# THE ECONOMICS OF PARTISAN GERRYMANDERING

Anton Kolotilin
School of Economics, UNSW Business School

Alexander Wolitzky
Department of Economics, MIT

We study the problem of a partisan gerrymanderer who assigns voters to equipopulous districts to maximize his party's expected seat share. The designer faces both aggregate, district-level uncertainty (how many votes his party will receive) and idiosyncratic, voter-level uncertainty (which voters will vote for his party). *Segregate-pair districting*, where weaker districts contain one type of voter, while stronger districts contain two, is optimal for the gerrymanderer. The optimal form of segregate-pair districting depends on the designer's popularity and the relative amounts of aggregate and idiosyncratic uncertainty. When idiosyncratic uncertainty dominates, a designer with majority support pairs all voters, while a designer with minority support segregates opposing voters and pairs more favorable voters; these plans resemble uniform districting and "packing-and-cracking," respectively. When aggregate uncertainty dominates, the designer segregates moderate voters and pairs extreme voters; this "matching slices" plan has received some attention in the literature. Estimating the model using precinct-level returns from recent US House elections shows that, in practice, idiosyncratic uncertainty dominates. We discuss implications for redistricting reform, political polarization, and detecting gerrymandering. Methodologically, we exploit a formal connection between gerrymandering—partitioning voters into districts—and information design—partitioning states of the world into signals.

Keywords: Gerrymandering, pack-and-crack, segregate-pair, information design

## 1. Introduction

Legislative district boundaries are drawn by political partisans under many electoral systems (Bickerstaff, 2020). In the United States, the significance of partisan districting has grown with the rise of computer-assisted districting (Newkirk, 2017), together with intense partisan efforts to gain and exploit control of the districting process. These trends culminated in "The Great Gerrymander of 2012" (McGhee, 2020), where the Republican party's Redistricting Majority Project (REDMAP), having previously targeted state-level elections that would give Republicans control of redistricting, aggressively redistricted several states, including Michigan, Ohio, Pennsylvania, and Wisconsin. The resulting districting plans are widely viewed as contributing to the outcome of the 2012 general election, where Republican congressional candidates won a 33-seat majority in

Anton Kolotilin: akolotilin@gmail.com
Alexander Wolitzky: wolitzky@mit.com

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the House of Representatives with 49.4% of the two-party vote (McGann, Smith, Latner, and Keena, 2016). In light of these developments—along with the Supreme Court ruling in *Rucho v. Common Cause* (2019) that partisan gerrymanders are not judiciable in federal court and the continued prominence of gerrymandering in the 2020 US redistricting cycle (Rakich and Mejia, 2022)—partisan gerrymandering is likely to remain an important feature of American politics for some time.

This paper studies the problem of a partisan gerrymanderer (the "designer") who assigns voters to a large number of equipopulous districts so as to maximize his party's expected seat share.[1] This problem approximates the one facing many partisan gerrymanderers in the United States, where the constraint that districts must be equipopulous is strictly enforced.[2] In practice, gerrymanderers also face additional constraints, such as the federal requirements that districts are contiguous and do not discriminate on the basis of race, and various state-level restrictions such as "compactness" requirements, requirements to respect political sub-divisions such as county lines, requirements to represent racial or ethnic groups or other communities of interest, and so on. While these complex constraints can be important, we believe that often they are not as binding as they might seem, and also that they are more productively considered on a case-by-case basis rather than as part of a general theoretical analysis.[3] We therefore follow much of the literature (discussed below) in focusing on the simpler problem with only the equipopulation constraint.

When the designer has perfect information, the solution to this problem is well-known. If the designer's party is supported by a minority of voters of size $m < 1/2$, he "packs" $1 - 2m$ opposing voters in districts where he receives zero votes and "cracks" the remaining $2m$ voters in districts which he wins with 50% of the vote. If the designer has majority support, he can win all districts by making them identical. Thus, under perfect information, *pack-and-crack* is optimal for a designer with minority support, while *uniform districting* is optimal for a designer with majority support. We instead consider the more general and realistic case where the designer must allocate a variety of types of voters (or, more realistically, groups of voters such as census blocks or precincts) under uncertainty. The goal of this paper is to characterize optimal partisan gerrymandering in this setting and to draw implications for broader legal and political economy issues surrounding gerrymandering.

In outline, our model and results are as follows. We assume that the designer faces both aggregate, district-level uncertainty (how many votes his party will receive) and idiosyncratic, voter-level uncertainty (which voters will vote for his party). Aggregate uncertainty is parameterized by a one-dimensional aggregate shock, while voters are parameterized by a one-dimensional type that determines a voter's probability of voting for the designer's party for each value of the aggregate shock. We assume that the

[1] We hasten to add that studying this problem does not endorse gerrymandering, any more than studying monopolistic behavior endorses monopoly.

[2] In *Karcher v. Daggett* (1983), the Supreme Court rejected a districting plan in New Jersey with less than a 1% deviation from population equality, finding that "there are no *de minimus* population variations, which could practically be avoided, but which nonetheless meet the standard of Article I, Section 2 [of the U.S. Constitution] without justification."

[3] An exception is the requirement to respect county lines, which we address in Section 6.2. See Friedman and Holden (2008) for discussion of the other constraints. For example, contiguity is not as severe a constraint as it might seem, because contiguous districts can have highly irregular shapes. The title of this paper, typeset in Gerry font (https://www.uglygerry.com/), contains many examples of irregularly shaped districts.

distributions of the aggregate and idiosyncratic shocks are symmetric and unimodal with log-concave densities drawn from the same location-scale family. This assumption lets us cleanly compare the "amounts" of aggregate and idiosyncratic uncertainty.

Our first result is that optimal districting takes a *segregate-pair* form. Under segregate-pair districting, the designer creates weaker districts that contain a single voter type (which are analogous to the packed districts under pack-and-crack) and stronger districts that contain two voter types (which are analogous to the cracked districts under pack-and-crack). The class of segregate-pair plans admits a tight characterization but is rich enough to cover a variety of districting plans, including refinements of all of the main plans proposed in the prior literature. The optimality of segregate-pair districting is thus a key organizing result.

We then turn to our main substantive results, which characterize optimal districting as a function of the designer's popularity and the relative amounts of idiosyncratic and aggregate uncertainty. First, we show that if the designer has strong support from all voter types, then a *negative assortative districting* (*NAD*) plan is optimal, where extreme left and right-wing voters are paired together. Conversely, if the designer has weak support from all voter types and idiosyncratic uncertainty is larger than aggregate uncertainty, then a *segregation* plan is optimal, where each district contains only a single voter type. Second, if aggregate uncetainty is very small, optimal districting for a designer with majority support approximates NAD, while optimal districting for a designer with minority support approximates a *segregate-opponents-and-pair* (*SOP*) plan, where unfavorable voters are segregated and more favorable voters are paired in a negatively assortative manner. The former result is analogous to the optimality of uniform districting for a designer with majority support without uncertainty, because NAD plans are versions of uniform districting that pair voter types rather than pooling all types together; similarly, the latter result is analogous to the optimality of pack-and-crack districting for a designer with minority support without uncertainty, because SOP plans are versions of pack-and-crack districting that segregate unfavorable voters and pair more favorable voter types rather than pooling them. Indeed, while exactly optimal districting with small aggregate uncertainty approximates NAD or SOP, much simpler uniform districting or pack-and-crack plans are approximately optimal (for the cases of majority and minority designer support, respectively). Third, if idiosyncratic uncertainty is very small, optimal districting approximates NAD with a 50-50 voter type split in each district.[4] Fourth, in the intermediate region where both the designer's support among voters and the ratio of aggregate and idiosyncratic uncertainty are balanced, mixed plans can be optimal (as well as a *segregate-moderates-and-pair* plan, where moderate voters are segregated and extreme left and right-wing voters are paired), and we can numerically trace out the boundaries of the parameter regions where each type of plan is optimal.

As we discuss in Section 6, the form of optimal partisan districting has significant implications for several political and legal issues, including redistricting reform, intra- and inter-district political polarization, and measuring gerrymandering. Since our results show that the ratio of idiosyncratic and aggregate uncertainty is a key determinant of the form of optimal districting, it is therefore important to understand whether idiosyncratic or aggregate uncertainty is larger in practice. We answer this question using precinct-level returns from the 2016, 2018, and 2020 US House elections. The data clearly show that idiosyncratic uncertainty is much larger than aggregate uncertainty.

[4]This result refines the main result of Friedman and Holden (2008).

Intuitively, this finding results from the simple observation that, in practice, most precinct vote splits are much closer to 50-50 (the vote split under high idiosyncratic uncertainty) than 100-0 or 0-100 (the vote splits under high aggregate uncertainty).[5] Therefore, for the realistic parameter range, exactly optimal districting approximates NAD (for a designer with majority voter support) or SOP (for a designer with minority support), while uniform districting or pack-and-crack is approximately optimal. This finding can help explain why actual gerrymandering usually resembles uniform districting or pack-and-crack.

Methodologically, we establish a formal connection between gerrymandering—partitioning voters into districts—and information design—partitioning states of the world into signals. The partisan gerrymandering problem we study is mathematically equivalent to a non-linear Bayesian persuasion problem with a one-dimensional state, a one-dimensional action for the receiver, and state-independent sender preferences. Our results are novel in the context of this persuasion problem, so this paper directly contributes to information design as well as gerrymandering. More importantly, we establish a strong connection between these topics.[6]

**Related Literature.** The closest prior papers on optimal partisan gerrymandering are Owen and Grofman (1988), Friedman and Holden (2008), and Gul and Pesendorfer (2010). Owen and Grofman's model is equivalent to the special case of our model with two voter types. Gul and Pesendorfer study competition between two designers who each control districting in some area and aim to win a majority of seats.[7] A simplified version of their model with a single designer is equivalent to the special case of our model with uniform idiosyncratic shocks. Friedman and Holden consider a model similar to ours (although with finitely many districts, rather than a continuum as in our model and Gul and Pesendorfer), but their main results concern the case where idiosyncratic uncertainty is much smaller than aggregate uncertainty. In contrast, we do not restrict the relative amounts of aggregate and idiosyncratic uncertainty, and we show empirically that the practically relevant case is the one where idiosyncratic uncertainty dominates (i.e., the opposite of the case emphasized by Friedman and Holden).

The broader literature on gerrymandering and redistricting addresses a wide range of issues, including geographic constraints on gerrymandering (Sherstyuk, 1998, Shotts, 2001, Puppe and Tasnádi, 2009), gerrymandering with heterogeneous voter turnout (Bouton, Genicot, Castanheira, and Stashko, 2024, Gomberg, Pancs, and Sharma, 2024), socially optimal districting (Gilligan and Matsusaka, 2006, Coate and Knight, 2007, Bracco, 2013), measuring district compactness (Chambers and Miller, 2010, Fryer and Holden, 2011, Ely, 2022), the interaction of redistricting and policy choices (Shotts, 2002, Besley and Preston, 2007, Groll and O'Halloran, 2024), measuring gerrymandering (King and Browning, 1987, McGhee, 2014, Stephanopoulos and McGhee, 2015, Deford, Duchin, and Solomon, 2021, Gomberg, Pancs, and Sharma, 2023), and assessing the consequences of redistricting (among many: Gelman and King, 1994b, McCarty,

[5]This observation also implies that simpler models with only two types of voters or precincts (e.g., Owen and Grofman 1988) cannot closely approximate the problem facing actual gerrymanderers, who must assign many different types of precincts.

[6]Contemporaneous papers by Lagarde and Tomala (2021) and Gomberg, Pancs, and Sharma (2023) also emphasize connections between gerrymandering and information design, albeit in less general models: Lagarde and Tomala assume two voter types, while Gomberg, Pancs, and Sharma assume no aggregate uncertainty. The closest paper in the persuasion literature is our companion paper, Kolotilin, Corrao, and Wolitzky (2024), which we discuss later on. In turn, Kolotilin, Corrao, and Wolitzky was greatly influenced by Friedman and Holden (2008).

[7]Friedman and Holden (2020) study designer competition in the model of their 2008 paper.

Poole, and Rosenthal, 2009, Hayes and McKee, 2009, Jeong and Shenoy, 2024, Sabet and Yuchtman, 2024). As the partisan gerrymandering problem interacts with many of these issues, our analysis may facilitate future research in these areas.

**Outline.** The paper is organized as follows: Section 2 presents the model. Section 3 establishes general properties of optimal districting plans that hold regardless of the designer's popularity or the amount of aggregate and idiosyncratic uncertainty. Section 4 contains our main theoretical and numerical results, which characterize optimal districting as a function of these parameters. Section 5 contains our empirical results, which estimate which parameters are the practically relevant ones. Section 6 discusses policy implications. Section 7 concludes. All proofs are deferred to the appendix.

## 2. MODEL

We consider a standard electoral model with one-dimensional voter types (parameterizing voter partisanship) and one-dimensional aggregate uncertainty in each district-level race (parameterizing the vote share for the designer's party).

**Voters and Vote Shares.** There is a continuum of voters. A voter votes for the designer's party (for short, "votes for the designer") iff $s \geq r + t$, where

- $s \in [\underline{s}, \overline{s}]$, with $\underline{s} < \overline{s}$, is the voter's type, which is observed by the designer and is the object of districting. The population distribution of $s$ is denoted by $F$.
- $r \in \mathbb{R}$ is the aggregate shock in the voter's district, which realizes after districting. The distribution of $r$ in each district is denoted by $G$.[8]
- $t \in \mathbb{R}$ is an idiosyncratic, voter-specific "taste shock," which also realizes after districting. The distribution of $t$ is denoted by $Q$.

Thus, the share of type-$s$ voters who vote for the designer in a district where the aggregate shock takes value $r$ equals $Q(s-r)$.

Note that the designer faces two kinds of uncertainty at the time of districting: aggregate, district-level uncertainty, $r$, and idiosyncratic, voter-level uncertainty, $t$. Many of our results turn on a comparison of the "amount" of each kind of uncertainty. To facilitate this comparison, we assume that $G$ and $Q$ have the same shape, in that there exists $\eta > 0$ such that $G(r) = Q(\eta r)$.[9] We also define $\gamma = \eta^2/(1+\eta^2) \in (0,1)$, so the ratio of the variances of $r$ and $t$ is $(1-\gamma)/\gamma$. The parameter $\gamma$ thus captures the *share of idiosyncratic uncertainty*. We say that aggregate uncertainty is *larger* than idiosyncratic uncertainty if $\gamma < 0.5$, while idiosyncratic uncertainty is larger if $\gamma > 0.5$.

The model is now fully parameterized by the distributions $F$ and $Q$ and the parameter $\gamma \in (0,1)$. We assume that $F$ and $Q$ admit strictly positive densities $f$ and $q$ that are three-times differentiable. We also assume that $q$ is symmetric about 0 and strictly log-concave: $d^2 \ln(q(t))/dt^2 < 0$ for all $t$. This implies that $Q$ is strictly convex below 0 and strictly concave above 0, with $Q(0) = 1/2$.

Log-concavity of $q$ is a key assumption. This standard property is satisfied by many distributions (Bagnoli and Bergstrom, 2005) and is similar to Friedman and Holden's (2008) "Informative Signal Property" assumption. Substantively, it captures the real-

[8] The correlation among district-level aggregate shocks is irrelevant for our analysis. However, we do estimate it empirically.

[9] Mathematically, this says that $G$ and $Q$ lie in the same location-scale family. An earlier version of this paper, Kolotilin and Wolitzky (2020), contains additional results where $G$ and $Q$ have different shapes.

istic feature that moderate voters are more sensitive to the aggregate shock than more extreme voters.[10]

**Districting Plans.** The designer assigns voters to a continuum of equipopulous districts based on their types $s$, and thus determines the distribution $P$ of $s$ in each district.[11] A district is characterized by the distribution $P$ of voter types $s$ it contains. Thus, a *districting plan*—which specifies the measure of districts with each voter type distribution $P$—is a distribution $\mathcal{H}$ over distributions $P$ of $s$, such that the population distribution of $s$ is given by $F$: that is, $\mathcal{H} \in \Delta\Delta[\underline{s}, \overline{s}]$ and $\int P(s) d\mathcal{H}(P) = F(s)$ for all $s$.[12] For example, under *uniform districting*, where all districts are the same, $\mathcal{H}$ assigns probability 1 to $P = F$. In the opposite extreme case of *segregation*, where each district consists entirely of one type of voter, every distribution $P$ in the support of $\mathcal{H}$ takes the form $P = \delta_s$ for some $s \in [\underline{s}, \overline{s}]$, where $\delta_s$ denotes the degenerate distribution on voter type $s$. Finally, the best-known districting plan is *pack-and-crack*, where there is a cutoff voter type $s^* \in (\underline{s}, \overline{s})$ such that $\text{supp}(\mathcal{H}) = \{P, P'\}$ and $P$ and $P'$ are the lower and upper truncations of $F$ at $s^*$.

