability under strategy profile σ . Recall that $H^{<j'}$ is the set of module history profiles at the beginning of $\mathbb{T}(j')$ that are σ -consistent for some $\sigma \in \Sigma$, and let $H_j^{\mathbb{T}(j')}|_{h^{<j'}}$ be the set of module histories during $\mathbb{T}(j')$ that are $(\sigma_j, \sigma_{-j}^*)$ -consistent for some $\sigma_j \in \Sigma_j$ given $h^{<j'}$. We assume that, for every player $j, j' \in I$, module strategy σ_j , $h^{<j'} \in H^{<j'}$, and $h_j \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, player j believes that all other players are jamming players with probability at least $\exp(-T^{\frac{1}{2}})$:

$$\Pr_j(n \in \mathcal{I}_{\text{jam}} \forall n \neq j | h^{<j'}, h_j) \geq \exp(-T^{\frac{1}{2}}). \quad (43)$$

LEMMA 13: Suppose that T is sufficiently large such that

$$KNb(A^{4b(M_i)}) \exp(-\bar{\eta}T + 2T^{\frac{1}{2}}) \leq 1. \quad (44)$$

If the premise for verified communication with magnitude K and (43) hold for each $j \in I$, then with payoff functions (42) the verified communication protocol is a sequential equilibrium. In addition, if there exists $i^* \in I \setminus \{i\}$ such that $\mathcal{I}_{\text{jam}} = I \setminus \{i^*\}$ and $v_j^E = v_j^{m_i}$ for all $j \neq i^*$ and $m_i \in M_i \cup 0$, while for player i^* the premise for verified communication and (43) hold, then with payoff functions (42) the verified communication protocol is an i^* -QBFE.

Intuitively, if the prior probability that players jam is not too low, whenever player j observes an erroneous history she believes that JAM is played and $\theta_j(h_{-j}, \zeta) = E$. Otherwise, she believes that all other players match with the sender at least once in each half-interval. Hence, if she deviates and changes some player's inference, this induces $\text{susp}(h_n) = 1$ and yields the punishment payoff v_j^0 . It will be useful to remember that all the lemmas in this section hold if Conditions (38), (43), and (44) are satisfied.

APPENDIX I: REDUCTION LEMMAS: PHASE (final, 1, i)

This section further simplifies Lemma 9, using phase (final, 1, i) $_{i \in I}$.

I.1. Reduction Lemma 14: Letting Rewards Depend on Other Players' Main Phase Histories

Recall that, for each main phase $l = 1, \dots, L$, player i randomly selects a monitoring period $t_i(l) \in \mathbb{T}(l, \text{main})$. We show that player i 's reward function in the T^* -period repeated game can be made to depend on players $-i$'s histories in periods in $\mathbb{L}_{i-1} = (t_i(l))_{l=1}^L$, that is, on

$$h_{-i}^{\mathbb{L}_{i-1}} := (a_{-i, t_{i-1}(l)}, \omega_{-i, t_{i-1}(l)})_{l=1, \dots, L}. \quad (45)$$

Recall that $\mathbb{T}' := \{1, \dots, T^*\} \setminus \bigcup_{l=1}^L \mathbb{T}(l, \text{main})$. The reward function takes the form $\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i})$, where $\chi_n \in \{0, 1\}$ was defined in Section E.²⁸ We wish to replace

²⁸Relative to Lemma 9, the argument $h_{-i}^{\mathbb{L}_{i-1}}$ has been added to the reward function and the argument $h_{i-1}^{T^*}$ has been removed, as $h_{-i}^{\mathbb{L}_{i-1}}$ contains enough information about player $i-1$'s main phase history to provide incentives for player i .

$\pi_i^*(x_{i-1}, h_{i-1}^{T*}, h_{-i}^{\mathbb{T}'})$ with $\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i})$ in Conditions (35)–(37). In the following conditions, we also cancel the instantaneous utilities outside of the main phases (which can be accomplished by using the reward function (7)):

1. [Range Restriction] The range of the reward function is bounded by $7\bar{u}T^*$:

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} |\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i})| \leq 7\bar{u}T^*. \quad (46)$$

2. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) | h_i^{t-1} \right]. \quad (47)$$

3. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) - 2\varepsilon^* = \frac{1}{L(T_0)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) \right]. \quad (48)$$

4. [Self-Generation] For all x_{-i} , $h_{-i}^{\mathbb{T}'}$, and $h_{-i}^{\mathbb{L}_{i-1}}$,

$$\operatorname{sign}(x_{i-1}) \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) \geq -2\varepsilon^*T^*. \quad (49)$$

LEMMA 14: Suppose that, in the T^* -period repeated game, there exist strategies $(\sigma_i^*(x_i))_{i, x_i}$, consistent ex post belief system β^* , and reward functions $(\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}))_{i, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}}$ such that Conditions (46)–(49) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

I.2. Reduction Lemma 15: “Ignoring” Other Players’ Deviations

We further simplify Lemma 14. Consider the following conditions:

1. [$t_i(l)$ Not Revealed Until End of Main Phase l] For all $x_i \in \{G, B\}$, $l \in \{1, \dots, L\}$, $t \in \{1, \dots, T^*\}$, $(\mathbb{L}_i, h_i^{t-1})$, and $(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})$, if $t \leq \tau$ for some $\tau \in \mathbb{T}(\text{main}(l))$, $t_i(\hat{l}) = \tilde{t}_i(\hat{l})$ for each $\hat{l} = 1, \dots, l-1$, and $h_i^{t-1} = \tilde{h}_i^{t-1}$, then

$$\sigma_i^*(x_i)|_{(\mathbb{L}_i, h_i^{t-1})} = \sigma_i^*(x_i)|_{(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})}. \quad (50)$$

2. [Reward Bound]

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} |\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i})| \leq 5\bar{u}T^*. \quad (51)$$

3. [Incentive Compatibility] Let $H_i(x_{-i})$ denote the set of histories that arise with positive probability under some strategy profile $(\sigma_i, \sigma_{-i}^*(x_{-i}))$ with $\sigma_i \in \Sigma_i^{T^*}$. For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i(x_{-i})$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}^*(x_{-i}))} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) | h_i^{t-1} \right]. \quad (52)$$

Note that we do not need to define “trembles” to define $\mathbb{E}[\cdot | \cdot]$ in (52).

4. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$\frac{v_i(G) - 2\varepsilon^*}{v_i(B) + 2\varepsilon^*} \leq \left\} \frac{1}{L(T_0)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{l-1}}) \right]. \quad (53)$$

5. [Self-Generation] The same as (49).

LEMMA 15: Suppose that, in the T^* -period repeated game, there exist strategies $(\sigma_i^*(x_i))_{i, x_i}$ and reward functions $(\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{l-1}}))_{i, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{l-1}}}$ such that Conditions (49)–(53) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

As in Lemma 14, players $-i$ communicate their history profile in \mathbb{L}_{l-1} , χ_{-i} . Since \mathbb{L}_{l-1} is random and is not revealed until main phase l is over, by giving a reward based on the history profile in \mathbb{L}_{l-1} , player i can be made indifferent over actions after another player “confesses” that she deviated in or before main phase l .