We say that a districting plan is *pure* if (almost) each voter type $s$ is assigned to only one kind of district (so there is a unique $P \in \text{supp}(\mathcal{H})$ such that $s \in \text{supp}(P)$) and *mixed* otherwise. Since the distribution of voter types $F$ is continuous, it is natural to expect pure districting to be optimal, but we will see that this is not always the case.

**Designer's Problem.** The designer wins a district iff he receives a majority of the district vote. Thus, the designer wins a district with voter type distribution $P$ (henceforth, "district $P$") iff the district's aggregate shock $r$ satisfies $\int Q(s-r) dP(s) \geq 1/2$. Since $Q(s-r)$ is decreasing in $r$ and $Q(0) = 1/2$, the designer wins district $P$ iff

$$r \leq r^*(P) := \left\{ \tilde{r} : \int Q(s - \tilde{r}) dP(s) = Q(0) \right\}.$$

Note that the threshold shock $r^*(P)$ to win a segregated district $P = \delta_s$ is simply $s$, while in general $r^*(P)$ lies somewhere in the convex hull of $\text{supp}(P)$. We say that a district $P$ is *weaker* than another district $P'$ if $r^*(P) < r^*(P')$. Since the aggregate shock has the same distribution in all districts, the designer wins weaker districts with lower probability.

We assume that the designer maximizes his party's expected seat share.[13] Thus, the designer's problem is to maximize $\int G(r^*(P)) d\mathcal{H}(P)$ over $\mathcal{H} \in \Delta\Delta[\underline{s}, \overline{s}]$ subject to $\int P d\mathcal{H}(P) = F$. This problem is similar to that of Friedman and Holden (2008), which in turn nests Owen and Grofman (1988) and a single-designer version of Gul and

[10]As Rakich and Silver (2018) put it in describing the "elasticity scores" in FiveThirtyEight.com's forecasting model, "Voters at the extreme end of the spectrum—those who have close to a 0 percent or a 100 percent chance of voting for one of the parties—don't swing as much as those in the middle."

[11]We follow Gul and Pesendorfer (2010) in assuming a continuum of districts. Since districting plans in the US are drawn at the state level, this implicitly assumes that each state contains a large number of districts. Of course, this is a better approximation for state legislative districts and for congressional districts in large states than it is for congressional districts in small states. Introducing integer constraints on the number of districts, while interesting and realistic, would complicate the analysis and obscure our insights.

[12]Throughout, for any compact metric space $X$, $\Delta X$ denotes the set of probability measures on $X$, endowed with the weak* topology. For any $\mu \in \Delta X$, its support $\text{supp}(\mu)$ is the smallest compact set of measure one.

[13]See Section 7 and Kolotilin and Wolitzky (2020) for discussion of more general designer objectives.

Pesendorfer (2010).[14] It is also equivalent to a Bayesian persuasion problem where the designer splits a prior distribution $F$ into posterior distributions $P$ and obtains utility $G(r^*(P))$ from inducing $P$.[15]

## 3. OPTIMAL PARTISAN GERRYMANDERING: GENERAL PROPERTIES

We first establish two general properties of optimal districting plans that hold regardless of the designer's popularity or the amount of aggregate and idiosyncratic uncertainty. The first of these, *single-dippedness*, is the key property identified by Friedman and Holden (2008)—we just re-establish their result in our continuum-district model. The second property, *segregate-pairedness*, is novel.

### 3.1. *Single-Dippedness*

We first show that optimal districting plans are *strictly single-dipped*, in that more extreme voters are assigned to stronger districts: formally, any district $P \in \text{supp}(\mathcal{H})$ containing any two voter types $s < s''$ is stronger than any district $P' \in \text{supp}(\mathcal{H})$ containing any intervening voter type $s' \in (s, s'')$, in that $r^*(P) > r^*(P')$.[16] Note that if districting is strictly single-dipped then each district contains at most two distinct voter types. Thus, any district $P$ in the support of a strictly single-dipped districting plan $\mathcal{H}$ is either *segregated* (if $|\text{supp}(P)| = 1$) or *paired* (if $|\text{supp}(P)| = 2$). For example, segregation is strictly single-dipped, but uniform districting and pack-and-crack are not.

LEMMA 1: *Any optimal districting plan is strictly single-dipped.*

Lemma 1 recapitulates Lemma 1 of Friedman and Holden (2008) in our continuum-district model.[17] To see the intuition, suppose a districting plan creates two districts, 1 and 2, with the same threshold aggregate shock $r^*$, but where District 1 contains moderate voters and District 2 contains a mix of left-wing and right-wing extremists. Since $q$ is log-concave, the vote share is more sensitive to the aggregate shock in District 1 than in District 2, which implies that a marginal voter is more likely to be pivotal in District 2 than in District 1. The designer can then profitably exploit this asymmetry by re-assigning some unfavorable voters to District 1 and re-assigning some favorable voters to District 2, thus weakening the moderate District 1 and strengthening the extreme District 2. Breaking all ties in favor of extreme distticts in this manner leads to strictly single-dipped districting.

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[14] Friedman and Holden assume a finite number of districts rather than a continuum and do not assume that $G$ and $Q$ have the same shape. Owen and Grofman assume binary voter types. Gul and Pesendorfer consider a majoritarian objective with both state-level and district-level aggregate shocks; however, after conditioning on the pivotal value of the state-level shock, their problem reduces to maximizing expected seat share with only district-level shocks.

[15] Specifically, the designer's problem lies in the translation-invariant subcase of the *state-independent sender* case of the persuasion problem studied in Kolotilin, Corrao, and Wolitzky (2024), which specializes the general Bayesian persuasion problem of Kamenica and Gentzkow (2011) by assuming that the state and the receiver's action are one-dimensional, the receiver's utility is supermodular and concave in her action, and the sender's utility is independent of the state and increasing in the receiver's action. In the gerrymandering context, the designer's preferences are state-independent because he only cares about how many districts he wins and not directly about the districts' composition.

[16] We say that a district $P$ "contains" a voter type $s$ if $s \in \text{supp}(P)$.

[17] Kolotilin, Corrao, and Wolitzky (2024) give sufficient conditions for single-dippedness in a more general model that allows state-dependent designer preferences.

### 3.2. *Segregate-Pairedness*

A strictly single-dipped districting plan can contain a mix of segregated and paired districts of varying strengths. If such a plan $\mathcal{H}$ has the further property that every segregated district is weaker than every paired district (i.e., there do not exist $P, P' \in \text{supp}(\mathcal{H})$ such that $|\text{supp}(P)| = 1$, $|\text{supp}(P')| = 2$, and $r^*(P) > r^*(P')$), we say that $\mathcal{H}$ is *segregate-pair*.

A segregate-pair plan $\mathcal{H}$ can be described in a simple way. There is a *bifurcation point* $r^b \in (\underline{s}, \overline{s}]$ that divides the segregated and paired districts, so that $r^*(P) \leq r^b$ for all segregated districts $P \in \text{supp}(\mathcal{H})$, and $r^*(P) > r^b$ for all paired districts $P \in \text{supp}(\mathcal{H})$. The assignment of voters to paired districts is then described by a decreasing function $s_1$ and an increasing function $s_2$, where the two types in a paired district $P$ are $s_1(r^*(P))$ and $s_2(r^*(P)) > s_1(r^*(P))$. Stronger paired districts thus contain more extreme voters, as single-dippedness requires.[18]

Despite this tight characterization, a range of interesting districting plans are segregate-pair, including the following:

**Segregation,** where all voters are segregated: $P = \delta_{r^*(P)}$ for all $P \in \text{supp}(\mathcal{H})$, or equivalently $r^b = \overline{s}$.

**Segregate-Moderates-and-Pair (SMP),** where moderate voters are segregated and extreme voters are paired in a negatively assortative manner: $\mathcal{H}$ is pure, $r^b \in (\underline{s}, \overline{s})$, and there exists $\hat{s} \in (\underline{s}, r^b)$ such that a district $P \in \text{supp}(\mathcal{H})$ is segregated iff $r^*(P) \in [\hat{s}, r^b]$.

**Segregate-Opponents-and-Pair (SOP),** where unfavorable voters are segregated and more favorable voters are paired in a negatively assortative manner: $\mathcal{H}$ is pure, $r^b \in (\underline{s}, \overline{s})$, and there exists $\hat{s} \in (\underline{s}, r^b)$ such that a district $P \in \text{supp}(\mathcal{H})$ is segregated iff $r^*(P) \in [\underline{s}, \hat{s}]$.

**Negative Assortative Districting (NAD),** where all voters are paired in a negatively assortative manner: $r^b = \inf_{P \in \text{supp}(\mathcal{H})} r^*(P)$.

These four plans feature prominently in our results (as illustrated in Figures 1–3) and warrant some discussion. First, segregation and NAD are the extreme segregate-pair plans where all voter types are segregated and where only a single type is segregated. There is a unique segregatation plan, but there is a continuum of NAD plans, depending on the weights on the different voter types in each paired district. (Similarly, there is also a continuum of SMP and SOP plans.) NAD plans can be viewed as "strictly single-dipped versions" of uniform districting: starting from uniform districting and splitting the pool of voters into pairs in a strictly single-dipped manner yields NAD.

Similarly, SOP plans are strictly single-dipped version of Gul and Pesendorfer's (2010) "$p$-segregation" plan, where unfavorable voters are segregated and more favorable voters are pooled: starting from $p$-segregation and splitting the pool into pairs yields SOP. SOP can also be obtained from pack-and-crack districting by first splitting the weak districts into segregated ones (yielding $p$-segregation) and then splitting the strong districts into pairs.

Finally, SMP is the same as Friedman and Holden's (2008) "matching slices" plan, with the difference that Friedman and Holden assume a finite number of districts and do not mention the possibility of segregating a non-trivial interval of moderate voter types.

[18] Lemma 5 in Appendix A formalizes the description of a segregate-pair plan by a bifurcation point $r^b$ and functions $s_1$ and $s_2$. In particular, we define the bifurcation point as the infimum of $r^*(P)$ over all paired districts $P \in \text{supp}(\mathcal{H})$.

An instructive example of a plan that can be strictly single-dipped but not segregate-pair is "Segregate-Supporters-and-Pair," where favorable voters are segregated and less favorable voters are paired. This plan can be obtained from pack-and-crack by splitting weak districts into pairs and splitting strong districts into segregated ones.

Our first main result is that segregate-pair districting is optimal if idiosyncratic uncertainty is larger than aggregate uncertainty.

THEOREM 1: *If idiosyncratic uncertainty is larger than aggregate uncertainty, there is a unique optimal districting plan, which is segregate-pair.*

Numerically, segregate-pair districting is also optimal when idiosyncratic uncertainty is smaller than aggregate uncertainty, but we were not able to prove this.[19] However, Theorem 1 covers the empirically relevant case, as we will estimate that $\gamma$ is much greater than 0.5.

The intuition for Theorem 1 is as follows. First, log-concavity of $g$ implies that $G$, the distribution of the aggregate shock, is first convex and then concave. Second, convexity of $G$ favors segregation (as splitting a district with threshold aggregate shock $r^*$ into districts with threshold shocks $r^* - \varepsilon$ and $r^* + \varepsilon$ increases expected seat share when $G$ is convex and $Q$ is linear), while concavity of $G$ favors pairing. Thus, when $G$ is first convex and then concave, weak districts should be segregated and strong districts should be paired. This is precisely segregate-pair.[20]

Theorem 1 is a fundamental result: it is optimal to make paired districts stronger than segregated ones. However, while segregate-pair plans admit a tight characterization, we have seen that a wide variety of plans are segregate-pair. The next section characterizes optimal plans as a function of $F$, $Q$, and $\gamma$.

As an aside, we note that Theorem 1 also contributes to the Bayesian persuasion literature. An important disclosure policy that frequently arises in this literature is *upper censorship*, where states below a cutoff are disclosed and states above the cutoff are pooled. Upper censorship is often optimal in "linear" persuasion problems, where a posterior can be summarized by its mean (Kolotilin, 2018, Kolotilin, Mylovanov, and Zapechelnyuk, 2022). However, in non-linear persuasion problems, a version of strict single-dippedness often holds, so disclosure polices that pool more than two states (like upper censorship) cannot be optimal (Kolotilin, Corrao, and Wolitzky, 2024). This raises the question of when a strictly single-dipped version of upper censorship—such as a segregate-pair policy—is optimal. Theorem 1 is the first result in the literature to give sufficient conditions for such policies to be optimal.

## 4. OPTIMAL PARTISAN GERRYMANDERING IN DIFFERENT PARAMETER REGIMES

We now present our results on optimal districting as a function of the designer's popularity and the ratio of idiosyncratic and aggregate uncertainty. First, NAD is optimal if the designer has strong support from all voter types, and segregation is optimal if the designer has weak support from all voter types and idiosyncratic uncertainty is larger

[19]See Figures 1–3, where all optimal plans are segregate-pair. In these figures, $G$ and $Q$ are normal. Using Lemma 6 in Appendix A, we have checked numerically that segregate-pair is also optimal when $G$ and $Q$ are logistic. The normal and logistic families are the only standard location-scale families we are aware of with symmetric and strictly log-concave densities on $\mathbb{R}$.

[20]A complication is that log-concavity of $q$ always favors pairing. In Section 4.1, we explain how $\gamma > 0.5$ ensures that the log-concavity of $g$ "dominates" that of $q$.

than aggregate uncertainty (Theorem 2). Second, optimal plans approximate NAD or SOP with equally strong paired districts if aggregate uncertainty is small (Theorem 3). Since we will estimate that aggregate uncertainty is small empirically, Theorem 3 is our most practically relevant result. However, while exactly optimal plans approximate NAD or SOP, uniform districting or pack-and-crack districting are also approximately optimal. Third, optimal plans approximate NAD with a 50-50 voter type split in each district if idiosyncratic uncertainty is small (Theorem 4). Fourth, in the intermediate region where both the designer's support among voters and the ratio of idiosyncratic and aggregate uncertainty are balanced, mixed versions of SOP and SMP can be optimal, and we can numerically trace out the boundaries of the parameter regions where each type of plan emerges (Theorem 5 and the subsequent numerical results). Overall, we give a fairly complete picture of how optimal districting varies with the designer's support and the ratio of idiosyncratic and aggregate uncertainty, which we illustrate in Figure 3 at the end of this section.

### 4.1. *Optimal Districting with Imbalanced Voter Support*

We first investigate optimal districting when voter support is highly imbalanced between the parties. This case is relatively simple and is not too unrealistic: we will estimate that in around half of US states, voter support is sufficiently imbalanced that NAD (an optimal plan in the high imbalanced case) is optimal for our estimated parameters.[21]

We say that the designer has *uniformly strong support* if $\underline{s} \geq 0$. This means that, when the aggregate shock takes its modal value of 0, *all* voter types vote for the designer with probability at least 50%. Thus, a designer with uniformly strong support can only lose a district when the aggregate shock lands in the right (unfavorable) tail. Conversely, the designer has *uniformly weak support* if $\overline{s} \leq 0$, so he can only win a district when the aggregate shock lands in the left tail. Finally, the designer has *balanced support* if $r^*(F) = 0$, so the overall vote is 50-50 when the aggregate shock takes its modal value.

THEOREM 2: *The following hold:*

1. *If the designer has uniformly strong support, there is a unique optimal districting plan, which is NAD.*
2. *If the designer has uniformly weak support and idiosyncratic uncertainty is larger than aggregate uncertainty, there is a unique optimal districting plan, which is segregation.*
3. *If the designer has balanced support, NAD and segregation are both suboptimal.*

Since NAD plans are strictly single-dipped versions of uniform districting, the optimality of NAD in case 1 is akin to the optimality of uniform districting for a designer with majority support in the absence of aggregate uncertainty. To see why NAD is optimal, recall that any strictly single-dipped plan that never segregates two distinct voter types is NAD. So, since $\underline{s} \geq 0$, it suffices to show that it is sub-optimal for the designer to segregate any two voter types $s < s'$ that lie in a region where $G$ is concave. To see this, suppose the designer pools a few type-$s$ voters in with the type-$s'$ voters.

[21] However, the estimated parameters are *not* extreme enough to satisfy the conditions in Theorem 2, which are sufficient but not necessary for NAD to be optimal.

The marginal effect of this change on the designer's expected seat share among type-$s$ voters is

$$G(s') - G(s),$$

which is the increased probability of winning a type-$s$ voter's district when she moves from the weak district $\delta_s$ to the strong district $\delta_{s'}$. On the other hand, the marginal effect of this change on the designer's expected seat share among type-$s'$ voters is

$$\frac{Q(s-s') - Q(0)}{q(0)} g(s').$$

This follows because the first term is the marginal effect on the threshold shock to win the strong district, where this comes from using the implicit function theorem to calculate $dr/d\rho$ at $\rho = 0$ from the equation $\rho Q(s-r) + (1-\rho)Q(s'-r) = Q(0)$, and the second term is the density of the aggregate shock at $r^*(\delta_{s'}) = s'$. Finally, the sum of the two effects is positive, because

$$\frac{G(s') - G(s)}{g(s')} > s' - s > \frac{Q(0) - Q(s-s')}{q(0)},$$

where the first inequality is by strict concavity of $G$ on $[s, s']$, and the second inequality is by strict convexity of $Q$ on $[s-s', 0]$.

The intuition for why segregation is optimal in case 2 is that, for any two voter types $s$ and $s'$ that lie in a region where $G$ is "sufficiently convex" relative to $Q$, we have

$$\frac{G(s') - G(s)}{g(s')} < \frac{Q(0) - Q(s-s')}{q(0)},$$

which by a similar logic as above implies that it is optimal for the designer to separate any two voter types rather than pooling them. Note that it is not enough for $G$ to be just slightly convex, because now (in contrast to case 1) the convexity of $G$ and the convexity of $Q$ compete in comparing the two effects above: intuitively, log-concavity of $Q$ favors pooling, because a few unfavorable voters are unlikely to be pivotal and thus can be safely added to a stronger district. The proof of Theorem 2 shows that the convexity of $G$ "wins" if $\gamma \geq 0.5$. Intuitively, when $\gamma \geq 0.5$, $G$ is more convex than $Q$, which suffices for the above inequality.

In case 3, neither NAD nor segregation are optimal, so the optimal plan creates a mix of segregated and paired districts.[22] The intuition is that the designer prefers pooling any two positive voter types, so segregation is suboptimal; but at the same time, for any strictly single-dipped NAD plan, there exist nearby voter types that are paired in a district $P$ with $r^*(P) < 0$, and the designer is better-off segregating these types.

We remark that the logic of Theorem 1 is a more intricate variant of Theorem 2's. Note that a strictly single-dipped plan is *not* segregate-pair iff there exist $s < r < s' \leq s''$ such that voter types $s < s'$ are paired in a district $P$ with $r^*(P) = r \in (s, s')$ and voter type $s''$ is segregated. Suppose toward a contradiction that such a plan is optimal. As in case 2 of Theorem 2, if idiosyncratic uncertainty is larger than aggregate uncertainty

[22]Proposition 1 of Friedman and Holden (2008) shows that SMP ("matching slices") is optimal when idiosyncratic uncertainty is sufficiently small, but their discussion focuses on NAD. In contrast, Proposition 2 shows that NAD is never optimal with symmetric parties and a continuum of districts.

then pooling type-$s$ voters in district $P$ is worse than segregating them if $r \leq 0$ (the range where $G$ is convex), so we must have $r > 0$. But then, it can be shown that the planner would be better-off removing a few type-$s$ voters from district $P$ and pooling them in with the type-$s''$ voters, by a similar argument as in Theorem 2.

To turn the above arguments into rigorous proofs of Theorems 1 and 2, we rely on duality and complementary slackness theorems developed in Kolotilin, Corrao, and Wolitzky (2024), which we restate as Lemma 2 in Appendix A. The key implication of Lemma 2 is that there is a well-defined Lagrange multiplier $\lambda(r^*(P))$ on the constraint $\int Q(s - r^*(P))dP(s) = Q(0)$, which is given by the formula

$$\lambda(r^*(P)) = \frac{g(r^*(P))}{\displaystyle\int q(s - r^*(P))dP(s)}$$

for all districts $P$ in the support of an optimal plan $\mathcal{H}$, and that the designer only assigns type-$s$ voters to districts $P$ that maximize

$$G(r^*(P)) + \lambda(r^*(P))(Q(s - r^*(P)) - Q(0)).$$

Intuitively, $\lambda(r^*(P))$ is the designer's value of an extra vote in district $P$, which equals the product of the designer's marginal utility of increasing $r^*(P)$ (which equals $g(r^*(P))$) and the derivative of $r^*(P)$ with respect to $\varepsilon$ in the equation $\int Q(s - r^*(P))dP(s) = Q(0) - \varepsilon$ (which equals $1/\int q(s - r^*(P))dP(s)$ by the implicit function theorem). The designer's "total value" of assigning a type-$s$ voter to district $P$ then equals the sum of $G(r^*(P))$ (the probability of winning district $P$) and $\lambda(r^*(P))(Q(s - r^*(P)) - Q(0))$ (the product of the designer's value of an extra vote in district $P$ and the number of net votes provided by a type-$s$ voter at the pivotal aggregate shock $r^*(P)$). The proofs of Theorems 1 and 2 combine Lemma 2 and the above arguments.

### 4.2. *Optimal Districting with Small Aggregate or Idiosyncratic Uncertainty*

We now consider optimal districting when either aggregate or idiosyncratic uncertainty is small. We will see that the small aggregate uncertainty case is the empirically relevant one. We include the small idiosyncratic uncertainty case for completeness and also to show how the main result of Friedman and Holden (2008) fits in our framework.[23]

Aggregate uncertainty is small when $F$ and $Q$ are fixed and $\gamma \to 1$, so the aggregate shock $r$ is close to 0 with high probability. When aggregate uncertainty is small and $r^*(F) > 0$ (so the designer has majority support at the modal aggregate shock), the designer's expected seat share is close to 1 under uniform districting. But since uniform districting is not strictly single-dipped, it cannot be exactly optimal for any $\gamma < 1$, by Lemma 1. Instead, we show that optimal districting approximates NAD with equally strong paired districts. Intuitively, when aggregate uncertainty is small and $r^*(F) > 0$, optimal districting starts from uniform districting and then splits pooled districts into equally strong paired districts in a negatively assortative manner.

When aggregate uncertainty is small and $r^*(F) < 0$, the designer's optimal expected seat share is approximately $1 - F(s^*(0))$, where, for any $r \in (r^*(F), \bar{s})$, $s^*(r)$ is defined

[23]The results in this subsection, Theorems 3 and 4, do not require the assumption that $G$ and $Q$ lie in the same location-scale family, although this assumption facilitates the exposition.

so that the designer's vote share among voter types $s \geq s^*(r)$ at aggregate shock $r$ is 50%.[24] This expected seat share can be approximated by a pack-and-crack plan where, for a small $\varepsilon > 0$, voter types $s < s^*(\varepsilon)$ are assigned to identical weak districts that the designer loses with high probability, and voter types $s \geq s^*(\varepsilon)$ are assigned to identical strong districts that the designer wins with a vote share close to 50% with high probability.[25] However, since pack-and-crack districting is not strictly single-dipped, it cannot be exactly optimal. Instead, we show that optimal districting approximates SOP with equally strong paired districts. Intuitively, when aggregate uncertainty is small and $r^*(F) < 0$, optimal districting starts from pack-and-crack and then splits the packed districts into segregated districts and splits the cracked districts into equally strong paired districts in a negatively assortative manner.

To state our result, let $\mathcal{H}^*$ be the unique districting plan that segregates types below $s^*(0)$ (if $s^*(0) > \underline{s}$, which holds when $r^*(F) < 0$) and pairs types above $s^*(0)$ in equally strong districts in a negatively assortative manner. Formally, letting $r_+^*(F) = \max\{0, r^*(F)\}$, $\mathcal{H}^*$ is the unique plan $\mathcal{H}$ such that, for any $P \in \text{supp}(\mathcal{H})$, either (a) $\text{supp}(P) = \{s(P)\}$ such that $s(P) \in [\underline{s}, s^*(0)] \cup \{r_+^*(F)\}$, or (b) $\text{supp}(P) = \{s_1(P), s_2(P)\}$ such that $r^*(P) = r_+^*(F)$, $s^*(0) \leq s_1(P) < r_+^*(F) < s_2(P) \leq \overline{s}$, and

$$\int_{[s^*(0), s_1(P)] \cup [s_2(P), \overline{s}]} (Q(s - r_+^*(F)) - Q(0)) dF(s) = 0.$$

THEOREM 3: *As aggregate uncertainty vanishes ($\gamma \to 1$ with $F$ and $Q$ fixed), the optimal expected seat share converges to $1 - F(s^*(0))$, and the optimal districting plan converges to $\mathcal{H}^*$.*[26] *Thus, when aggregate uncertainty is small, optimal districting approximates NAD with equally strong paired districts if $r^*(F) \geq 0$ and approximates SOP with equally strong paired districts if $r^*(F) < 0$.*

The intuition for why paired districts are approximately equally strong is that, when aggregate uncertainty is small, it is approximately optimal for the designer to assign voters among the paired districts so as to make the weakest of these districts as strong as possible. Mathematically, this follows from log-concavity. This simple and intuitive property is the basis for the gerrymandering test we propose in Section 6.4.

To appreciate how optimal districting with small aggregate uncertainty differs from uniform districting (when $r^*(F) \geq 0$) or $p$-segregation (when $r^*(F) < 0$), consider the difference between pooling an interval of voter types and splitting the pool into equally strong paired districts in a negatively assortative manner. This splitting does not affect the designer's expected seat share. Indeed, it does not affect the joint distribution over voter types $s$ and threshold shocks $r^*(P)$ in districts to which they are assigned, because if $P = \alpha P' + (1 - \alpha) P''$ and $r^*(P') = r^*(P'')$, then $r^*(P) = r^*(P') = r^*(P'')$. Thus, in terms of *outcomes* (joint distributions of $s$ and $r^*(P)$), NAD with equally strong paired districts is equivalent to uniform districting, and SOP with equally strong paired districts is equivalent to $p$-segregation. However, viewed as *districting plans* (distributions of $P$), NAD with equally strong paired districts is quite different from uniform districting, and SOP with equally strong paired districts is quite different

[24]Formally, we define $s^*(r)$ as the smallest $\tilde{s} \in [\underline{s}, \overline{s}]$ such that $\int_{\tilde{s}}^{\overline{s}} (Q(s - r) - Q(0)) dF(s) \geq 0$. Note that $s^*(r) = \underline{s}$ if $r \leq r^*(F)$, $s^*(r) = \overline{s}$ if $r \geq \overline{s}$, and $s^*(r) \in (\underline{s}, \overline{s})$ if $r \in (r^*(F), \overline{s})$.

[25]This is shown formally in Lemma 11 in the appendix.

[26]The latter convergence is in the weak* topology.

from $p$-segregation. Thus, Theorem 3 implies that optimal districting plans with small aggregate uncertainty are quite different from uniform districting and $p$-segregation; but also that, at the same time, these differences are not very consequential for the joint distribution of $s$ and $r^*(P)$ or the designer's expected seat share.

We now turn to the case where idiosyncratic uncertainty is small. Here, $F$ and $G$ are fixed and $\gamma \to 0$, so each idiosyncratic shock $t$ is close to 0 with high probability. In this case, whether the designer wins a district $P$ at aggregate shock $r$ is essentially determined by the median voter type $s^P$ in district $P$: with a unique median $s^P$, the designer loses districts where $s^P < r - \varepsilon$ and wins districts where $s^P > r + \varepsilon$. Therefore, any optimal districting plan must approximate the highest feasible distribution of district median voters, which is attained by pairing each voter type $s$ above the population median $s^m = F^{-1}(1/2)$ with below-median types, with 50% weight on the above-median type. Under such a plan with an extra $\varepsilon$ weight on the above-median type in each district, the designer's expected seat share is approximately $2\int_{s^m}^{\overline{s}} G(r)dF(r)$. Moreover, for such a plan to be strictly single-dipped, all voter types must be paired in a negatively assortative manner. The resulting districting plan approximates NAD with a 50-50 voter type split in each district.

Let $\mathcal{H}^{**}$ be NAD with a 50-50 split in each district. Formally, $\mathcal{H}^{**}$ is the unique plan $\mathcal{H}$ such that, for any $P \in \text{supp}(\mathcal{H})$, we have either (a) $\text{supp}(P) = \{s^m\}$, or (b) $\text{supp}(P) = \{s_1(P), s_2(P)\}$ such that $\underline{s} \leq s_1(P) < s^m < s_2(P) \leq \overline{s}$, and $F(s_1(P)) = 1 - F(s_2(P))$.

THEOREM 4: *As idiosyncratic uncertainty vanishes ($\gamma \to 0$ with $F$ and $G$ fixed), the optimal expected seat share converges to $2\int_{s^m}^{\overline{s}} G(r)dF(r)$, and the optimal districting plan converges to $\mathcal{H}^{**}$. Thus, when idiosyncratic uncertainty is small, optimal districting approximates NAD with a 50-50 voter type split in each district.*

Theorem 4 is similar to Friedman and Holden's (2008) main result. With finitely many districts, Friedman and Holden show that, when idiosyncratic uncertainty is sufficiently small, optimal districting is a discrete version of SMP.[27] Theorem 4 adds that, in the limit, only one district is segregated, and the voter type split in all other districts is 50-50.

Comparing Theorems 3 and 4, we see that varying the ratio of aggregate and idiosyncratic uncertainty leads to completely different districting plans. When aggregate uncertainty is small, pack-and-crack is approximately optimal, and the exactly optimal plan is close to NAD or SOP with equally strong paired districts. When idiosyncratic uncertainty is small, pack-and-crack is far from optimal, and the optimal plan is close to NAD with a 50-50 voter type split in each district. In particular, while a NAD plan can arise in either case, the plans are extremely different: NAD with equally strong paired districts is outcome-equivalent to uniform districting, while NAD with a 50-50 split (or, away from the limit, a $50 - \varepsilon$-$50 + \varepsilon$ split in favor of the higher type) in each district is very different from uniform districting with small idiosyncratic uncertainty, as $r^*(P)$ is much higher in $50 - \varepsilon$-$50 + \varepsilon$ districts with more extreme voter types. (For example, compare Figures 2(d) and 2(f).) The critical role of the ratio of aggregate and idiosyncratic uncertainty motivates estimating this parameter in Section 5.