APPENDIX J: EQUILIBRIUM STRATEGIES: REMAINING DETAILS

We now complete the construction of the equilibrium strategies $(\sigma_i^*(x_i))_{i \in I}$ in sub-block $0, \dots, L$. From now on, we abbreviate “the verified communication protocol with repetition T_0 ” to simply “the communication protocol.” Recall the different phases of each sub-block defined in Section E. We let λ represent a generic phase. That is,

$$\lambda \in \{0 \times (\{\text{jam}\} \cup I \cup (I \times \{\text{con}\}))\} \cup \{ \{1, \dots, L\} \times \{\text{main}\} \cup I \cup I^2 \cup (I \times \{\text{con}\}) \}.$$

In this notation, the first coordinate of λ is l throughout sub-block $l \in \{0, \dots, L\}$. The second coordinate of λ is (i) jam for the jamming coordination phase (for $l = 0$), (ii) $i \in I$ for phase (l, i) (for $l \geq 0$), (iii) (i, con) for phase (l, i, con) (for $l \geq 0$), (iv) main for main phase l (for $l \geq 1$), or (v) (i, n) for phase (l, i, n) (for $l \geq 1$).

For $l \in \{0, \dots, L\}$ we write $\lambda \leq l$ (resp., $\lambda < l$) if the first coordinate of λ is $\leq l$ (resp., $< l$), and similarly for $\lambda \geq l$ and $\lambda > l$. Similarly, for two phases λ and λ' , we say $\lambda \leq \lambda'$ if and only if phase λ precedes or equals phase λ' .

Given λ , let h_i^λ be player i 's history $(a_{i,t}, \omega_{i,t})_{t \in \mathbb{T}(\lambda)}$ within phase λ . Let $h_i^{<\lambda}$ and $h_i^{\leq\lambda}$ be player i 's history at the beginning and the end of phase λ , respectively. Define $h^{<\lambda}$, $h^{\leq\lambda}$, $h_{-i}^{<\lambda}$, and $h_{-i}^{\leq\lambda}$ similarly. We now define equilibrium strategies in each phase.

J.1. Sub-Block 0

J.1.1. Jamming Coordination Phase

At the beginning of the block, player i randomly selects a period $t_i(l) \in \mathbb{T}(\text{main}(l))$ for each $l = 1, \dots, L$. This is encoded in \mathbb{L}_i as defined in Section I.1.

Then the jamming coordination protocol is played in phase $(0, \text{jam})$. Denote player i 's protocol history by $h_i^{(0, \text{jam})} = (a_{i,t}, \omega_{i,t})_{t=1}^2$. Recall from Section D.4 that $\zeta_i(h_i^{(0, \text{jam})}) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$; otherwise, $\zeta_i(h_i^{(0, \text{jam})}) = \text{reg}$. In subsequent communication protocols, let $i \in \mathcal{I}_{\text{jam}}$ if and only if $\zeta_i(h_i^{(0, \text{jam})}) = \text{jam}$.

J.1.2. Initial Communication Phase

For each $i \in I$, in phase $(0, i)$, player i sends x_i by the communication protocol. As a result, for each $j \in I$, player j 's history $h_j^{(0,i)}$ in phase $(0, i)$ determines an inference $x_i(j) \in \{G, B, 0\}$ and a realization $\text{susp}(h_j^{(0,i)}) \in \{0, 1\}$. After phase $(0, i)$ is concluded for all $i \in I$, the history of each player $j \in I$ determines an inferred state profile $x(j) = (x_i(j))_{i \in I} \in \{G, B, 0\}^N$. Further, for $i \in I$, given $h^{\leq(0,i)}$, let

$$I^D(h^{\leq(0,i)}) := \{j \in I : \text{susp}(h_j^\lambda) = 1 \text{ for some phase } \lambda \leq (0, i)\}$$

be the set of players who reach suspicious histories by the end of the phase $(0, i)$.²⁹

J.1.3. Contagion Phase 0

For each $i \in I$, in phase $(0, i, \text{con})$, player i communicates whether her history is suspicious. In particular, given $I^D(h^{<(0,1,\text{con})})$ (which equals $I^D(h^{\leq(0,N)})$), in phase $(0, i, \text{con})$ player i sends $m_i^{(0,i,\text{con})} = 1$ if $i \in I^D(h^{<(0,i,\text{con})})$ and $m_i^{(0,i,\text{con})} = 0$ otherwise. For each $j \in I$, player j 's history $h_j^{(0,i,\text{con})}$ determines an inference $m_i^{(0,i,\text{con})}(j) \in \{0, 1\}$ and a realization $\text{susp}(h_j^{(0,i,\text{con})}) \in \{0, 1\}$. For the history $h^{\leq(0,i,\text{con})}$ at the end of phase $(0, i, \text{con})$, let

$$I^D(h^{\leq(0,i,\text{con})}) := I^D(h^{<(0,i,\text{con})}) \cup \{j \in I : m_i^{(0,i,\text{con})}(j) = 1 \text{ or } \text{susp}(h_j^{(0,i,\text{con})}) = 1\}. \quad (54)$$

J.2. Sub-Block l

For $l = 1, \dots, L$, strategies in sub-block l depend on the variables $I^D(h^{<(l,\text{main})}) \subset I$. We have already defined $I^D(h^{<(l,\text{main})})$ for $l = 1$. As we will see, the outcome of sub-block l together with $I^D(h^{<(l,\text{main})})$ determines $I^D(h^{<(l+1,\text{main})})$. This inductively determines $I^D(h^{<(l,\text{main})})$ for each l .

J.2.1. Main Phase l

If $i \in I^D(h^{<(l,\text{main})})$, player i plays a^{\min} in every period. If $i \notin I^D(h^{<(l,\text{main})})$, then $x_j(i) \in \{G, B\}$ for all $j \in I$, and hence the action profile $\mathbf{a}^l(x(i))$ is well-defined. In this case, in every period player i plays $a_i^l(x(i))$, the i th component of action profile $\mathbf{a}^l(x(i))$. Given a history profile $h^{\leq(l,\text{main})}$ at the end of main phase l , let $I^D(h^{\leq(l,\text{main})}) = I^D(h^{<(l,\text{main})})$. That is, I^D remains constant in main phase l .

J.2.2. Communication Phase l , Part 1

For each $i \in I$, player $i-1$ sends the number $t_{i-1}(l)$ by the communication protocol in phase (l, i) . For each $j \in I$, player j 's history $h_j^{(l,i)}$ in phase (l, i) determines $t_{i-1}(l)(j) \in \mathbb{T}(l, \text{main}) \cup \{0\}$ and $\text{susp}(h_j^{(l,i)}) \in \{0, 1\}$.

J.2.3. Communication Phase l , Part 2

For each $i \in I$ and $n \in I$, player i sends the message $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)})$ by the communication protocol in phase (l, i, n) . (If $t_{n-1}(l)(i) = 0$, she sends $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)}) = (a^0, a^0)$.) For each $j \in I$, player j 's history $h_j^{(l,i,n)}$ in phase (l, i, n) determines an inference $(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j)) \in A^2 \cup \{0\}$ and a realization $\text{susp}(h_j^{(l,i,n)}) \in \{0, 1\}$.

²⁹If $\lambda = (0, \text{jam})$, define $\text{susp}(h_j^\lambda) = 0$.

After phase (l, i, n) has concluded for each $i \in I$ and $n \in I$, the history of each player $j \in I$ determines an inferred vector of outcomes $(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j))_{i \in I} \in \prod_{i \in I} (A^2 \cup \{0\})$.