The distinction between optimal districting under small aggregate uncertainty and small idiosyncratic uncertainty relates to results in the probabilistic voting literature. When aggregate uncertainty is very small, the probability that the designer wins a

[27] Friedman and Holden's proof relies on perturbation arguments, while our proofs use duality.

district is approximately determined by the mean voter type in the district, as in probabilistic voting models with partisan taste shocks (e.g., Hinich 1977, Lindbeck and Weibull 1993). Optimizing the distribution of district means against a unimodal aggregate shock then requires segregating opposing voters and pooling more favorable voters, as in $p$-segregation or SOP or NAD with equally strong paired districts. In contrast, when aggregate uncertainty is very small, the probability that the designer wins a district is approximately determined by the median voter type in the district, as in probabilistic voting models with an uncertain median bliss point (e.g., Wittman 1983, Calvert 1985). The distribution of district medians is then optimized by pairing above-population-median and below-population-median voter types, as in NAD with a 50-50 voter type split in each district.[28]

### 4.3. *The Balanced Case and Regime Transitions*

Finally, we analyze optimal districting in the intermediate case where neither the parties' supporters nor the amounts of aggregate and idiosyncratic uncertainty are highly imbalanced. Here, optimal districting will take the form of either SOP, SMP, or a mixed versions of these districting plans that we call "Y-districting." We say that a segregate-pair plan $\mathcal{H}$ is *Y-districting* if there exists a positive number $\varepsilon > 0$ such that

1. For all $r \in [r^b - \varepsilon, r^b + \varepsilon]$ (where $r^b$ is the bifurcation point), there exists $P \in \text{supp}(\mathcal{H})$ such that $r^*(P) = r$.
2. The functions $s_1$ and $s_2$ describing the voter types in paired districts are twice differentiable and satisfy $\lim_{r \downarrow r^b} s_1(r) = \lim_{r \downarrow r^b} s_2(r)$.[29]

Note that Y-districting encompasses a mixed version of SOP, where there exists $\hat{s} \in (\underline{s}, r^b)$ such that voter types in $[\underline{s}, \hat{s})$ are always segregated and types in $(\hat{s}, r^b)$ are sometimes segregated and sometimes paired, as well as a mixed version of SMP, where there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always paired and types in $(\hat{s}, r^b)$ are sometimes segregated and sometimes paired.[30] We will show that, with balanced voter support, SOP is optimal when idiosyncratic uncertainty is "substantially" larger than aggregate uncertainty, SMP is optimal when aggregate uncertainty is larger than idiosyncratic uncertainty, and Y-districting (and, in particular, mixed SOP or mixed SMP) is optimal in the intermediate range.

To analyze these cases, we let $J$ be the distribution with variance 1 satisfying $Q(t) = J(t/\sqrt{\gamma})$ and $G(r) = J(r/\sqrt{1-\gamma})$, so that the variances of $t$ and $r$ are $\gamma$ and $1-\gamma$. For example, if $Q$ and $G$ are normal then $J$ is the standard normal distribution. By varying $\gamma$ while fixing $J$, we can simultaneously approximate the low-aggregate uncertainty and low-idiosyncratic uncertainty limits analyzed in Theorems 3 and 4, as $Q$ is almost constant as $\gamma \to 1$ and $G$ is almost constant as $\gamma \to 0$.

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[28]The distinction between mean and median-dependence applies to several related strands of the literature. In gerrymandering, Owen and Grofman (1988) and Gul and Pesendorfer (2010) study the mean-dependent case, while Friedman and Holden (2008) study an approximately median-depedent case. In persuasion, Gentzkow and Kamenica (2016), Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017), Kolotilin (2018), Dworczak and Martini (2019), and Kleiner, Moldovanu, and Strack (2021) study the mean-depedent case, while Kolotilin, Corrao, and Wolitzky (2024) study a general case nesting both the mean and quantile (e.g., median)-dependent case, and Yang and Zentefis (2024) and Kolotilin and Wolitzky (2024) study the quantile-dependent case.

[29]Differentiability is used in the proof of Theorem 5. It may be possible to drop it.

[30]In contrast, under SOP there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always segregated and types in $(\hat{s}, r^b)$ are always paired, while under SMP there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always paired and types in $(\hat{s}, r^b)$ are always segregated.

Our analytic result in this section is modest: if Y-districting is optimal, then the ratio of idiosyncratic and aggregate uncertainty must fall in an intermediate range. However, numerically it appears that this result actually fully characterizes optimal districting when voter support is balanced: at least when $J$ is normal and $F$ is uniform, our necessary conditions for optimality of Y-districting are also approximately sufficient, and when the ratio of idiosyncratic uncertainty to aggregate uncertainty is below (resp., above) the range where Y-districting is optimal, then SMP (resp., SOP) is optimal.

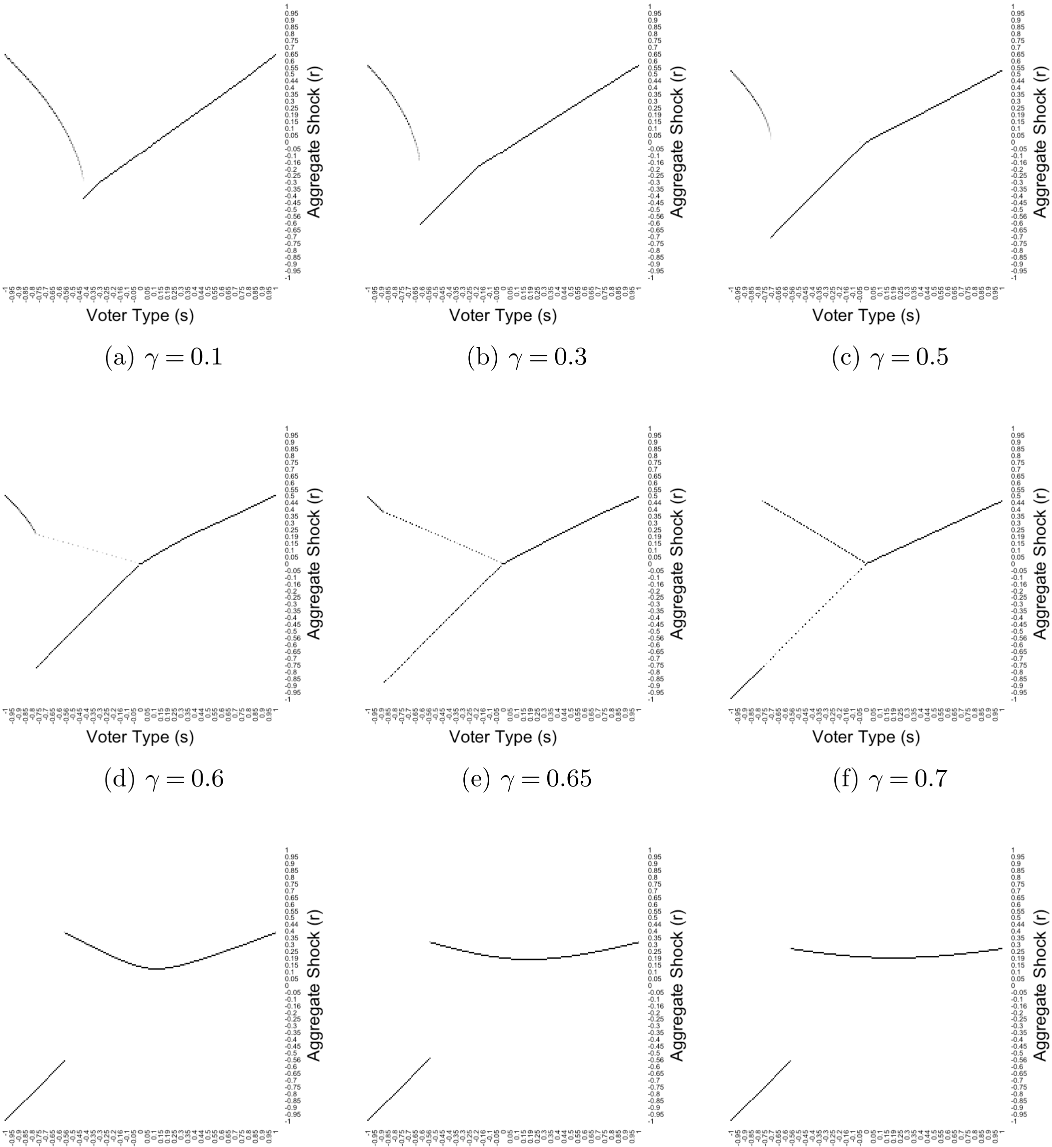
THEOREM 5: *If Y-districting is optimal, then* $r^b = 0$ *and* $\gamma \in (0.5, \sqrt{3}-1 \approx 0.732]$.

The proof of Theorem 5 proceeds by deriving three necessary conditions for optimal Y-districting to involve a bifurcation point at $r$ and showing that these conditions imply that $r$ must equal 0 and $\gamma$ must lie in an intermediate range. The first condition (equation (15) in Appendix A) says that it is optimal to pair voter types just below and just above $r$. The second condition (equation (16)) says that it is optimal to segregate types just below $r$. The third condition (equation (17)) says that the proportions of favorable and unfavorable voters in each district $P$ with $r^*(P) = r'$ just above $r$ actually induce the desired cutoff $r'$. Intuitively, for it to be optimal to pair nearby voter types around $r$, $G$ must be weakly concave at $r$; and for it to be optimal to segregate voter types just below $r$, $G$ must be weakly convex at $r$. Hence, bifurcation can occur only at 0, the inflection point of $G$. Moreover, if we take parameters where Y-districting is optimal and increase aggregate uncertainty, it eventually becomes optimal to always segregate voter types just below 0 rather than pairing them with higher voter types, at which point optimal districting becomes SMP (with a bifurcation point below 0). On the other hand, if we take parameters where Y-districting is optimal and decrease aggregate uncertainty, it eventually becomes optimal to always pair voter types just below 0 with higher voter types rather than segregating them, at which point optimal districting becomes SOP (with a bifurcation point above 0).

Supposing that the condition $\gamma \in (0.5, 0.732)$ is sufficient as well as necessary for Y-districting to be optimal, this intuition suggests that, with balanced voter support, SMP is optimal when $\gamma \leq 0.5$, Y-districting is optimal when $\gamma \in (0.5, 0.732)$, and SOP is optimal when $\gamma \geq 0.732$. Figure 1 presents numerical solutions that verify this heuristic. In the figure, $J$ is standard normal and $F$ is uniform on $[-1, 1]$.[31] Voter types are on the $x$-axis, and the threshold shocks to win the districts to which each voter type is assigned are on the $y$-axis. (Thus, segregated districts lie on the 45° line, while paired districts straddle the 45° line.) For mixed districting plans (i.e., Y-districting, the middle row of the figure), the shading intensity indicates the probability that a voter type is assigned to each district. We see that optimal districting takes the conjectured form: SMP is optimal for $\gamma \in \{0.1, 0.3, 0.5\}$, Y-districting is optimal for $\gamma \in \{0.6, 0.65, 0.7\}$, and SOP is optimal for $\gamma \in \{0.8, 0.9, 0.95\}$. The highest value of $\gamma$ in the figure, $\gamma = 0.95$, is the value closest to our empirical estimates. When $\gamma = 0.95$, SOP remains optimal but now closely resembles $p$-segregation. Thus, for what we will see is the empirically relevant parameter range, $p$-segregation is approximately optimal.

We now explain how optimal districting transitions from SMP to SOP as $\gamma$ increases. First, consider the extreme cases where $\gamma \approx 0$ (small idiosyncratic uncertainty) and

[31] To create the figure, we approximated the designer's problem by a finite-dimensional linear program and solved it using Gurobi Optimizer. Our approximation specifies that $s$ is uniformly distributed on $\{-1, -0.99, \ldots, 0.99, 1\}$ and that the designer is constrained to create districts $P$ satisfying $r^*(P) \in \{-1, -0.99, \ldots, 0.99, 1\}$.




(a) $\gamma = 0.1$ (b) $\gamma = 0.3$ (c) $\gamma = 0.5$

(d) $\gamma = 0.6$ (e) $\gamma = 0.65$ (f) $\gamma = 0.7$

(g) $\gamma = 0.8$ (h) $\gamma = 0.9$ (i) $\gamma = 0.95$

FIGURE 1.—Optimal Districting with Balanced Support as Share of Idiosyncratic Uncertainty Varies
*Notes:* The optimal districting plan is SMP for $\gamma \in \{0.1, 0.3, 0.5\}$, Y-districting for $\gamma \in \{0.6, 0.65, 0.7\}$ (and, specifically, mixed SMP for $\gamma \in \{0.6, 0.65\}$ and mixed SOP for $\gamma = 0.7$), and SOP for $\gamma \in \{0.8, 0.9, 0.95\}$. Our empirical estimates of $\gamma$ in Section 5 are above 0.96 for all US states.

$\gamma \approx 1$ (small aggregate uncertainty). When $\gamma \approx 0$, SMP is optimal; moreover, optimal districting approximates NAD with a 50-50 split in each district, which implies that the bifurcation point is below 0 and the range of values of $r^*(P)$ across paired districts

$P$ is large.[32] When $\gamma \approx 1$, SOP is optimal; moreover, paired districts are almost equally strong (the range of $r^*(P)$ across paired districts $P$ is small), which implies that the bifurcation point is above 0.[33] Now, as $\gamma$ increases from 0 toward 0.5, the range of $r^*(P)$ across paired districts decreases (since the range of probable aggregate shocks decreases), and the proportion of segregated districts increases. When $\gamma$ reaches 0.5, it becomes optimal to segregate voters with $s = 0$. Since it cannot be optimal to segregate voters with $s > 0$, once $\gamma$ crosses 0.5 it becomes optimal to pair voters with $s$ just above 0 with a few slightly less favorable voters. At this point, districting takes the form of mixed SMP. As $\gamma$ increases farther above 0.5, the range of $r^*(P)$ across paired districts continues to decrease, and the left arm of the "Y" gets steeper as the right arm gets flatter.[34] At some point, the right arm of the Y becomes flatter than the left arm, so that the most extreme left-wing voters have no right-wing voters to match with, at which point these voters are segregated: this point marks the transition from mixed SMP to mixed SOP, which occurs at $\gamma = 2/3$.[35] The $\gamma = 0.65$ and $\gamma = 0.7$ panels in Figure 1 illustrate points just before and just after this transition. As $\gamma$ increases further, more and more mixed unfavorable voters are assigned to paired districts, until all such voters are assigned to paired districts, at which point optimal districting becomes SOP and the bifurcation point becomes positive. This occurs when $\gamma \approx 0.732$. Finally, as $\gamma$ increases beyond 0.732, the range of $r^*(P)$ across paired districts continues to decrease, and the optimal SOP plan eventually approximates $p$-segregation.

Figure 2 illustrates optimal districting for the same parameters as Figure 1, except that now voter types are uniform on $[x-1, x+1]$ where $x$ is scaled to give an expected vote share of 40% (top panels) or 60% (bottom panels). The figure shows that a less popular designer segregates more unfavorable voters, while a more popular designer pools more voters. The last panel shows that NAD (approximating uniform districting) is optimal for a designer with a 60% expected vote share and $\gamma = 0.9$.

Figure 3 illustrates the form of optimal districting as a function of the designer's expected vote share and $\gamma$. The figure continues to assume that $J$ is standard normal and $F$ is uniform on $[x-1, x+1]$, where $x$ is scaled so that the designer's expect vote share ranges from 0 to 1. The figure shows that segregation is optimal for an unpopular designer (unless aggregate uncertainty dominates), NAD is optimal for a popular one, and optimal districting ranges from SMP to Y-districting to SOP as $\gamma$ ranges from 0 to 1 with balanced voter support. These results match Theorems 2–5.[36]

Figure 3 also plots our point estimates of a Republican designer's expected vote share and $\gamma$ for every US state. (The data and estimation procedure is described in the next section.) The most important observation is that $\gamma$ is close to 1 in every state: the mean estimate of $\gamma$ is 0.986, and the lowest estimate (for North Carolina) is 0.962. These estimates are all far above the cutoff of 0.732 above which SOP is optimal with

[32] Another property of optimal SMP plans is that the left arm of the "Y" is infinitely steep at the bifurcation point: $\lim_{r \downarrow r^b} s_1'(r) = 0$.

[33] Another property of optimal SOP plans is that pairing at the bifurcation point is smooth: $\lim_{r \downarrow r^b} s_1'(r) = -\infty$ and $\lim_{r \downarrow r^b} s_2'(r) = \infty$.

[34] The proof of Theorem 5 shows that, for all sufficiently small positive $r$, $|s_1'(r)|$ is decreasing in $\gamma$ (i.e., the left arm gets steeper) and $s_2'(r)$ is increasing in $\gamma$ (i.e., the right arm gets flatter).

[35] The transition point is the unique value of $\gamma$ at which $\lim_{r \downarrow 0} |s_1'(r)| = \lim_{r \downarrow 0} s_2'(r)$.

[36] Due to numerical error, it is difficult to confidently classify optimal plans within one or two grid points of the boundaries between the regions where different plans are optimal in Figure 3. (By continuity, plans of different forms are both approximately optimal near the boundary.) The boundaries should thus be viewed as approximations.