Players identify deviations as follows: Given $n \in I$, $x \in \{G, B\}^N$, and $(\mathbf{a}, \boldsymbol{\omega}) \in A^{2N}$, let $\text{dev}_n^l(x, \mathbf{a}, \boldsymbol{\omega}) = 1$ denote the event that either $(a_n, \omega_n) \neq \varphi(a_{-n}, \omega_{-n})$ (Lemma 2 implies (a_n, ω_n) is infeasible given players $-n$'s history) or $a_n \neq a_n^l(x)$. In addition, let $\text{dev}_n^l(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)) = 1$ if $x(i) \notin \{G, B\}^N$ or $(\mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)) \notin A^{2N}$. Thus, $\text{dev}_n^l(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)) = 1$ means that the outcome of the communication in phases $(l, j, n)_{j \in I}$ implies that either player n deviated in the main phase, some player deviated in the communication phase, or the players failed to coordinate on some message.

Let h be a history at the end of phase (l, i) or (l, i, n) . Let $I^D(h)$ be the set of players who infer $\text{susp} = 1$ or $\text{dev} = 1$ by the end of the phase; that is, for phase (l, i) , we define

$$I^D(h) := I^D(h^{\leq(l, \text{main})}) \cup \left\{ j \in I : \max_{\lambda \leq(l, i)} \text{susp}(h_j^\lambda) = 1 \right\},$$

and for phase (l, i, n) , the set $I^D(h)$ is defined as

$$I^D(h^{\leq(l, \text{main})}) \cup \left\{ j \in I : \max \left\{ \max_{\lambda \leq(l, i, n)} \text{susp}(h_j^\lambda), \max_{(l, N, n') \leq(l, i, n)} \text{dev}_{n'}^l(x(j), \mathbf{a}_{t_{n'-1}(l)}(j), \boldsymbol{\omega}_{t_{n'-1}(l)}(j)) \right\} = 1 \right\}.$$

J.2.4. Contagion Phase l

For each $i \in I$, in phase (l, i, con) , player i sends whether $i \in I^D(h^{\leq(l, i, \text{con})})$, as in phase $(0, i, \text{con})$. We define $I^D(h^{\leq(l, i, \text{con})})$ as in phase $(0, i, \text{con})$.

Finally, for a general h , let $I_{-i}^D(h_{-i}) = I^D(h) \setminus \{i\}$. Note that I_{-i}^D is a function of players $-i$'s histories only, since whether $j \in I^D(h)$ is determined by h_j .

APPENDIX K: REWARD FUNCTION

This section constructs the reward function (ignoring for the moment the jamming coordination phase, which is addressed in Lemma 19).

K.1. Statistics Used to Construct the Reward Functions

We first define some statistics, $(\theta_i)_{i \in I}$. For phase $(0, \text{jam})$, since Lemma 2 implies that $h_{-i}^{(0, \text{jam})}$ uniquely identifies $h_i^{(0, \text{jam})}$, we can equally view $(\zeta_n)_{n \in I}$ as a function of $h_{-i}^{(0, \text{jam})}$, denoted by $\zeta(h_{-i}^{(0, \text{jam})})$. Let $\theta_i(h_{-i}^{(0, \text{jam})}) = R$ if $\zeta_i(h_{-i}^{(0, \text{jam})}) = \text{reg}$ and $\theta_i(h_{-i}^{(0, \text{jam})}) = E$ if $\zeta_i(h_{-i}^{(0, \text{jam})}) = \text{jam}$. By Lemma 14, player i 's reward function can be conditioned on $\zeta(h_{-j}^{(0, \text{jam})})$ and $\theta_i(h_{-j}^{(0, \text{jam})})$.

For non-main phases $\lambda > (0, \text{jam})$, players follow the verified communication module. Define $\theta_j(h_{-j}^\lambda, \zeta(h_{-j}^{(0, \text{jam})})) \in \{E, R\}$ as in Section H.1. Given the history $h^{\leq \lambda}$ at the end of phase λ , define $\theta_j(h_{-j}^{\leq \lambda}) = E$ if there exists a phase $\lambda' \leq \lambda$ such that $\theta_j(h_{-j}^{\lambda'}, \zeta(h_{-j}^{(0, \text{jam})})) = E$. (If $\lambda = (0, \text{jam})$, define $\theta_j(h_{-j}^\lambda, \zeta(h_{-j}^{\text{jam}})) = \theta_i(h_{-j}^{(0, \text{jam})})$.) Otherwise, define $\theta_j(h_{-j}^{\leq \lambda}) = R$.

For main phase (l, main) , let $\theta_j(h_{-j}^{\leq(l, \text{main})}) = \theta_j(h_{-j}^{\leq(l, \text{main})})$. That is, θ_j remains constant.

We make some immediate observations. For each player $i \in I$, regardless of her strategy, either all her opponents successfully infer the state x , or they all become suspicious,

or $\theta_i(h_{-i}) = E$. In addition, if some player became suspicious in one sub-block, then either everyone becomes suspicious or $\theta_i(h_{-i}) = E$ in the next sub-block. Finally, a deviation by player i from $a_i(x(i))$ in period $t_{i-1}(l)$ is detected for sure.

LEMMA 16: For any $i \in I$, $x \in \{G, B\}$, $\sigma_i \in \Sigma_i$, $l \in \{1, \dots, L\}$, $l \leq \lambda < l + 1$, and $(\sigma_i, \sigma_{-i}^*(x_{-i}))$ -consistent history $h^{<\lambda}$ at the beginning of phase λ , the following claims hold:

1. Either (i) $x(n) = x(i - 1) \forall n \in I$ with $x_j(n) = x_j$ for each $j \neq i$, (ii) $I_{-i}^D(h_{-i}^{<\lambda}) = I \setminus \{i\}$, or (iii) $\theta_i(h_{-i}^{<\lambda}) = E$.
2. If $I_{-i}^D(h_{-i}^{<(\tilde{l}, \text{main})}) \neq \emptyset$ for some $\tilde{l} \leq l - 1$, then either $I_{-i}^D(h_{-i}^{<\lambda}) = I \setminus \{i\}$ or $\theta_i(h_{-i}^{<\lambda}) = E$.
3. If $a_{i, t_{i-1}(l)} \neq a_i(x(i))$, then either $I_{-i}^D(h_{-i}^{<(\tilde{l}+1, \text{main})}) = I \setminus \{i\}$ or $\theta_i(h_{-i}^{<(\tilde{l}+1, \text{main})}) = E$.

PROOF: Claims 1 and 2: By Claims 1 and 2 of Lemma 11, either (i) $x(n) = \hat{x} \in \{G, B\}^N \forall n \in I$ with $\hat{x}_j = x_j$ for each $j \neq i$, (ii) $\text{susp}_n(h_n^{(0,j)}) = 1$ for some $n \neq i$ and $j \in I$, or (iii) $\theta_i(h_{-i}^{(0,j)}, \zeta(h_{-i}^{(0,j), \text{jam}})) = E$ for some $j \in I$. By the same claim applied to the contagion phase, if $I_{-i}^D(h_{-i}^{<(\tilde{l}, \text{main})}) \neq \emptyset$ for some $\tilde{l} \leq l - 1$, then $I_{-i}^D(h_{-i}) = I \setminus \{i\}$ or $\theta_i(h_{-i}) = E$ at the end of contagion phase \tilde{l} .