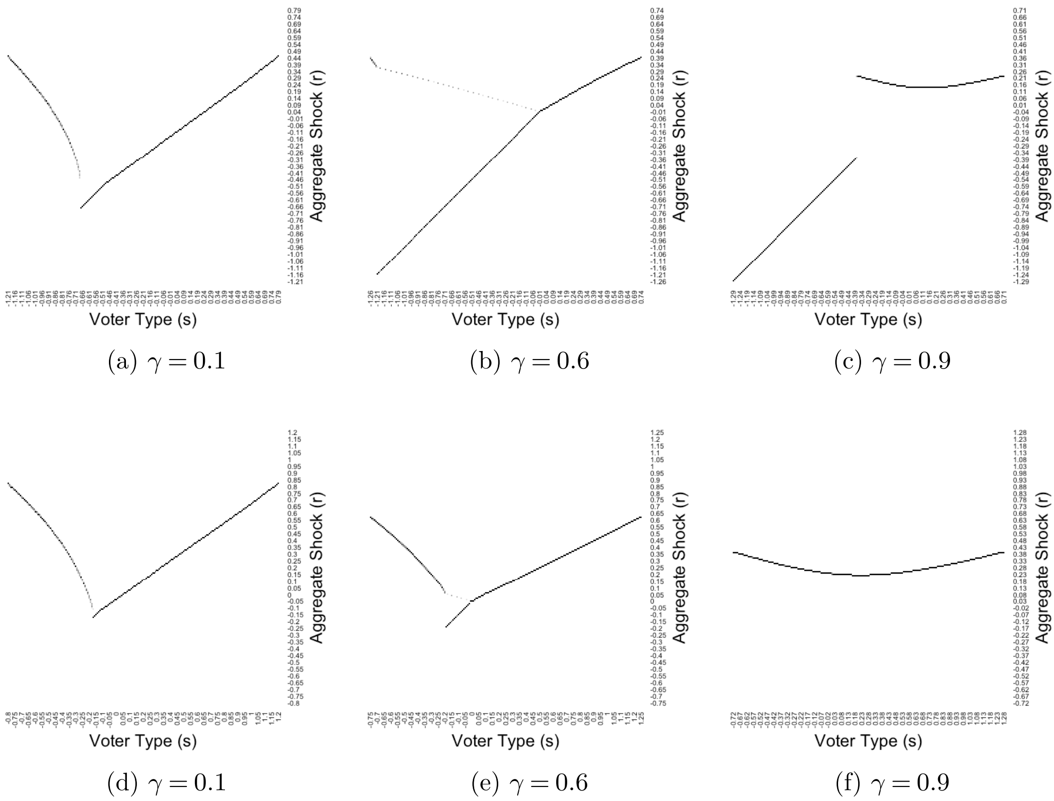



FIGURE 2.—Optimal Districting with Imbalanced Support as Share of Idiosyncratic Uncertainty Varies
*Notes:* In the top and bottom panels, the designer's expected vote share is 40% and 60%, respectively.

balanced voter support. Thus, NAD (approximating uniform districting) is optimal for a Republican designer in Republican states like Oklahoma and Louisiana, while SOP (approximating pack-and-crack) is optimal for a Republican designer in swing states like Michigan and North Carolina, as well as in Democratic states like New York and Maryland (in the fanciful event that the Republicans found themselves controlling districting in such states).[37]

REMARK 1—**Approximate Optimality of Pack-and-Crack**: Lemma 11 in the appendix shows that uniform districting (for a designer with majority support) or pack-and-crack districting (for a designer with minority support) is approximately optimal with small aggregate uncertainty. The intuition is simple: with small aggregate uncertainty, a designer with minority support can obtain an expected seat share of approximately $1 - F(s^*(0))$ by creating slightly fewer than $1 - F(s^*(0))$ identical districts each with an expected vote share slightly greater than $1/2$, and $1 - F(s^*(0))$ is the optimal expected seat share in the limit. Indeed, pack-and-crack districting is approximately

[37]A caveat is that Figure 3 is a 2-dimensional plot and thus neglects heterogeneity in the variance of $s$ across states, which we also estimate. It turns out that assuming that the variance of $s$ is $1/\sqrt{3} \approx 0.577$ in all states—which is implicitly what Figure 3 does—yields the correct classification of optimal districting for every state except Hawaii, where Republican-optimal districting is actually segregation (see Table I in Section 5).

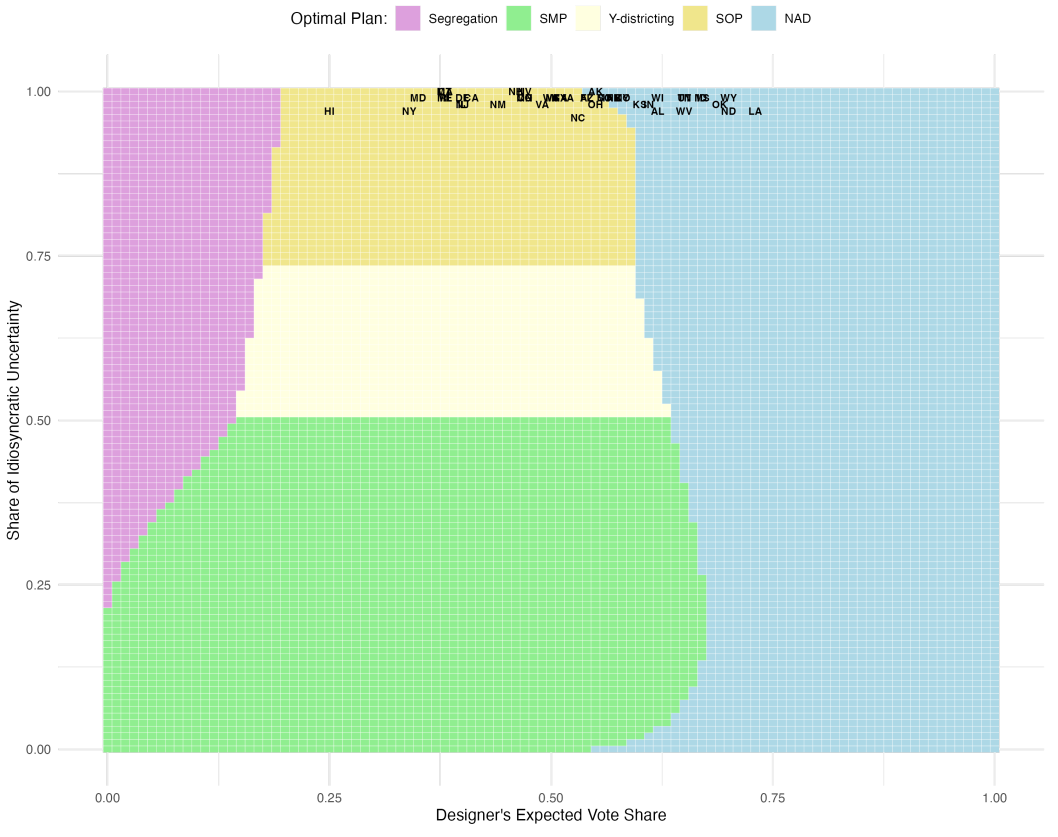



FIGURE 3.—Optimal Districting as Designer's Popularity and Share of Idiosyncratic Uncertainty Vary
*Notes:* Each US state is located at its point estimate in Table I in Section 5.

optimal for realistic parameters. For the same parameters as in Figure 1, Figure 4 plots the expected seat share under the optimal pack-and-crack plan and under the unconstrained optimal plan. The figure shows that the unconstrained expected seat share never exceeds the pack-and-crack expected seat share by more than 0.1% for any value of $\gamma$ above 0.95. (Recall that our lowest estimate of $\gamma$ for any US state is 0.962.) We also estimate that the maximum loss from pack-and-crack relative to optimal districting in any US state (accounting for unbalanced voter support) is 0.56% (see Table I in Section 5).[38]

## 5. ESTIMATION

We have seen that the form of optimal districting depends on the designer's expected vote share and the parameter $\gamma$ (the share of idiosyncratic uncertainty). We now estimate these parameters using precinct-level returns from recent US House elections.

[38]The maximum loss is attained by Rhode Island, a Democratic state where Republicans are very unlikely to ever control districting. If we exclude the Democratic states of Rhode Island, Massachusetts, and Maine, the maximum estimated loss from pack-and-crack relative to optimal districting in any US state is 0.09%.

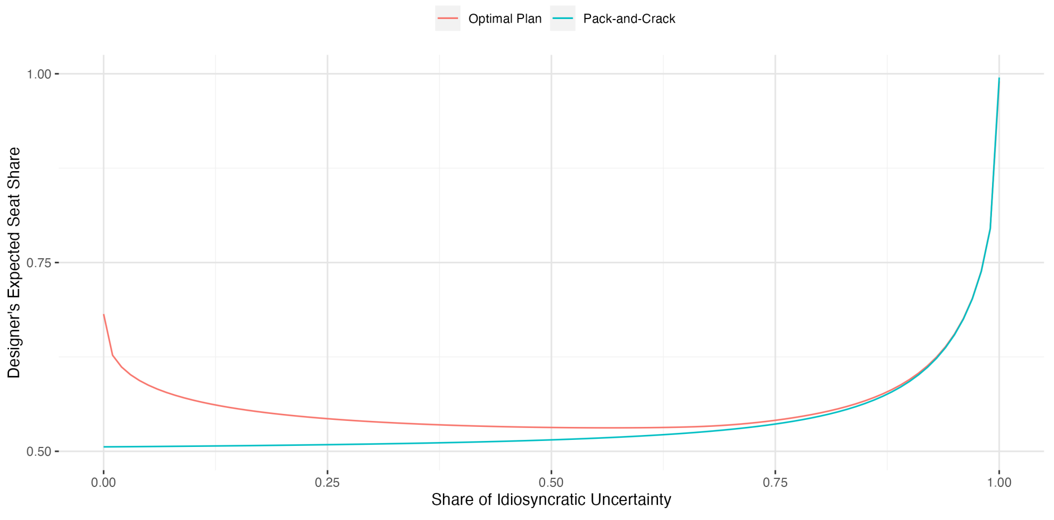



FIGURE 4.—Expected Seat Share under Optimal Districting and Optimal Pack-and-Crack Districting

### 5.1. *Data and Empirical Model*

Our data are the precinct-level returns from the US House elections in 2016, 2018, and 2020, which were recently standardized by Baltz et al. (2022). For each precinct $n$ and election year $y \in \{2016, 2018, 2020\}$, we observe the total two-party vote $k_{ny}$ and the share of the two-party vote for the Republican candidate $v_{ny}$.[39] The data are a repeated cross-section rather than a panel, because there is no general way to match precincts across elections (Baltz et al. 2022, p. 6). We drop all districts with an uncontested House race in any of 2016, 2018, or 2020 (which drops 25% of all districts).[40] This results in dropping South Dakota and Vermont, as in these states the single at-large district was uncontested in 2020 and 2016, respectively; we also drop Pennsylvania, as it was redistricted between 2016 and 2018. For each election, we also drop precincts with fewer than 50 total votes (which drops 0.14% of all votes) or where the Republican vote share is 0 or 1 (which drops an additional 0.015% of votes).

To take the model to these data, we assume that $s$ indexes precincts, so that $Q(s-r)$ is the designer's vote share in a type-$s$ precinct at aggregate shock $r$. Formally, this

[39] A "precinct" is the smallest election-reporting unit in a state, which typically corresponds to a geographic area where all voters vote at the same polling place. Maine and New Jersey report election returns only at the township level, so for these states $n$ indexes townships rather than precincts. For some elections where a nominally third-party candidate runs in place of an official Democratic or Republican candidate, we manually re-label this candidate as a Democrat or Republican. For example, in New York, we re-assign Working Families Party candidates as Democrats and re-assign Conservative Party candidates as Republicans. Throughout, we focus on the two-party vote $k_{ny}$ and the Republican share of the two-party vote $v_{ny}$, ignoring third parties.

[40] Keeping these districts would bias our estimate of $\gamma$, because the relevant vote shares are for contested elections, and if these districts were contested their vote shares would be different from 0 or 1. Keeping a district with one or two uncontested elections only for the elections where it is contested would also bias our estimate of $\gamma$, by distorting the estimated swing across elections. Dropping uncontested districts does likely bias our estimate of the voter type distribution $F$, as uncontested districts are presumably more extreme; however, this bias is irrelevant for our main goal of estimating $\gamma$.

is equivalent to assuming that all voters in a precinct have the same type. (As we clarify below, this does not mean that all voters in a precinct vote the same way.) We also assume that precincts are relatively large (in the data, the mean precinct vote count is 794 with standard deviation 1,434, after dropping precincts with fewer than 50 total votes or a 0 or 1 vote share), and idiosyncratic voter taste shocks are normally distributed.[41] By the law of large numbers, this implies that the designer's vote share in a precinct $n$ with type $s_n$ in district $d$ and election $y$ is given by

$$Q(s_n - r_{dy}) = \Phi\left(\frac{s_n - r_{dy}}{\sqrt{\gamma}}\right), \tag{1}$$

where $\Phi$ is the standard normal distribution. To see this, recall that each voter $i$ in precinct $n$ votes for the designer's party iff $s_n \geq r_{dy} + t_{iy}$, where $t_{iy}$ is the voter's normally distributed idiosyncratic taste shock, and hence votes for the designer's party with probability $\Phi((s_n - r_{dy})/\sqrt{\gamma})$.[42] Applying the law of large numbers at the precinct level gives (1).

We emphasize that this empirical model does not allow precinct-level aggregate shocks: the vote share $Q(s_n - r_{dy})$ in precinct $n$ in district $d$ and election $y$ is given by (1), which is a deterministic function of the persistent precinct type $s_n$ and the district-level aggregate shock in election $y$, $r_{dy}$.

To interpret the assumption that all voters in a precinct have the same type, note that a voter's type and taste shock enter only through their difference $s_n - t_{iy}$. If we call this difference the voter's "preference," our assumption is that voter preferences in precinct $s_n$ are normally distributed with mean $s_n$ and variance $\gamma$. Also, while voter preferences must be independent across voters in each district to justify (1), the correlation of each voter's preference across elections is arbitrary. Thus, voters in a precinct can differ in their persistent tastes for the parties as well as in their election-specific tastes.

Remark 2—**What if Precincts Can be Split?**: Our estimation assumes that the smallest "districtable unit" is a precinct, which is the smallest election-reporting unit in a state. In practice, the smallest district able unit is usually not a precinct but a census block, which is the smallest geographic unit for which the US Census tabulates complete data. Census blocks are usually much smaller than precincts. However, Bouton, Genicot, Castanheira, and Stashko (2024) report that only 2% of precincts are split across proposed congressional districts in their sample. In addition, in Section 6.2 we redo our estimation under the assumption that designers can only assign counties rather than precincts and find that that this increases our estimate of $\gamma$ by only about 0.001. The difference in size between a county and precinct is roughly similar to that between a precinct and a census block: there are around 50 times as many precincts as counties in the US, and around 50 times as many census blocks as precincts. This suggests that our estimates are reasonably robust to letting designers split precincts.

However, precincts (and even census blocks) sometimes are split, and some designers strive to split them as finely as possible. For example, in *Dickson v. Rucho* (2014),

[41] Our estimates are not sensitive to assuming normality: because we will find that $\gamma$ is very large, the taste shock distribution is approximately uniform over the relevant range, so specifying any smooth taste shock distribution leaves our estimates almost unchanged. For example, our point estimate of $\gamma$ for the US as a whole is 0.986 with normal taste shocks, 0.987 with logistic taste shocks, 0.989 with Laplace taste shocks, and 0.981 with uniform taste shocks.

[42] In this section, as in Section 4.3, we assume that $Q(t) = \Phi(t/\sqrt{\gamma})$ and $G(r) = \Phi(r/\sqrt{1-\gamma})$.

plaintiffs alleged that a Republican-drawn map in North Carolina "divides 563 of the state's 2,692 precincts into more than 1,400 sections," (Newkirk, 2017). If designers can split precincts extremely finely, so as to isolate individual voters or very small groups of voters, this could substantially affect our estimate of $\gamma$ and our conclusions about the form of gerrymandering. We are not aware of evidence that fine precinct-splitting or census block-splitting is widespread, but we acknowledge that, to the extent that this is the case or may become so in the future, our analysis would have to be redone at the level of the smallest districtable unit (subject to data limitations, as our estimation already uses the finest currently available data).

### 5.2. *Descriptive Figures and Summary Statistics*

We first present a histogram (Figure 5(a)) showing the number of voters in the United States who live in a precinct with Republican vote share $v$, with bin breaks $\{0, 0.05, \ldots, 0.95, 1\}$, averaging over elections $y \in \{2016, 2018, 2020\}$. The histogram shows that the distribution of $v_{ny}$ is unimodal, with a large majority (74%) of the mass on $v \in [0.25, 0.75]$. This pattern has two simple, but important, implications for our model. First, the distribution of voter/precinct types is far from bimodal: there is a continuum of types, with most mass "toward the middle." A designer choosing how to assign precincts to districts thus faces a continuum of types, as in our model. Second, idiosyncratic uncertainty appears large relative to aggregate uncertainty. To see this, note that as if idiosyncratic uncertainty dominates ($\gamma \to 1$), Figure 5(a) would show a unimodal density with mode around $v = 1/2$ (as the distribution of the precinct vote share $v$ is $F(Q^{-1}(v))$), while if aggregate uncertainty dominates ($\gamma \to 0$), it would show a bimodal distribution with all mass at 0 and 1. The former case is a much better approximation, as the distribution in Figure 5(a) is unimodal, with 74% of the mass on $v \in [0.25, 0.75]$. While we quantitatively estimate $\gamma$ in the next subsection, this observation already suggests what we will find, which is that $\gamma$ is much greater than 0.5.

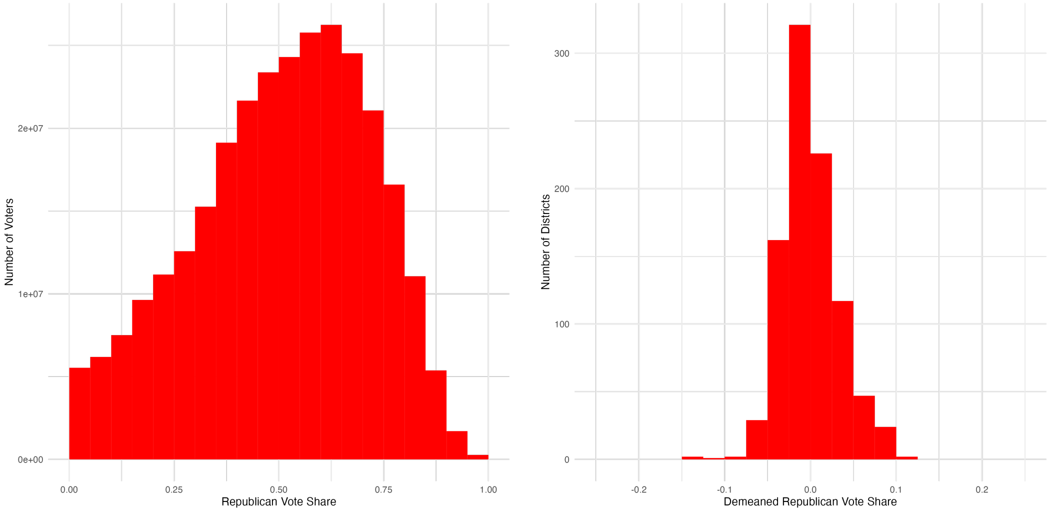
Next we present another histogram (Figure 5(b)), which shows the number of *(district, election)* pairs where the district-wide Republican vote share deviated from its mean over the three elections we consider by $x$, with bin breaks $\{-0.25, -0.225, \ldots, 0.225, 0.25\}$.[43] This histogram gives another way of showing that aggregate shocks are small: the distribution is centrally unimodal, and most of the mass (59%) is on $x \in [-0.025, 0.025]$. In contrast, if aggregate shocks were very large, we would again have a bimodal distribution with all mass far from 0.

### 5.3. *Estimates*

We now estimate the key parameter $\gamma$, as well as the other parameters. Since districting plans in the US are drawn at the state level, we estimate parameters separately for each US state.[44] We assume that aggregate shocks are jointly normally distributed across districts and independent across elections, so that the variance of $r_{dy}$ is $1-\gamma$; the correlation between $r_{dy}$ and $r_{d'y}$ is $\rho$ for each $d \neq d'$ and $y$; and the correlation between $r_{dy}$ and $r_{d'y'}$ is 0 for each $d$, $d'$, and $y \neq y'$. Recall that the results in Section 4.3 show

[43]This histogram is compiled at the district level because precincts are not matched across elections.

[44]While our model assumes a large number of districts, we estimate parameters for all states (including ones with only one congressional district) to give as complete parameter estimates as possible.




(a) Precinct Vote Shares (b) District Vote Swings

FIGURE 5.—Distributions of Precinct Vote Shares and District Vote Swings

that, with balanced voter support, SMP is optimal if $\gamma \leq 0.5$, Y-districting is optimal if $\gamma \in (0.5, 0.732)$, and SOP is optimal if $\gamma \geq 0.732$. Thus, a key question of interest is which of these three regions contains our estimate of $\gamma$.

We estimate $\gamma$ for each state by method of moments. Recall that $v_{ny}$ is the Republican share of the two-party vote in precinct $n$ and election $y$. Let $w_{ny} = \Phi^{-1}(v_{ny})$, the corresponding standard normal quantile. Let $T = 3$ denote the number of elections, $D$ the number of districts in the state, and $\mathcal{N}_{dy}$ the set of precincts in district $d$ and election $y$. Next, define

$$w_{dy} = \sum_{n \in \mathcal{N}_{dy}} k_{ny} w_{ny} \Big/ \sum_{n \in \mathcal{N}_{dy}} k_{ny} \quad \text{and} \quad w_{d\bullet} = \frac{1}{T} \sum_{y} w_{dy}.$$

That is, $w_{dy}$ is the average value of $w_{ny}$ over precincts in district $d$, weighted by the number of votes in each precinct; and $w_{d\bullet}$ is the average value of $w_{dy}$ over elections $y$. It can then be shown that a consistent estimator of $\gamma$ is given by

$$\widehat{\gamma} = 1 \Big/ \left(1 + \tfrac{1}{D(T-1)} \sum_{d,y} (w_{dy} - w_{d\bullet})^2\right).$$

In the Online Appendix, we also construct a confidence interval for $\gamma$, as well as a point estimator of the correlation among the district-level aggregate shocks, and point estimators of the mean and standard deviation of the distribution of precinct types.

Table I displays the resulting estimates for each US state, as well as for the state average weighted by the number of districts included in the analysis (row WS) and the US as a whole (row US). The states are ordered by column $v$, the designers expected vote share in the districts included in the estimation. Columns $D_T$ and $D_A$ are the total number of districts and the number of districts included in the analysis.

Columns $\gamma$ and $\underline{\gamma}$ are our point estimate and the lower bound of a 95% one-sided confidence interval for $\gamma$. The confidence interval is wide because we only have data from three elections: $T = 3$. However, it is clear that $\gamma$ is far above the critical value of 0.732. The lowest point estimate for $\gamma$ for any state is 0.962 in North Carolina, and the weighted mean estimate for $\gamma$ and the estimate for $\gamma$ for the US as a whole are both 0.986. Moreover, even with $T = 3$, the lower bound of a 95% one-sided confidence interval is above 0.732 for all available states except North Dakota, where the lower endpoint is 0.619. If we expand our dataset to include the returns from the 2012 and 2014 elections (thus covering all five congressional elections held under the 2010 districting plans), the lower endpoints of the 95% confidence interval exceeds 0.732 for all states, including North Dakota.[45] Together with the results in Section 4.3 (including Figure 3, which accounts for imbalances in voter support), this provides strong evidence that optimal gerrymandering is given by SOP (for a designer with minority support) or NAD (for a designer with majority support) for realistic parameters. Moreover, our estimates for $\gamma$ are high enough that the optimal SOP plan approximates $p$-segregation and the optimal NAD plan approximates uniform districting (recall Figures 1–4), and that pack-and-crack (with minority support) or uniform districting (with majority support) is approximately optimal.

Columns $v$ and $\sigma_s$ are the designer's expected vote share and the standard deviation of $s$. The latter estimates are similar to those in Figure 1. However, our estimates of $v$ and $\sigma_s$ may be biased by dropping uncontested elections (unlike our estimates of $\gamma$, which remain unbiased after dropping any set of districts).[46] Column $\sigma_s^c$ is the standard deviation of $s$ across counties rather than precincts. We discuss county-level estimates in Section 6.2.

Columns $V$ and $\underline{V}$ are the designer's expected seat share under optimal unconstrained districting and optimal pack-and-crack districting, respectively. As illustrated in Figure 4, the shares are very similar. Column $V^c$ is the expected seat share under optimal districting where the designer assigns counties rather than precincts: see Section 6.2.

Finally, Column $\mathcal{H}$ is the form of the optimal districting plan at the estimated parameters. We estimate that if Republicans somehow found themselves in charge of districting Hawaii, they would segregate the state. Otherwise, SOP is optimal (and pack-and-crack is approximately optimal) in states where the expected Republican vote share is less than 55%, and NAD is optimal (and uniform districting is approximately optimal) in states where the expected Republican vote share is greater than 55%. This reflects the fact that, for our estimated value of $\gamma$, the optimal pack-and-crack plan creates cracked districts where the designer's expected seat share is around 55%.

## 6. DISCUSSION: WHY DOES THE FORM OF GERRYMANDERING MATTER?

We briefly discuss potential political and legal implications of our results. We consider three areas: implications for how regulations and restrictions on districting affect partisan representation; implications for how gerrymandering affects political competition and polarization; and implications for detecting and measuring gerrymandering.

[45] Precinct-level returns for 2012 and 2014 have been compiled by Ansolabehere, Palmer, and Lee (2014) but are less complete and less standardized than the Baltz et al. (2022) data we use, which only cover 2016, 2018, and 2020. We have checked that all of our empirical results are robust to including the 2012 and 2014 data.

[46] We also estimate the correlation $\rho$ among the district-level aggregate shocks to be 0.317 (at the country level). Since this estimate is not close to either 0 or 1, estimating a simpler empirical model where district-level shocks are either uncorrelated or perfectly correlated would yield biased estimates of $\gamma$.

| US | $D_T$ | $D_A$ | $\gamma$ | $\underline{\gamma}$ | $\upsilon$ | $\sigma_s$ | $\sigma_s^c$ | $V$ | $\underline{V}$ | $V^c$ | $\mathcal{H}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| HI | 2 | 2 | 0.972 | 0.839 | 0.250 | 0.181 | 0.076 | 0.001 | 0.001 | 0.000 | Seg |
| NY | 27 | 19 | 0.966 | 0.937 | 0.342 | 0.826 | 0.659 | 0.416 | 0.415 | 0.356 | SOP |
| MD | 8 | 8 | 0.990 | 0.978 | 0.346 | 0.728 | 0.624 | 0.456 | 0.456 | 0.410 | SOP |
| RI | 2 | 1 | 0.990 | 0.833 | 0.375 | 0.302 | 0.259 | 0.199 | 0.194 | 0.145 | SOP |
| CT | 5 | 5 | 0.995 | 0.987 | 0.379 | 0.377 | 0.327 | 0.328 | 0.328 | 0.263 | SOP |
| ME | 2 | 1 | 0.992 | 0.866 | 0.385 | 0.311 | 0.304 | 0.246 | 0.244 | 0.236 | SOP |
| MA | 9 | 1 | 0.998 | 0.956 | 0.385 | 0.233 | 0.211 | 0.142 | 0.139 | 0.098 | SOP |
| DE | 1 | 1 | 0.990 | 0.836 | 0.397 | 0.488 | 0.268 | 0.456 | 0.456 | 0.228 | SOP |
| IL | 18 | 13 | 0.984 | 0.962 | 0.399 | 0.737 | 0.545 | 0.560 | 0.560 | 0.463 | SOP |
| NJ | 12 | 12 | 0.981 | 0.962 | 0.402 | 0.590 | 0.445 | 0.492 | 0.492 | 0.411 | SOP |
| CA | 53 | 35 | 0.992 | 0.987 | 0.412 | 0.483 | 0.337 | 0.508 | 0.508 | 0.382 | SOP |
| NM | 3 | 3 | 0.979 | 0.925 | 0.436 | 0.543 | 0.412 | 0.548 | 0.548 | 0.476 | SOP |
| NH | 2 | 2 | 0.997 | 0.960 | 0.463 | 0.263 | 0.256 | 0.575 | 0.575 | 0.568 | SOP |
| NV | 4 | 4 | 0.998 | 0.992 | 0.467 | 0.449 | 0.310 | 0.741 | 0.741 | 0.662 | SOP |
| MN | 8 | 8 | 0.987 | 0.973 | 0.470 | 0.436 | 0.342 | 0.626 | 0.626 | 0.569 | SOP |
| CO | 7 | 7 | 0.989 | 0.953 | 0.470 | 0.527 | 0.429 | 0.691 | 0.691 | 0.649 | SOP |
| OR | 5 | 4 | 0.987 | 0.961 | 0.471 | 0.498 | 0.377 | 0.662 | 0.662 | 0.603 | SOP |
| VA | 11 | 8 | 0.985 | 0.934 | 0.491 | 0.548 | 0.428 | 0.726 | 0.726 | 0.684 | SOP |
| WA | 10 | 5 | 0.987 | 0.960 | 0.496 | 0.375 | 0.264 | 0.689 | 0.689 | 0.624 | SOP |
| MI | 14 | 13 | 0.990 | 0.980 | 0.501 | 0.596 | 0.503 | 0.803 | 0.802 | 0.772 | SOP |
| GA | 14 | 7 | 0.985 | 0.959 | 0.509 | 0.718 | 0.525 | 0.821 | 0.820 | 0.781 | SOP |
| TX | 36 | 23 | 0.989 | 0.978 | 0.514 | 0.645 | 0.472 | 0.841 | 0.841 | 0.811 | SOP |
| IA | 4 | 4 | 0.986 | 0.949 | 0.519 | 0.372 | 0.270 | 0.763 | 0.763 | 0.726 | SOP |
| NC | 13 | 11 | 0.962 | 0.933 | 0.526 | 0.560 | 0.373 | 0.740 | 0.739 | 0.699 | SOP |
| AZ | 9 | 6 | 0.990 | 0.974 | 0.537 | 0.402 | 0.286 | 0.868 | 0.868 | 0.852 | SOP |
| FL | 27 | 20 | 0.994 | 0.987 | 0.545 | 0.444 | 0.291 | 0.949 | 0.948 | 0.948 | SOP |
| OH | 16 | 16 | 0.984 | 0.967 | 0.552 | 0.635 | 0.469 | 0.908 | 0.908 | 0.896 | NAD |
| AK | 1 | 1 | 0.996 | 0.922 | 0.554 | 0.396 | 0.298 | 0.987 | 0.987 | 0.983 | NAD |
| MT | 1 | 1 | 0.993 | 0.884 | 0.556 | 0.490 | 0.325 | 0.973 | 0.973 | 0.973 | NAD |
| SC | 7 | 7 | 0.994 | 0.988 | 0.559 | 0.622 | 0.402 | 0.990 | 0.990 | 0.988 | NAD |
| AR | 4 | 1 | 0.985 | 0.773 | 0.566 | 0.629 | 0.391 | 0.950 | 0.949 | 0.945 | NAD |
| NE | 3 | 2 | 0.990 | 0.945 | 0.575 | 0.446 | 0.297 | 0.981 | 0.981 | 0.979 | NAD |
| KY | 6 | 4 | 0.991 | 0.968 | 0.584 | 0.548 | 0.408 | 0.994 | 0.994 | 0.992 | NAD |
| MO | 8 | 8 | 0.995 | 0.981 | 0.584 | 0.702 | 0.579 | 1.000 | 1.000 | 1.000 | NAD |
| KS | 4 | 4 | 0.978 | 0.905 | 0.598 | 0.463 | 0.355 | 0.968 | 0.968 | 0.971 | NAD |
| IN | 9 | 8 | 0.983 | 0.963 | 0.608 | 0.524 | 0.351 | 0.991 | 0.991 | 0.990 | NAD |
| WI | 8 | 5 | 0.989 | 0.970 | 0.617 | 0.301 | 0.228 | 0.998 | 0.998 | 0.999 | NAD |
| AL | 7 | 2 | 0.971 | 0.846 | 0.624 | 0.674 | 0.409 | 0.988 | 0.988 | 0.992 | NAD |
| WV | 3 | 3 | 0.971 | 0.883 | 0.646 | 0.340 | 0.252 | 0.989 | 0.989 | 0.989 | NAD |
| UT | 4 | 4 | 0.989 | 0.947 | 0.647 | 0.585 | 0.472 | 1.000 | 1.000 | 1.000 | NAD |
| TN | 9 | 8 | 0.991 | 0.976 | 0.650 | 0.691 | 0.526 | 1.000 | 1.000 | 1.000 | NAD |
| MS | 4 | 2 | 0.993 | 0.934 | 0.672 | 0.671 | 0.340 | 1.000 | 1.000 | 1.000 | NAD |
| ID | 2 | 2 | 0.987 | 0.930 | 0.673 | 0.462 | 0.365 | 1.000 | 1.000 | 1.000 | NAD |
| OK | 5 | 4 | 0.983 | 0.922 | 0.685 | 0.454 | 0.318 | 1.000 | 1.000 | 1.000 | NAD |
| ND | 1 | 1 | 0.969 | 0.619 | 0.696 | 0.426 | 0.335 | 0.999 | 0.999 | 0.999 | NAD |
| WY | 1 | 1 | 0.990 | 0.835 | 0.701 | 0.478 | 0.375 | 1.000 | 1.000 | 1.000 | NAD |
| LA | 6 | 4 | 0.974 | 0.898 | 0.725 | 0.595 | 0.288 | 1.000 | 1.000 | 1.000 | NAD |
| WS | 18 | 13 | 0.986 | 0.959 | 0.497 | 0.561 | 0.415 | 0.755 | 0.754 | 0.709 | SOP |
| US | 417 | 311 | 0.986 | 0.979 | 0.497 | 0.643 | 0.508 | 0.777 | 0.776 | 0.745 | SOP |