Claim 3: Suppose $a_{i, t_{i-1}(l)} \neq a_i(x(i))$. By Claim 1, either $a_{i, t_{i-1}(l)} \neq a_i(x(i - 1))$, $I_{-i}^D(h_{-i}^{<(\tilde{l}, \text{main})}) = I \setminus \{i\}$, or $\theta_i(h_{-i}^{<(\tilde{l}, \text{main})}) = E$. If $a_{i, t_{i-1}(l)} \neq a_i(x(i - 1))$, then by Claim 1 of Lemma 11, at the beginning of contagion phase l , either (i) $\text{dev}_i^l(x(i - 1), \mathbf{a}_{t_{i-1}(l)}(i - 1), \boldsymbol{\omega}_{t_{i-1}(l)}(i - 1)) = 1$, (ii) $\text{susp}_n(h_n^{\tilde{\lambda}}) = 1$ for some $n \neq i$ and $\tilde{\lambda} \in (l, i) \cup \{(l, n'), i\}_{n' \in I}$, or (iii) $\theta_i(h_{-i}) = E$. Since the former two conditions imply $I_{-i}^D(h_{-i}) \neq \emptyset$ at the beginning of contagion phase l , we have $I_{-i}^D(h_{-i}^{<(\tilde{l}+1, \text{main})}) = I \setminus \{i\}$ or $\theta_i(h_{-i}^{<(\tilde{l}+1, \text{main})}) = E$ as a result of contagion phase l by Claim 1 of Lemma 11. Q.E.D.

K.2. Construction of the Reward Function

Let $u^G = \min_{(a, a') \in A^2} u(a, a')$ and $u^B = \max_{(a, a') \in A^2} u(a, a')$. By (5), for all $i \in I$, we have

$$\max\{v_i(G), u^B\} - \min\{u^G, v_i(B)\} \leq 2\bar{u}. \quad (55)$$

Recall that, by Lemma 2, the history (a_{-i}, ω_{-i}) perfectly identifies \mathbf{a} . So, we define $\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i}) = u^{x_{i-1}} - \hat{u}_i(\mathbf{a})$, $\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}) = v_i(x_{i-1}) - \hat{u}_i(\mathbf{a})$, and $\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} | \alpha^{\min}) = v_i(x_{i-1}) - u(a_i, \alpha^{\min})$. Given this, for each $\mathbf{a} \in A^N$, we have

$$\begin{aligned} \mathbb{E}[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) | \mathbf{a}] &= u^{x_{i-1}}, \\ \mathbb{E}[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}) | \mathbf{a}] &= v_i(x_{i-1}), \\ \mathbb{E}[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} | \alpha^{\min}) | a_i, \alpha_{-i}^{\min}] &= v_i(x_{i-1}). \end{aligned} \quad (56)$$

Moreover, since $u^{x_{i-1}}$ and $v_i(x_{i-1})$ are feasible payoffs,

$$\begin{aligned} \text{sign}(x_{i-1}) \pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i}) &\geq 0, \\ \max_{x_{i-1}, a_{-i}, \omega_{-i}} \max\{|\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i})|, |\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})|, \\ &\quad |\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} | \alpha^{\min})|\} \leq 2\bar{u}. \end{aligned} \quad (57)$$

Moreover, letting $\varphi_A(a_{-i}, \omega_{-i})$ be the unique action $a_i \in A$ such that $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$ for some $\omega_i \in A$, we have, by (5),

$$\begin{aligned} & \text{sign}(x_{i-1}) \frac{1}{K_v} \sum_{k=1}^{K_v} \pi_i^{v_i}(a_{-i}^k(x), \omega_{-i,k}) \\ & \geq 0 \quad \text{if } \varphi_A(a_{-i}^k(x), \omega_{-i,k}) = a_i^k(x) \quad \forall k \in \{1, \dots, K_v\}, \\ & 2\bar{u} \geq \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} | \alpha^{\min}) \\ & \geq 0 \quad \text{for all } (x_{i-1}, a_{-i}, \omega_{-i}). \end{aligned} \quad (58)$$

The reward function is the sum of rewards for the main phases, π_i^{main} , and rewards for the communication and contagion phases, $\pi_i^{\text{non-main}}$. Define

$$\pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}) = \mathbf{1}_{\{\xi_i(h_{-i}^{(0,\text{jam})}) = \text{reg}\}} \sum_{t \in \mathbb{T}'} \pi_{i,t}(h_{-i}^{\mathbb{T}'}), \quad (59)$$

where $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ is the reward for the verified communication module in (42). Next, define

$$\pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) = \sum_{l=1}^L \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}),$$

where, for each l , we define

$$\begin{aligned} & \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \\ & = \sum_{t \in \mathbb{T}(\text{main}(l))} \mathbf{1}_{\{t_{i-1}(l)=t\}} (T_0)^3 (\mathbf{1}_{\{\theta_i(h_{-i}^{<(l,\text{main})})=E\}} \pi_i^{\theta=E}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ & \quad + \mathbf{1}_{\{\theta_i(h_{-i}^{<(l,\text{main})})=R\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq I \setminus \{i\}\}} \pi_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ & \quad + \mathbf{1}_{\{\theta_i(h_{-i}^{<(l,\text{main})})=R\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \setminus \{i\}\}} \pi_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t} | \alpha^{\min}) \\ & \quad - \mathbf{1}_{\{\theta_i(h_{-i}^{<(l,\text{main})})=R\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset\}} \mathbf{1}_{\{x_{i-1}=G\}} 2\bar{u}). \end{aligned} \quad (60)$$

In total, the reward function following the jamming coordination phase is defined as

$$\pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) = \pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) + \pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}).$$

Note that we have

$$|\pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})| \leq 4\bar{u}L(T_0)^3 + |\mathbb{T}'| \leq 4\bar{u}T^*. \quad (61)$$

APPENDIX L: REDUCTION LEMMA: PHASE (0, jam)

L.1. Jamming Coordination Module

We consider payoffs and rewards for the jamming coordination protocol. For each $i \in I$, payoff functions take the form

$$\sum_{t=1}^2 \pi_{i,t}^{\text{indiff}}(h_{-i}) + w_i(h). \quad (62)$$

Again, as in (30), we ignore player i 's instantaneous payoffs.

We say that *the premise for jamming coordination with magnitude K is satisfied* if there exist $K \geq 1$ and $(v_i(\mathcal{I}_{\text{jam}}))_{\mathcal{I}_{\text{jam}} \subset I} \in \mathbb{R}^{2^N}$ satisfying the following conditions:

1. $w_i(h) = v_i(\mathcal{I}_{\text{jam}})$ for every history h such that $\mathcal{I}_{\text{jam}} = \{n \in I : \zeta_n(h_n) = \text{jam}\}$.
2. $v_i(\mathcal{I}_{\text{jam}}) = v_i(\widetilde{\mathcal{I}}_{\text{jam}})$ for all \mathcal{I}_{jam} and $\widetilde{\mathcal{I}}_{\text{jam}}$ such that $i \in \mathcal{I}_{\text{jam}} \cap \widetilde{\mathcal{I}}_{\text{jam}}$.
3. For \mathcal{I}_{jam} such that $i \notin \mathcal{I}_{\text{jam}}$, the range of $v_i(\mathcal{I}_{\text{jam}})$ is at most K :

$$\max_{i \in I, \mathcal{I}_{\text{jam}}, \widetilde{\mathcal{I}}_{\text{jam}} : i \notin \mathcal{I}_{\text{jam}}, i \notin \widetilde{\mathcal{I}}_{\text{jam}}} |v_i(\mathcal{I}_{\text{jam}}) - v_i(\widetilde{\mathcal{I}}_{\text{jam}})| \leq K. \quad (63)$$

LEMMA 17: *Take $(w_i(h))_{i \in I}$ and K such that the premise for jamming coordination with magnitude K is satisfied. There exists a function $(\pi_{i,t}^{\text{indiff}}(h_{-i}))_{i \in \{1,2\}}$ such that (i) we have $\max_{h_{-i}} |\sum_{t=1}^2 \pi_{i,t}^{\text{indiff}}(h_{-i})| \leq 2K$ and (ii) with payoffs (62), the jamming coordination protocol is a sequential equilibrium.*

L.2. Equilibrium Condition: Final Statement

The main remaining step in the proof is verifying the equilibrium conditions given each history in the jamming coordination phase. It suffices to establish incentive compatibility and promise keeping, as self-generation is addressed in the proof of Lemma 19.