TABLE I: Estimates

### 6.1. *Effects of Districting Reforms on Seat Shares I: Majority-Minority Districts*

The key US federal laws regulating gerrymandering are the Equal Protection Clause of the Fourteenth Amendment and the Voting Rights Act of 1965. These laws have been interpreted as not only prohibitting adverse racial gerrymandering, but also as affirmatively requiring states to create electoral districts where racial or ethnic minority voters form either a majority (a so-called "majority-minority district") or a large enough minority so as to have a strong opportunity to elect their candidate of choice (often called a "minority opportunity district"; e.g., Canon 2022). The creation of such districts played a significant role in increasing Black representation in state legislatures and the US Congress from the 1970's onward, especially in the South (Grofman and Handley 1991, Cox and Holden 2011). However, the overall partisan impact of majority-minority and minority opportunity districts has long been contested, with some observers arguing that these districts effectively pack strong Democratic supporters and thus resemble a component of a Republican-optimal districting plan. This issue came to a head following the 1994 Republican takeover of the US House, which many journalists and political scientists blamed in part on the creation of majority-minority districts in the 1990 redistricting cycle (but see Cox and Holden 2011, Washington 2012).

Previous studies have observed that the impact of a requirement to create majority-minority or minority opportunity districts on overall partisan representation hinges on the form of optimal gerrymandering. The convential view in the 1990's (what Cox and Holden 2011 call the "pack-and-crack consensus") was that optimal gerrymandering packs opponents, and hence that a requirement to create majority-minority districts that pack strong Democratic supporters is likely to increase overall Republican representation.[47] Shotts (2001) adds an important caveat by noting that, since uniform districting is optimal for a designer with majority support (without aggregate uncertainty), majority-minority mandates hurt Republican designers in strongly Republican states. More dramatically, building on the results of Friedman and Holden (2008), Cox and Holden (2011) challenge the pack-and-crack consensus by arguing that optimal districting is given by NAD, and thus packs moderates rather than opponents. Since NAD does not create districts packed with strong Democratic supporters, Cox and Holden argue that a requirement to create such districts precludes NAD and is therefore likely to reduce overall Republican representation.

Our results contribute to this debate as follows. Cox and Holden's argument that NAD is optimal in practice rests on an implicit assumption that the low-idiosyncratic-uncertainty case studied by Friedman and Holden (2008) is representative. For example, Cox and Holden write, "In a world with diverse voter types, however, there is no plausible distribution of African American voters that would make it optimal for Republican redistricting authorities to create districts in which African Americans make up a supermajority of voters. Within the model, packing one's opponents is never the optimal strategy," (p. 574). We instead show that, empirically, idiosyncratic uncertainty is much larger than aggregate uncertainty, and that this implies that packing opponents is optimal for a designer with minority voter support, while NAD is optimal for a designer with majority support. Thus, majority-minority mandates can increase Republican representation in closely divided states where SOP is optimal and pack-and-crack is approximately optimal (as in the pack-and-crack consensus), but are likely

[47] Minority opportunity districts may or may not raise similar issues, depending on the share of strong Democratic supporters in these districts (Lublin, Handley, Brunell, and Grofman, 2020).

to decrease Republican representation in strongly Republican states where NAD is optimal and uniform districting is approximately optimal (as argued by Shotts (2001) in a model without aggregate uncertainty). Overall, by analyzing a general model that does not restrict the relative amounts of idiosyncratic and aggregate uncertainty, we reach a conclusion similar to that of Shotts (2001) and quite different from that of Cox and Holden (2011).

### 6.2. *Effects of Districting Reforms on Seat Shares II: Respecting Political Subdivisions*

Among the practical restrictions on districting beyond equipopulation, one that is amendable to our analysis is a requirement not to split counties or other political subdivisions. Preserving counties or other subdivisions is one of the six traditional redistricting criteria according to the National Conference of State Legislators and is currently required in 29 of the 50 US states.[48] From the perspective of partisan gerrymandering, a requirement to preserve counties constrains the designer to choose among a coarser set of districting plans, where counties rather than census blocks or precincts become the object of districting.

We can assess the impact of a requirement to preserve counties by re-running our estimation of $\gamma$ and $F$, taking the unit of districting as counties rather than precincts. Our estimates of $\gamma$ are similar in both cases but are slightly higher with counties, because precinct vote shares swing slightly more from election to election than county vote shares: our mean precinct-based estimate of $\gamma$ is 0.986, while our mean county-based estimate is 0.987. More importantly, our estimate of the standard deviation of $F$ is considerably smaller with counties: the mean precinct-based estimate is 0.561, while the mean county-based estimate is 0.415. This gap is the key consequence of constraining the designer to assigning coarser units. Finally, this constraint significantly affects the designer's optimal expected seat share in closely divided states where SOP is optimal, as now fewer highly unfavorable units can be packed; however, it has only a small effect on the optimal seat share in states where the designer has strong support and NAD is optimal, as uniform districting (which is unaffected by a requirement to preserve counties) is approximately optimal in these states. In particular, our estimate of the reduction in a Republican designer's seat share from requiring him to preserve counties ranges from essentially 0 in strongly Republican states to 23% in Delaware, with a weighted average across states of 4.5%.[49]

### 6.3. *Effects of Gerrymandering on Political Competition and Polarization*

An important debate concerns the impact of gerrymandering on the intensity of electoral competition (e.g., the fraction of "competitive" districts or the extent of incumbency advantage) and political polarization. Popular discourse often blames gerrymandering for reducing competition and increasing polarization (but see Gelman and

[48]The other criteria are compactness, contiguity, preservation of communities of interest, preservation of the "cores" of previous districts, and avoiding incumbent pairing (https://www.ncsl.org/elections-and-campaigns/2020-redistricting-criteria).

[49]Our weighted mean estimate of a Republican designer's optimal seat share is 70.9% under county-based districting and 75.4% under precinct-based districting. A limitation of this comparison is that our assumption that the designer assigns a continuum of units is more accurate when units are precincts rather than counties. The omitted integer constraint would bind more harshly for county-based districting, which biases our estimates of the designer's loss from being restricted to assigning counties downward.

King 1994a, Abramowitz, Alexander, and Gunning 2006, McCarty, Poole, and Rosenthal 2009, Friedman and Holden 2009). Regardless of the size of the overall effects of gerrymandering on competition and polarization, the nature of these effects depends on the form of gerrymandering. In particular, under SOP, intra-district polarization is relatively low while inter-district polarization is relatively high; while under NAD or SMP, intra-district polarization is high and inter-district polarization is low. To see this, note that, with a right-wing designer, SOP or pack-and-crack creates a few strongly left-leaning districts and many slightly right-leaning districts, with a gap between the left-leaning and right-leaning districts: formally, there is a gap between the highest value of $r^*(P)$ among segregated districts and the lowest value of $r^*(P)$ among paired districts (see the last three panels in Figure 1). SOP also involves relatively low intra-district polarization within each district, since the lowest voter types in paired districts are "moderates" rather than extreme left-wingers. In contrast, NAD or SMP creates a continuum of districts ranging from left-leaning to right-leaning—formally, the set $\{r : r = r^*(P) \text{ for some } P \in \text{supp}(\mathcal{H})\}$ is an interval (see the first three panels in Figure 1)—with less extreme left-leaning districts than under SOP. NAD or SMP also involves greater intra-district polarization than SOP, in that the maximum range of voter types that are pooled together under SMP is greater than under SOP.

Our model does not encompass endogenous political responses to districting, such as effects of districting on which politicians run for office and on what platforms. With this caveat, we can draw some tentative implications of the above features of optimal districting for political competition and polarization. First, since the distribution of threshold shocks $r^*(P)$ has a gap under SOP or pack-and-crack but not under NAD or SMP, SOP or pack-and-crack may lead to more polarized legislatures, where the packed districts elect left-wing representatives and the cracked districts elect right-leaning ones. Indeed, the possibility that packing opponents can increase polarization in this manner is a long-standing concern (e.g., Cox and Holden 2011, p. 595). In contrast, NAD or SMP may lead to less polarized legislatures. Second, SOP or pack-and-crack may produce more "uncompetitive," far-left districts. Creating uncompetitive districts is usually viewed as a socially undesirable feature of a districting plan, but see Buchler (2005) and Brunell (2008) for opposing views. Finally, lower intra-district polarization under SOP or pack-and-crack may be socially desirable if voters benefit from being ideologically close to their representative, as in Besley and Preston (2007) and Gomberg, Pancs, and Sharma (2023). These and other implications of optimal districting for political processes and outcomes can be studied more fully in models that endogenize additional aspects of political competition.

### 6.4. *Detecting and Measuring Gerrymandering*

A large literature proposes metrics that attempt to detect and measure gerrymandering. Most metrics compare a party's seat share and its vote share, with a high seat share viewed as indicative of gerrymandering.[50] However, a limitation of this approach is that one can debate what range of seat shares is "reasonable" for a given vote share. Indeed, the Supreme Court has objected that this class of metrics encodes a form of

[50]Such measures include the partisan bias (King and Browning, 1987), efficiency gap (Stephanopoulos and McGhee, 2015), mean-median gap (Wang, 2016), and declination (Warrington, 2018). An alternative approach relies on statistical analysis of an ensemble of simulated maps (Deford, Duchin, and Solomon, 2021).

proportionality between seat and vote shares: as Justice Roberts wrote in *Rucho v. Common Cause*, “Partisan gerrymandering claims rest on an instinct that groups with a certain level of political support should enjoy a commensurate level of political power and influence. Such claims invariably sound in a desire for proportional representation, but the Constitution does not require proportional representation.”

Our results suggest an alternative test for gerrymandering that compares vote shares across districts, rather than comparing seat and vote shares. A novel and robust prediction of our analysis is that, in the realistic case of small aggregate uncertainty, optimal plans make favorable districts equally strong: a designer with majority support creates equally strong districts under NAD or uniform districting, while a designer with minority support creates some packed districts that are lost with high probability and creates equally strong districts that are won with high probability under SOP or pack-and-crack. In contrast, there is no reason to expect favorable districts to be equally strong under non-gerrymandered districting. Thus, a proposed test for gerrymandering is whether a districting plan displays an unusually low variance in vote shares among districts won by the designer’s party. This test can be operationalized in future work.

## 7. Conclusion

This paper has developed a simple and general model of optimal partisan gerrymandering. Our main message has four parts. First, optimal districting is “segregate-pair”: weak districts are segregated; strong districts are paired. Second, the optimal form of segregate-pair depends on the gerrymandering party’s popularity and—more subtly—on the relative amounts of aggregate and idiosyncratic uncertainty facing the gerrymanderer. Packing opposing voters is optimal when idiosyncratic uncertainty dominates, while packing moderate voters is optimal when aggregate uncertainty dominates. Third, empirically, idiosyncratic uncertainty dominates, implying that segregate-opponents-and-pair (SOP) districting is optimal for a designer with minority support, while negative assortative districting (NAD) is optimal for a designer with majority support. This finding also establishes that the relevant parameter range for future research on gerrymandering (and electoral competition more generally) is that where aggregate uncertainty is much smaller than idiosyncratic uncertainty. Fourth, estimated aggregate uncertainty is so small that a simple pack-and-crack plan is approximately optimal for a deisgner with minority support, while uniform districting is approximately optimal for a designer with majority support. This last observation helps rationalize the observed use of simple districting plans.

Methodologically, we develop and exploit a tight connection between gerrymandering and information design. We show that a general model of partisan gerrymandering is equivalent to a general Bayesian persuasion problem where the state of the world and the receiver’s action are both one-dimensional and the sender’s preferences are state-independent. This common framework nests the important prior contributions of Owen and Grofman (1988), Friedman and Holden (2008), and Gul and Pesendorfer (2010), and facilitates a more general and realistic analysis that allows diverse voter types and non-linear vote swings without restricting the relative amounts of aggregate and idiosyncratic uncertainty.

We hope our model can inform future research on various aspects of redistricting. We mention a few directions for future research.

First, we have assumed that the designer maximizes his party’s expected seat share. It may be more realistic to assume that the designer’s utility is non-linear in seat shares, for example because of a premium on winning a majority of seats. We examined this case in an earlier version of this paper (Kolotilin and Wolitzky, 2020). While non-linear designer utility introduces new complications, the extreme case where the designer simply maximizes the probability of winning a majority is straightforward: here, optimal districting maximizes seats conditional on the threshold aggregate shock at which the designer is barely able to attain a majority, and hence reduces to optimal districting without aggregate uncertainty.

Second, we have assumed that all voters always vote, or at least always vote at the same rate (as is equivalent). It would be interesting to incorporate heterogeneous turnout in the analysis. Recently, Bouton, Genicot, Castanheira, and Stashko (2024) consider voters with a binary partisan type (as in Owen and Grofman 1988), uniform aggregate shocks, and a continuous "turnout type," which captures fixed turnout heterogeneity across voters. It is promising to explore mutual generalizations of our models that allow more general forms of aggregate uncertainty as well as heterogeneous turnout. An alternative model, which captures variable turnout heterogeneity, would retain one-dimensional voter types but assume that voters abstain when they are close to indifferent between the parties. It is interesting to compare these models, as in practice turnout heterogeneity has both exogenous sources (e.g., education, race) and endogenous ones (e.g., almost-indifferent voters turn out less).

Finally, further questions include: What does the model imply for political competition and policy choices? What are the model's comparative statics—for example, what factors determine the proportion of packed and cracked districts?[51] And, what does the model imply about how gerrymandering should be measured and regulated? Understanding the form of optimal partisan gerrymandering can contribute to the study of these questions and related ones.

## REFERENCES

Abramowitz, A. I., B. Alexander, and M. Gunning (2006): "Incumbency, Redistricting, and the Decline of Competition in US House Elections," *Journal of Politics*, 68, 75–88. [29]

Ansolabehere, S., M. Palmer, and A. Lee (2014): "Precinct-Level Election Data," *Harvard Dataverse*. [25]

Bagnoli, M., and T. Bergstrom (2005): "Log-Concave Probability and Its Applications," *Economic Theory*, 26(2), 445–469. [5]

Baltz, S., et al. (2022): "American Election Results at the Precinct Level," *Scientific Data*, 9(651). [21, 25]

Besley, T., and I. Preston (2007): "Electoral Bias and Policy Choice: Theory and Evidence," *Quarterly Journal of Economics*, 122, 1473–1510. [4, 29]

Bickerstaff, S. (2020): *Election Systems and Gerrymandering Worldwide*. Springer. [1]

Bouton, L., G. Genicot, M. Castanheira, and A. Stashko (2024): "Pack-Crack-Pack: Gerrymandering with Differential Turnout," Working Paper. [4, 22, 31]

Bracco, E. (2013): "Optimal Districting with Endogenous Party Platforms," *Journal of Public Economics*, 104, 1–13. [4]

Brunell, T. (2008): *Redistricting and Representation: Why Competitive Elections Are Bad for America*. Routledge. [29]

Buchler, J. (2005): "Competition, Representation, and Redistricting: The Case against Competitive Congressional Districts," *Journal of Theoretical Politics*, 17, 431–463. [29]

Calvert, R. L. (1985): "Robustness of the Multidimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence," *American Journal of Political Science*, 29(1), 69–95. [15]

Canon, D. (2022): "Race and Redistricting," *Annual Review of Political Science*, 25, 509–528. [27]

[51] Kolotilin and Wolitzky (2020) analyze comparative statics with binary voter types.