LEMMA 18: *For all $i \in I$, all $x \in \{G, B\}^N$, and all jamming coordination phase histories $h_i^{(0,\text{jam})}$, we have:*

1. [Incentive Compatibility] For each $t \geq 3$ and $h_i^{t-1} \in H_i(x_{-i})$,

$$\sigma_i^*(x_i) \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}^*(x_{-i}))} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | h_i^{(0,\text{jam})}, h_i^{t-1} \right]. \quad (64)$$

2. [Promise Keeping after $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$] If $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$ and

$$v_i(x_{-i}, \mathcal{I}_{\text{jam}} \setminus \{i\}) := \frac{1}{L(T_0)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | \mathcal{I}_{\text{jam}} \right], \quad (65)$$

then, for all $\mathcal{I}_{\text{jam}} \setminus \{i\}, \widetilde{\mathcal{I}}_{\text{jam}} \setminus \{i\} \subset I \setminus \{i\}$, we have

$$v_i(x_{-i}, \mathcal{I}_{\text{jam}} \setminus \{i\}) \begin{cases} \geq v_i(x_{i-1}) - \varepsilon^* & \text{if } x_{i-1} = G, \\ \leq v_i(x_{i-1}) + \varepsilon^* & \text{if } x_{i-1} = B, \end{cases} \quad \text{and} \quad (66)$$

$$|v_i(x_{-i}, \mathcal{I}_{\text{jam}} \setminus \{i\}) - v_i(x_{-i}, \widetilde{\mathcal{I}}_{\text{jam}} \setminus \{i\})| \leq N \#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}}) 2\bar{u}T^*, \quad (67)$$

where $\#_{\text{half}} = 2Nb(A^{4b(2)}) + 2Nb(A^{4b(2)}) + L(2Nb(A^{4b(T_0^3)}) + 2N^2b(A^{4b(LA^2)}) + 2N^2 \times b(A^{4b(2)}))$ is the number of half-intervals in sub-blocks from 0 to L .

The theorem now follows easily from Lemmas 15, 17, and 18.

LEMMA 19: *Suppose Lemma 18 holds. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

PROOF: By definition of $\sigma^*(x)$ in Section J, (50) holds. Hence, putting together Lemmas 6–15, it suffices to construct reward functions π_i^* that, together with $\sigma^*(x)$, satisfy equations (49) and (51)–(53). We first construct the reward for the jamming coordination phase, denoted $\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0, \text{jam})})$, using Lemma 17. So, we verify the premise for jamming coordination.

The probability that any jamming player other than i plays JAM during sub-blocks $0, \dots, L$ is at most $N\#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}})$. (i) The range of $\pi_i^{\geq 3}$ is at most $4\bar{u}T^*$ (by (61)), (ii) once a jamming player takes a jamming strategy, the reward is bounded by $2\bar{u}T^*$, and (iii) per-period payoffs are bounded by $[-\bar{u}, \bar{u}]$. Hence, we have

$$\max_{i \in I, \mathcal{T}_{\text{jam}}, \widetilde{\mathcal{T}}_{\text{jam}}: i \notin \mathcal{T}_{\text{jam}}, i \notin \widetilde{\mathcal{T}}_{\text{jam}}} |v_i(\mathcal{T}_{\text{jam}}) - v_i(\widetilde{\mathcal{T}}_{\text{jam}})| \leq N\#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}}) 6\bar{u}T^*.$$

Hence, by Lemma 17, there exists $\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0, \text{jam})})$ such that the jamming coordination protocol is incentive compatible and

$$\max_{x_{-i}, h_{-i}^{(0, \text{jam})}} |\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0, \text{jam})})| \leq N\#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}}) 12\bar{u}T^*. \quad (68)$$

We now define the total reward function as $\pi_i(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) = \pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0, \text{jam})}) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})$. It remains to verify (49)–(53).

First, the bound (51) follows from (61) and (68), since (17) implies that $\#_{\text{half}} \leq (T_0)^{0.1}$ and $(T_0)^{0.1} \exp(-(T_0)^{\frac{1}{2}}) 12\bar{u}T^* \leq \varepsilon^* T^*$.

Note that, by the construction of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ in (42), for all $x \in \{G, B\}^N$ and $h_{-i}^{\mathbb{T}'}$, we have

$$\text{sign}(x_{i-1}) \pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}) \geq -|\mathbb{T}'|. \quad (69)$$

To derive a similar equation for π_i^{main} , if $\theta_i(h_{-i}^{<(l, \text{main})}) = E$, then (57) implies that π_i^{main} is non-positive if $x_{i-1} = G$ and non-negative if $x_{i-1} = B$. If $\theta_i(h_{-i}^{<(l, \text{main})}) = R$ and $I_{-i}^D(h_{-i}^{<(l, \text{main})}) = I \setminus \{i\}$, then the same conclusion holds by (58).

We now show that, in all other cases, we have $\text{sign}(x_{i-1}) \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) < 0$ in at most $(1 + K_v)$ sub-blocks. To see this, note that if $I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq \emptyset$, then Lemma 16 implies that, as a result of contagion phase $l + 1$, either $I_{-i}^D(h_{-i}^{<((l+1), \text{main})}) = I \setminus \{i\}$ or $\theta_i(h_{-i}^{<((l+1), \text{main})}) = E$ (regardless of player i 's behavior). If both $\theta_i(h_{-i}^{<(l, \text{main})}) = R$ and $I_{-i}^D(h_{-i}^{<(l, \text{main})}) = \emptyset$, then Lemma 16 implies that, for each $n \in I$, we have $x(n) = \hat{x}$ for some $\hat{x} \in \{G, B\}^N$ with $\hat{x}_{i-1} = x_{i-1}$. Hence, by (58), we have $\text{sign}(x_{i-1}) \frac{1}{K_v} \sum_{k=1}^{K_v} \pi_i^{v_i}(x_{i-1}, a_{-i, t_{i-1}(l)}, \omega_{-i, t_{i-1}(l)}) \geq 0$ as long as $a_i^l(x(i-1)) = \varphi_A(a_{-i, t_{i-1}(l)}, \omega_{-i, t_{i-1}(l)}) = a_{i, t_{i-1}(l)}$. Moreover, if $a_{i, t_{i-1}(l)} \neq a_i^l(x(i-1))$, then Lemma 16 implies that either $I_{-i}^D(h_{-i}^{<((l+1), \text{main})}) = I \setminus \{i\}$ or $\theta_i(h_{-i}^{<((l+1), \text{main})}) = E$.

It follows that there exists a subset $\mathcal{L} \subset \{1, \dots, L\}$ with $|\mathcal{L}| \geq L - (K_v + 1)$ such that $\sum_{l \in \mathcal{L}} \text{sign}(x_{i-1}) \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \geq 0$. Since $\pi_i^{v_i}(\cdot | \alpha^{\min})$ and $\pi_i^{v_i}(\cdot)$ are bounded by (57), we have

$$\begin{aligned} \text{sign}(x_{i-1}) \pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) &\geq -2\bar{u}(1 + K_v)(T_0)^3 \\ &\geq_{\text{by (6)}} -\varepsilon^* L(T_0)^3 \quad \forall x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}. \end{aligned} \quad (70)$$

Now, by (68), (69), and (70), for all x_{-i} , $h_{-i}^{\mathbb{T}'}$, $h_{-i}^{\mathbb{L}_{i-1}}$, we have

$$\begin{aligned} & \text{sign}(x_{-i-1}) \left(\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0, \text{jam})}) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \right) \\ & \geq -N \#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}}) 12\bar{u}T^* - |\mathbb{T}'| - \varepsilon^* L(T_0)^3. \end{aligned}$$

By (17), $\#_{\text{half}} \leq (T_0)^{0.1}$ and $N(T_0)^{0.1} \exp(-(T_0)^{\frac{1}{2}}) 12\bar{u}T^* + |\mathbb{T}'| + \varepsilon^* L(T_0)^3 \leq 2\varepsilon^* T^*$. Combining these inequalities yields (49).

Next, Lemma 18 implies that there is no profitable deviation from $\sigma_i^*(x_i)$ after the jamming coordination phase. Given this, Lemma 17 implies that there is also no profitable deviation from $\sigma_i^*(x_i)$ during the jamming coordination phase. Hence, (52) holds.

Finally, since (i) $\mathcal{I}_{\text{jam}} \neq \emptyset$ with probability no more than $1 - (1 - \exp(-(T_0)^{\frac{1}{3}}))^{2N}$, (ii) $\pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})$ is bounded by $4\bar{u}T^*$, (iii) once a jamming player takes a jamming strategy, the reward is bounded by $2\bar{u}T^*$, and (iv) $\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t)$ is bounded by $2\bar{u}L(T_0)^3$, the total payoff satisfies

$$\begin{aligned} & \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \right] - v_i(x_{-i}, \emptyset) \\ & \leq (1 - (1 - \exp(-(T_0)^{\frac{1}{3}}))^{2N}) 6\bar{u}T^*. \end{aligned}$$

Since (17) implies $(1 - (1 - \exp(-(T_0)^{\frac{1}{3}}))^{2N}) 6\bar{u}T^* \leq \varepsilon^* L(T_0)^3$, this inequality together with (66) implies (53). *Q.E.D.*

APPENDIX M: PROOF OF LEMMA 18

M.1. Notation

In this section, for any strategy σ_i and history h , we assume h to be $(\sigma_i, \sigma_{-i}^*(x_{-i}))$ -consistent.

For $l \in \{0, \dots, L\}$ and $l \leq \lambda < l+1$, let $\mathbb{L}^{\leq \lambda} := (t_n(\tilde{l}))_{n \in I, \tilde{l} \leq l}$ be the randomizations that have been realized in phase λ . Similarly, let $\mathbb{L}^{< \lambda} := (t_n(\tilde{l}))_{n \in I, \tilde{l} \leq l}$ if $l < \lambda$ and $\mathbb{L}^{< \lambda} := (t_n(\tilde{l}))_{n \in I, \tilde{l} < l}$ if $\lambda = (l, \text{main})$. For each λ , at the end of phase λ , if player i knew $\mathbb{L}^{\leq \lambda}$ and $h^{\leq \lambda}$, she could attain a continuation payoff of

$$\begin{aligned} w_i(x_{-i}, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) &:= \max_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}^*(x_{-i}))} \left[\sum_{\tilde{l}=l+1}^L \sum_{t \in \mathbb{T}(\tilde{l}, \text{main})} \hat{u}_i(\mathbf{a}_t) + \sum_{\tilde{l}=l+1}^L \pi_i^{\text{main}}(\tilde{l}, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \right. \\ & \quad \left. + \mathbf{1}_{\{\zeta_i(h_{-i}^{(0, \text{jam})}) = \text{reg}\}} \sum_{t \in \mathbb{T}': t \succsim \lambda} \pi_{i,t}(h_{-i}^{\mathbb{T}'}) | \mathbb{L}^{\leq \lambda}, h^{\leq \lambda} \right], \end{aligned} \quad (71)$$

where $t \succsim \lambda$ means period t follows or is within phase λ . On the other hand, let $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$ denote player i 's continuation payoff from strategy $\sigma_i^*(x_i)$. We will show that, for any phase λ and history $(\mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$, $w_i(x_{-i}, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$.

M.2. Equilibrium Properties

First, we show that there is no instantaneous deviation gain from $\sigma_i^*(x_i)$:

LEMMA 20: For any $i \in I$, $x \in \{G, B\}^N$, $\sigma_i \in \Sigma_i$, $l \in \{1, \dots, L\}$, $\mathbb{L}^{<(l, \text{main})}$, and history $h^{<(l, \text{main})}$ at the beginning of phase (l, main) ,

$$\begin{aligned} & \max_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}^*(x_{-i}))} \left[\sum_{t \in \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \right. \\ & \quad \left. + \mathbf{1}_{\{\zeta_i(h_{-i}^{(0, \text{jam})}) = \text{reg}\}} \sum_{t \in \mathbb{T}': t \text{ in sub-block } l} \pi_{i,t}(h_{-i}^{\mathbb{T}'}) | \mathbb{L}^{<(l, \text{main})}, h^{<(l, \text{main})} \right] \\ &= \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | \mathbb{L}^{<(l, \text{main})}, h^{<(l, \text{main})} \right] \\ &= \begin{cases} (T_0)^3 (v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq \emptyset\}} 2\bar{u}) & \text{if } \theta_i(h_{-i}^{<(l, \text{main})}) = R, \\ (T_0)^3 u^{x_{i-1}} & \text{if } \theta_i(h_{-i}^{<(l, \text{main})}) = E. \end{cases} \end{aligned}$$

PROOF: Playing $\sigma_i^*(x_i)$ yields the highest value of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$: 0. Hence, we focus on $\sum_{t \in \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t)$ and π_i^{main} . If $\theta_i(h_{-i}^{<(l, \text{main})}) = R$, then, by (60), the reward function satisfies

$$\begin{aligned} & \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \\ &= (T_0)^3 \times \begin{cases} \pi_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq \emptyset\}} 2\bar{u} & \text{if } I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq I \setminus \{i\}, \\ \pi_i^{v_i}(x_{i-1,t}, a_{-i,t}, \omega_{-i,t} | \alpha^{\min}) - \mathbf{1}_{\{x_{i-1}=G\}} 2\bar{u} & \text{if } I_{-i}^D(h_{-i}^{<(l, \text{main})}) = I \setminus \{i\}, \end{cases} \end{aligned}$$

for $t = t_{i-1}(l)$ (and 0 for other t 's). For each $t \in \mathbb{T}(\text{main}(l))$ and $a_{i,t}$, the random variable $t_{i-1}(l)$ equals t with probability $(T_0)^{-3}$ (recall that $\mathbb{L}^{<(l, \text{main})}$ does not include $t_{i-1}(l)$ and the condition (50) holds), and players $-i$ play $a_{-i}(x(i-1))$ when $I_{-i}^D(h_{-i}^{<(l, \text{main})}) = \emptyset$ (by Lemma 16) and play α^{\min} when $I_{-i}^D(h_{-i}^{<(l, \text{main})}) = I \setminus \{i\}$. Hence, the per-period expected payoff is $v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\{I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq \emptyset\}} 2\bar{u}$, by (56). If instead $\theta_i(h_{-i}^{<(l, \text{main})}) = E$, then the result follows from (56) and (60). Q.E.D.

Second, for each phase λ , if $i \in I^D(h^\lambda)$, then $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.

LEMMA 21: For any $i \in I$, λ , and history $h^{<\lambda}$ at the beginning of phase λ , if $i \in I^D(h^{<\lambda})$, then $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.

PROOF: By definition, $i \in I^D(h^{<\lambda})$ only if $\text{susp}_i(h_i) = 1$ or $\text{dev}_n^l(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \omega_{t_{n-1}(l)}(i)) = 1$ for some $n \in I$ as the result of communication phases preceding λ . We show that both these cases imply $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$. In each communication phase, by Claims 1 and 2 of Lemma 11, if $\text{susp}_i(h_i) = 1$, then $\theta_i(h_{-i}^{<\lambda}) = E$ for each subsequent phase. In addition, we have, either all players infer the same message, $\text{susp}_n(h_n) = 1$ for some $n \neq i$, or $\theta_i(h_{-i}) = E$. If $\text{dev}_n^l(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \omega_{t_{n-1}(l)}(i)) = 1$ for some $n \in I$, then each of these three cases implies either $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$. Q.E.D.

Third, the distribution of $\theta_i(h_{-i})$ is independent of the history in previous phases, and $\theta_i(h_{-i}) = E$ is rare.

LEMMA 22: For any $i \in I$, λ , and $l \geq \lambda$, there exists $p(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), l)$ such that, for any $x \in \{G, B\}^N$, $\mathbb{L}^{\leq \lambda}$, and history $h^{\leq \lambda}$ at the end of phase λ , we have

$$\Pr^{\sigma^*(x)}(\theta_i(h_{-i}^{\leq(l, \text{main})}) = E | \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), l).$$

Moreover, for $\theta_i(h_{-i}^{\leq \lambda}) = R$, we have $p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), l) \leq \exp(-(T_0)^{\frac{1}{3}})$.

PROOF: By Claim 5 of Lemma 11, the distribution of θ_i in each communication phase is determined by $\mathcal{I}_{\text{jam}} \setminus \{i\}$, independent of the message sent. In addition, since $\theta_i(h_{-i}^{\leq \lambda}) = R$ implies $\zeta_i(h_{-i}^{(0, \text{jam})}) = \text{reg}$, in each communication phase the probability of $\theta_i(h_{-i}, \zeta_i(h_{-i}^{(0, \text{jam})})) = E$ is at most $\#_{\text{half}}(\exp(-(T_0)^{\frac{1}{2}}) + \exp(-\bar{\varepsilon}T_0))$ (by Claim 4 of Lemma 11). By (17), this probability is less than $\exp(-(T_0)^{\frac{1}{3}})$. Q.E.D.

M.3. Verification of Promise Keeping and Incentive Compatibility

In equilibrium, by Lemma 20, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq \lambda}$, and $h^{\leq \lambda}$, we have

$$\begin{aligned} v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) &= \sum_{\tilde{l} \geq l+1} (T_0)^3 \{ p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l}) u^{x_{i-1}} \\ &\quad + (1 - p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l})) (v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\{I_{-i}^{D_i}(h_{-i}^{\leq(\tilde{l}, \text{main})}) \neq \emptyset\}} 2\bar{u}) \}. \end{aligned} \quad (72)$$

By Claim 3 of Lemma 11, the event $I_{-i}^D(h_{-i}^{\leq \lambda}) \neq \emptyset$ implies $\theta_i(h_{-i}^{\leq \lambda}) = E$ on path. Since (17) implies $\#_{\text{half}} \leq (T_0)^{0.1}$ and $(T_0)^{0.1} \exp(-(T_0)^{\frac{1}{3}}) 3\bar{u} \leq \varepsilon^* L (T_0)^3$, with $\lambda = (0, \text{jam})$, by Lemma 22, we have (65)–(67). It remains to verify (64). This involves verifying the premise for verified communication, which requires a lower bound on the probability of JAM:

LEMMA 23: For any $i \in I$, $x_{-i} \in \{G, B\}^{N-1}$, \mathbb{L} , $\sigma_i \in \Sigma_i$, h_i^t , and history $h^{3:t}$ from period 3 to t , we have

$$\Pr(\zeta_j(h_j^{(0, \text{jam})}) = \text{jam} \forall j \neq i | \mathbb{L}, h^{3:t}, h_i^t) \geq \exp(-(T_0)^{\frac{1}{2}}). \quad (73)$$

PROOF: By iterated expectations, it suffices to prove the lemma for $t = T^*$. For any jamming coordination phase history $h_i^{(0, \text{jam})}$, let $p_i(h_i^{(0, \text{jam})})$ denote the conditional probability that each player $j \neq i$ observes a^1 during the jamming coordination phase. By (16), we have $p_i(h_i^{(0, \text{jam})}) \geq \bar{\varepsilon} \exp(-(N-2)T^{\frac{1}{3}})$. It remains to account for updating from $h^{3:t}$ between periods 3 and T^* (recall that the jamming coordination phase ends in period 2).

Suppose player i could perfectly observe whether her opponents play REG or JAM in every half-interval. (Note that the other information in $(\mathbb{L}, h^{3:t})$ does not update the probability of $\zeta_j(h_j^{(0, \text{jam})})$.) Then $\Pr(\zeta_j(h_j^{(0, \text{jam})}) = \text{jam} \forall j \neq i | h_i^{T^*})$ would be minimized when REG is always played. As the probability that REG is always played is at least $1 - N\#_{\text{half}} \exp(-T^{\frac{1}{2}})$ (conditional on any realization of $(\zeta_j(h_j^{(0, \text{jam})}))_{j \in I}$), we have

$$\begin{aligned} \Pr^{\sigma_{-i}^*(x_{-i})}(\zeta_j(h_j^{(0, \text{jam})}) = \text{jam} \forall j \neq i | h_i^{T^*}) &\geq \frac{\bar{\varepsilon} \exp(-(N-2)(T_0)^{\frac{1}{3}}) (1 - N\#_{\text{half}} \exp(-(T_0)^{-\frac{1}{2}}))}{\bar{\varepsilon} \exp(-(N-2)(T_0)^{\frac{1}{3}}) (1 - N\#_{\text{half}} \exp(-(T_0)^{-\frac{1}{2}})) + 1} \geq_{\text{by (17)}} \exp(-(T_0)^{\frac{1}{2}}). \end{aligned} \quad \text{Q.E.D.}$$

It will also be useful to simplify equation (72). By Lemma 22, there exists a payoff $v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D)$ (where D stands for “Deviation is Detected”) such that, for each $h_{-i}^{\leq \lambda}$ with $I_{-i}^D(h_{-i}^{\leq \lambda}) \neq \emptyset$, we have $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D)$; and for each $h_{-i}^{\leq \lambda}$ with $I_{-i}^D(h_{-i}^{\leq \lambda}) = \emptyset$, we have (since $v_i(G) - 2\bar{u} \leq u^G$ and $v_i(B) \leq u^B$ by (55))

$$v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) \geq v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D). \quad (74)$$

In addition, on the equilibrium path, either $I_{-i}^D(h_{-i}^{\leq (l, \text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{\leq (l, \text{main})}) = E$. Hence, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq \lambda}$, and $h^{\leq \lambda}$, on-path payoffs are given by

$$\begin{aligned} v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) &= v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), N) \\ &:= \sum_{\tilde{l} \geq l+1} (T_0)^3 \{p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l}) u^{x_{i-1}} \\ &\quad + (1 - p_i(\mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l})) v_i(x_{i-1})\}. \end{aligned}$$

M.3.1. Proof of (64) (Incentive Compatibility)

The proof is by induction. For $\lambda \geq L$, $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = w_i(x_{-i}, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = 0$, since there is no main phase following λ and playing $\sigma_i^*(x_i)$ yields $\pi_{i,t}(h_{-i}^{\text{TV}}) = 0$. Given this observation, it suffices to establish the following claim:

Inductive hypothesis: For each x , λ , $\mathbb{L}^{\leq \lambda}$, and $h^{\leq \lambda}$, if the equilibrium continuation payoff given $(\mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$ equals $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$, then $\sigma_i^*(x_i)$ is sequentially rational given $(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$.

If $\theta_i(h_{-i}^{\leq \lambda}) = E$, then the claim follows from Lemma 20 and the fact that $\theta_i(h_{-i}^{\leq \lambda}) = E$ implies $\theta_i(h_{-i}^{\leq (l, \text{main})}) = E$ for all $l \geq \lambda$. So assume $\theta_i(h_{-i}^{\leq \lambda}) = R$.

For communication phase λ , we use v_i^E , $(v_i^{m_i})_{m_i \in M_i}$, and v_i^0 as in Section H. By Lemma 23, we have (43). Moreover, in what follows, (17) implies (38) and (44) with relevant continuation payoffs. Hence, we focus on proving the premise. Note that (74) implies, for each x , $\mathbb{L}^{\leq \lambda}$, $h^{\leq \lambda}$, $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda, R, D)$.

Contagion phase (l, i, con) : For the equilibrium message m_i (equal to 0 if $i \notin I^D(h^{\leq \lambda})$ and 1 if $i \in I^D(h^{\leq \lambda})$) and the alternative message $\hat{m}_i \in \{0, 1\} \setminus \{m_i\}$, we have

- $v_i^{m_i} \geq v_i^{\hat{m}_i} = v_i^0$ if $I_{-i}^D(h_{-i}^{\leq \lambda}) = \emptyset$ and $i \notin I^D(h^{\leq \lambda})$ (by (74)),
- $v_i^{m_i} = v_i^{\hat{m}_i} = v_i^0$ if $I_{-i}^D(h_{-i}^{\leq \lambda}) \neq \emptyset$ or $i \in I^D(h^{\leq \lambda})$, and
- $K \leq 2\bar{u}$ (as $u^{x_{i-1}}$ and $v_i(x_{i-1})$ are feasible payoffs),

since the event $\{\theta_i(h_{-i}^{\leq \lambda}) = R \text{ and } i \in I^D(h^{\leq \lambda})\}$ implies $I_{-i}^D(h_{-i}^{\leq \lambda}) \neq \emptyset$ by Lemma 21. Given $v_i^E = u^{x_{i-1}}$, the premise holds. Hence, $\sigma_i^*(x_i)$ is sequentially rational.

Contagion phase (l, j, con) with $j \neq i$: Since $v_i^{m_j} \geq v_i^0$ for all $m_j \in M_j$ by (74), the premise holds.

Communication phase (l, i, n) with $n \neq i$: In phases (l, n) and (l, j, n) with $j < i$, Claim 1 of Lemma 11 implies that either players coordinate on both $t_n(l-1)$ and $(a_{j, t_n(l-1)}, \omega_{j, t_n(l-1)})_j$, or we have $I_{-i}^D(h_{-i}^{\leq \lambda}) \neq \emptyset$ (given $\theta_i(h_{-i}^{\leq \lambda}) = R$). By the inductive hypothesis, players will follow σ^* in later phases, and therefore, by Claim 4 of Lemma 11, either players coordinate on $(a_{j, t_n(l-1)}, \omega_{j, t_n(l-1)})_{j>i}$ or $\theta_i(h_{-i}^{\leq (l+1, \text{main})}) = E$. If $\theta_i(h_{-i}^{\leq \lambda}) = E$ in some later phase, then player i 's payoff is independent of the message in the current phase. If $\theta_i(h_{-i}^{\leq \lambda}) = R$ in all later phases, we have $\theta_i(h_{-i}^{\leq (l+1, \text{main})}) = R$. Given this event, for each message $\hat{m}_i \neq (a_{i, t_n(l-1)(i)}, \omega_{i, t_n(l-1)(i)})$, coordinating on \hat{m}_i induces $\text{dev}_n = 1$. Hence, $v_i^{m_i} \geq v_i^{\hat{m}_i} = v_i^0$. Since $v_i^E = u^{x_{i-1}}$, the premise holds.

Communication phase (l, j, n) with $j \neq i$: The same as phase (l, j, con) .

Communication phase (l, i) : If $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, then $v_i^{m_i} = v_i^0$ for each $m_i \in M_i$, so the premise holds. So assume $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$.

Suppose first that $a_{i,t_{i-1}(l)} = a_i^l(x(i))$. Given $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$ and $\theta_i(h_{-i}^{<\lambda}) = R$, by Claim 1 of Lemma 11, players coordinated on $t_j(l-1)$ with $j-1 < i$. Since players will follow $\sigma^*(x)$ in later phases, Claim 4 of Lemma 11 implies that either players coordinate on the true message or $\theta_i(h_{-i}^{<(l+1, \text{main})}) = E$ in later sub-phases. Hence, for any $t \in \mathbb{T}(l, \text{main})$, as long as $t_i(l-1)(n) = t$ for each $n \in I$, we have $I_{-i}^D(h_{-i}^{<(l+1, \text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{<(l+1, \text{main})}) = E$. Therefore, for each message m_i , the continuation payoff is $v_i^{m_i} = v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda+1, R, N) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}} \setminus \{i\}, \lambda+1, R, D)$, so the premise holds.

Suppose instead $a_{i,t_{i-1}(l)} \neq a_i^l(x(i))$. Then Lemma 16 implies that $I_{-i}^D(h_{-i}^{<(l+1, \text{main})}) \neq \emptyset$ or $\theta_i(h_{-i}^{<(l+1, \text{main})}) = E$, regardless of player i 's behavior. Hence, for each message m_i , the continuation payoff is $v_i^{m_i} = v_i^0$. Again, the premise holds.

Communication phase (l, j) with $j \neq i$: The same as phase (l, j, con) .

Main phase: If $I_{-i}^D(h_{-i}^{<(l, \text{main})}) \neq \emptyset$, then the continuation payoff is independent of player i 's main phase behavior, so Lemma 20 implies the result. If $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$, then Lemma 20 ensures there is no instantaneous deviation gain. It remains to show that the continuation payoff decreases if player i deviates. Given history profile $(\mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$ at the end of main phase l , by Lemma 16, the probability that $I_{-i}^D(h_{-i}^{<(l+1, \text{main})}) \neq \emptyset$ is determined by and increasing in $\frac{|\{t \in \mathbb{T}(\text{main}(l)): a_{i,t} \neq a_i^l(x(i))\}|}{(T_0)^3}$. Since the distribution of $\theta_i(h_{-i}^{<(l+1, \text{main})})$ is independent of i 's behavior in main phase l by Lemma 22, continuation payoff is maximized by playing $a_{i,t} = a_i^l(x(i))$ for each t .

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