Chambers, C. P., and A. D. Miller (2010): "A Measure of Bizarreness," *Quarterly Journal of Political Science*, 5, 27–44. [4]

Coate, S., and B. Knight (2007): "Socially Optimal Districting: A Theoretical and Empirical Exploration," *Quarterly Journal of Economics*, 122, 1409–1471. [4]

Cox, A. B., and R. Holden (2011): "Reconsidering Racial and Partisan Gerrymandering," *University of Chicago Law Review*, 78, 553–604. [27, 28, 29]

Deford, D., M. Duchin, and J. Solomon (2021): "Recombination: A Family of Markov Chains for Redistricting," *Harvard Data Science Review*, 3. [4, 29]

Dworczak, P., and G. Martini (2019): "The Simple Economics of Optimal Persuasion," *Journal of Political Economy*, 127(5), 1993–2048. [15]

Ely, J. (2022): "A Cake-Cutting Solution to Gerrymandering," Working Paper. [4]

Friedman, J. N., and R. Holden (2008): "Optimal Gerrymandering: Sometimes Pack, but Never Crack," *American Economic Review*, 98(1), 113–44. [2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 15, 27, 30]

——— (2009): "The Rising Incumbent Reelection Rate: What's Gerrymandering Got To Do with It?," *Journal of Politics*, pp. 593–611. [29]

——— (2020): "Optimal Gerrymandering in a Competitive Environment," *Economic Theory Bulletin*, pp. 1–21. [4]

Fryer, R. G., and R. Holden (2011): "Measuring the Compactness of Political Districting Plans," *Journal of Law and Economics*, 54, 493–535. [4]

Gelman, A., and G. King (1994a): "Enhancing Democracy through Legislative Redistricting," *American Political Science Review*, 88, 541–559. [29]

——— (1994b): "A Unified Method of Evaluating Electoral Systems and Redistricting Plans," *American Journal of Political Science*, 38, 514–554. [4]

Gentzkow, M., and E. Kamenica (2016): "A Rothschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review, Papers & Proceedings*, 106, 597–601. [15]

Gilligan, T. W., and J. G. Matsusaka (2006): "Public Choice Principles of Redistricting," *Public Choice*, 129, 381–398. [4]

Gomberg, A., R. Pancs, and T. Sharma (2023): "Electoral Maldistricting," 64(3), 1223–1264. [4, 29]

——— (2024): "Padding and Pruning: Gerrymandering under Turnout Heterogeneity," *Social Choice and Welfare*, 63(2), 401–415. [4]

Grofman, B., and L. Handley (1991): "The Impact of the Voting Rights Act on Black Representation in Southern State Legislatures," *Legislative Studies Quarterly*, 16, 111–128. [27]

Groll, T., and S. O'Halloran (2024): "Redistricting and Representation: The Paradox of Minority Power," Working Paper. [4]

Gul, F., and W. Pesendorfer (2010): "Strategic Redistricting," *American Economic Review*, 100(4), 1616–1141. [4, 6, 7, 8, 15, 30]

Hayes, D., and S. C. McKee (2009): "The Participatory Effects of Redistricting," *American Journal of Political Science*, 53, 1006–1023. [5]

Hinich, M. J. (1977): "Equilibrium in Spatial Voting: The Median Voter Result is an Artifact," *Journal of Economic Theory*, 16(2), 208–219. [15]

Jeong, D., and A. Shenoy (2024): "The Targeting and Impact of Partisan Gerrymandering: Evidence from a Legislative Discontinuity," *Review of Economics and Statistics*, 106(3), 814–828. [5]

Kamenica, E., and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615. [7]

King, G., and R. X. Browning (1987): "Democratic Representation and Partisan Bias in Congressional Elections," *American Political Science Review*, 81(4), 1251–1273. [4, 29]

Kleiner, A., B. Moldovanu, and P. Strack (2021): "Extreme Points and Majorization: Economic Applications," *Econometrica*, 89(4), 1557–1593. [15]

Kolotilin, A. (2018): "Optimal Information Disclosure: A Linear Programming Approach," *Theoretical Economics*, 13, 607–636. [9, 15]

Kolotilin, A., R. Corrao, and A. Wolitzky (2024): "Persuasion and Matching: Optimal Productive Transport," *Journal of Political Economy*, forthcoming. [4, 7, 9, 12, 15, 34, 37, 39]

Kolotilin, A., T. Mylovanov, and A. Zapechelnyuk (2022): "Censorship as Optimal Persuasion," *Theoretical Economics*, 17(2), 561–585. [9]

Kolotilin, A., T. Mylovanov, A. Zapechelnyuk, and M. Li (2017): "Persuasion of a Privately Informed Receiver," *Econometrica*, 85, 1949–1964. [15]

Kolotilin, A., and A. Wolitzky (2020): "The Economics of Partisan Gerrymandering," Working Paper. [5, 6, 31]

——— (2024): "Distributions of Posterior Quantiles via Matching," *Theoretical Economics*, forthcoming. [15]

Lagarde, A., and T. Tomala (2021): "Optimality and Fairness of Partisan Gerrymandering," *Mathematical Programming*, pp. 1–37. [4]

Lindbeck, A., and J. Weibull (1993): "A Model of Political Equilibrium in a Representative Democracy," *Journal of Public Economics*, 51, 195–209. [15]

Lublin, D., L. Handley, T. Brunell, and B. Grofman (2020): "Minority Success in Non-Majority Minority Districts: Finding the "Sweet Spot"," *Journal of Race, Ethnicity, and Politics*, 5, 275–298. [27]

McCarty, N., K. T. Poole, and H. Rosenthal (2009): "Does Gerrymandering Cause Polarization?," *American Journal of Political Science*, 53, 666–680. [4, 29]

McGann, A. J., C. A. Smith, M. Latner, and A. Keena (2016): *Gerrymandering in America: The House of Representatives, the Supreme Court, and the Future of Popular Sovereignty*. Cambridge University Press. [2]

McGhee, E. (2014): "Measuring Partisan Bias in Single-Member District Electoral Systems," *Legislative Studies Quarterly*, 39(1), 55–85. [4]

——— (2020): "Partisan Gerrymandering and Political Science," *Annual Review of Political Science*, 23, 171–185. [1]

Newkirk, V. R. (2017): "How Redistricting Became a Technological Arms Race," *The Atlantic*, 28 October. [1, 23]

Owen, G., and B. Grofman (1988): "Optimal Partisan Gerrymandering," *Political Geography Quarterly*, 7(1), 5–22. [4, 6, 7, 15, 30, 31]

Puppe, C., and A. Tasnádi (2009): "Optimal Redistricting under Geographical Constraints: Why "Pack and Crack" Does Not Work," *Economics Letters*, 105, 93–96. [4]

Rakich, N., and E. Mejia (2022): "Did Redistricting Cost Democrats the House?," https://fivethirtyeight.com/features/redistricting-house-2022/. [2]

Rakich, N., and N. Silver (2018): "Election Update: The Most (And Least) Elastic States And Districts," https://fivethirtyeight.com/features/election-update-the-house-districts-that-swing-the-most-and-least-with-the-national-mood/. [6]

Sabet, N., and N. Yuchtman (2024): "Identifying Partisan Gerrymandering and Its Consequences: Evidence from the 1990 US Census Redistricting," *Journal of Political Economy: Microeconomics*, forthcoming. [5]

Santambrogio, F. (2015): *Optimal Transport for Applied Mathematicians*, vol. 55. Springer. [43]

Sherstyuk, K. (1998): "How to Gerrymander: A Formal Analysis," *Public Choice*, 95, 27–49. [4]

Shotts, K. W. (2001): "The Effect of Majority-Minority Mandates on Partisan Gerrymandering," *American Journal of Political Science*, pp. 120–135. [4, 27, 28]

——— (2002): "Gerrymandering, Legislative Composition, and National Policy Outcomes," *American Journal of Political Science*, pp. 398–414. [4]

Stephanopoulos, N. O., and E. M. McGhee (2015): "Partisan Gerrymandering and the Efficiency Gap," *University of Chicago Law Review*, 82, 831–900. [4, 29]

Wang, S. (2016): "Three Tests for Practical Evaluation of Partisan Gerrymandering," *Stanford Law Review*, 69, 1263–1321. [29]

Warrington, G. (2018): "Quantifying Gerrymandering Using the Vote Distribution," *Election Law Journal*, 17, 39–57. [29]

Washington, E. (2012): "Do Majority-Black Districts Limit Blacks' Representation? The Case of the 1990 Redistricting," *Journal of Law and Economics*, 55, 251–274. [27]

Wittman, D. (1983): "Candidate Motivation: A Synthesis of Alternative Theories," *American Political Science Review*, 77(1), 142–157. [15]

Yang, K. H., and A. Zentefis (2024): "Monotone Function Intervals: Theory and Applications," *American Economic Review*, 114(8), 2239–2270. [15]

# ONLINE APPENDIX

## APPENDIX A: Proofs

### A.1. Duality Lemma

We start with a duality result restating Theorem 1 in Kolotilin, Corrao, and Wolitzky (2024).

Lemma 2: *There exists a bounded, measurable function $\lambda : [\underline{s}, \overline{s}] \to \mathbb{R}$ such that, for any optimal districting plan $\mathcal{H}$, the following hold:*

1. *For all $P \in \operatorname{supp}(\mathcal{H})$, all $s \in \operatorname{supp}(P)$, and all $r \in [\underline{s}, \overline{s}]$, we have*

$$G(r^*(P)) + \lambda(r^*(P))\left(Q(s - r^*(P)) - Q(0)\right) \geq G(r) + \lambda(r)\left(Q(s - r) - Q(0)\right).$$

2. *For all $P \in \operatorname{supp}(\mathcal{H})$, we have*

$$\lambda(r^*(P)) = \frac{g(r^*(P))}{\displaystyle\int q(s - r^*(P))dP(s)}.$$

3. *For all non-degenerate $P \in \operatorname{supp}(\mathcal{H})$, $\lambda$ has derivative $\lambda'(r^*(P))$ at $r^*(P)$ satisfying, for all $s \in \operatorname{supp}(P)$,*

$$g(r^*(P)) - \lambda(r^*(P))q(s - r^*(P)) + \lambda'(r^*(P))\left(Q(s - r^*(P)) - Q(0)\right) = 0.$$

As explained in the text, $\lambda(r^*(P))$ is the multiplier on the constraint $\int Q(s - r^*(P))dP(s) = Q(0)$. Part 2 of the lemma gives the formula for $\lambda(r^*(P))$ from the implicit function theorem. Part 1 says that the designer assigns a type-$s$ voter to a district $P$ to maximize $G(r^*(P)) + \lambda(r^*(P))\left(Q(s - r^*(P)) - Q(0)\right)$. Part 3 says that the first-order condition of this maximization problem with respect to $r$ holds for all non-degenerate $P \in \operatorname{supp}(\mathcal{H})$ and all $s \in \operatorname{supp}(P)$.

### A.2. Proof of Lemma 1

Lemma 1 follows from Theorem 4 in Kolotilin, Corrao, and Wolitzky (2024) for the translation-invariant subcase of the state-independent sender case. For completeness, we prove Lemmas 3 and 4, which immediately yield Lemma 1.

Lemma 3: *For any optimal $\mathcal{H}$ and any $P, P' \in \operatorname{supp}(\mathcal{H})$ such that $P$ contains types $s < s''$ and $P'$ contains a type $s' \in (s, s'')$, we have $r^*(P) \geq r^*(P')$.*

Proof of Lemma 3: Suppose by contradiction that there exist districts $P$ and $P'$ such that $P$ contains $s < s''$, $P'$ contains $s' \in (s, s'')$, and $r = r^*(P) < r^*(P') = r'$. Then, by part 1 of Lemma 2, we have

$$G(r)+\lambda(r)(Q(s-r)-Q(0))\geq G(r')+\lambda(r')(Q(s-r')-Q(0)), \tag{2}$$

$$G(r')+\lambda(r')(Q(s'-r')-Q(0))\geq G(r)+\lambda(r)(Q(s'-r)-Q(0)), \quad \text{and} \tag{3}$$

$$G(r)+\lambda(r)(Q(s''-r)-Q(0))\geq G(r')+\lambda(r')(Q(s''-r')-Q(0)). \tag{4}$$

This yields a contradiction because

$$\begin{aligned} 0 &\geq (Q(s''-r')-Q(s'-r'))(Q(s'-r)-Q(s-r)) \\ &\quad -(Q(s''-r)-Q(s'-r))(Q(s'-r')-Q(s-r')) \\ &= \int_{s'}^{s''}\int_{s}^{s'} q(\tilde{s}'-r')q(\tilde{s}-r)d\tilde{s}d\tilde{s}' - \int_{s'}^{s''}\int_{s}^{s'} q(\tilde{s}'-r)q(\tilde{s}-r')d\tilde{s}d\tilde{s}' \\ &= \int_{s'}^{s''}\int_{s}^{s'} \left(q(\tilde{s}'-r')q(\tilde{s}-r)-q(\tilde{s}'-r)q(\tilde{s}-r')\right) d\tilde{s}d\tilde{s}' > 0, \end{aligned}$$

where the first inequality holds by summing (2) multiplied by $Q(s''-r)-Q(s'-r)$, (3) multiplied by $Q(s''-r)-Q(s-r)$, and (4) multiplied by $Q(s'-r)-Q(s-r)$, and then dividing by $\lambda(r')$, which is strictly positive by part 2 of Lemma 2; and the second inequality holds because the integrand is strictly positive for $r<r'$ and $\tilde{s}<\tilde{s}'$ by strict log-concavity of $q$. *Q.E.D.*

LEMMA 4: *For any optimal $\mathcal{H}$ and any $P\in \operatorname{supp}(\mathcal{H})$, we have $|\operatorname{supp}(P)|\leq 2$.*

PROOF OF LEMMA 4: Suppose by contradiction that some district $P\in\operatorname{supp}(\mathcal{H})$ contains three types $s<s'<s''$ and $r^*(P)=r$. Then, by part 3 of Lemma 2, we have

$$g(r)-\lambda(r)q(s-r)+\lambda'(r)\left(Q(s-r)-Q(0)\right)=0, \tag{5}$$

$$g(r)-\lambda(r)q(s'-r)+\lambda'(r)\left(Q(s'-r)-Q(0)\right)=0, \quad \text{and} \tag{6}$$

$$g(r)-\lambda(r)q(s''-r)+\lambda'(r)\left(Q(s''-r)-Q(0)\right)=0. \tag{7}$$

This yields a contradiction because

$$\begin{aligned} 0 &= \det\begin{pmatrix} g(r) & q(s-r) & Q(s-r)-Q(0) \\ g(r) & q(s'-r) & Q(s'-r)-Q(0) \\ g(r) & q(s''-r) & Q(s''-r)-Q(0)\end{pmatrix} \\ &= g(r)(q(s'-r)-q(s-r))(Q(s''-r)-Q(s'-r)) \\ &\quad -g(r)(q(s''-r)-q(s'-r))(Q(s'-r)-Q(s-r)) \end{aligned}$$

$$= g(r)\left[\int_s^{s'} q'(\tilde{s}-r)d\tilde{s}\int_{s'}^{s''} q(\tilde{s}'-r)d\tilde{s}' - \int_{s'}^{s''} q'(\tilde{s}'-r)d\tilde{s}'\int_s^{s'} q(\tilde{s}-r)d\tilde{s}\right]$$

$$> \frac{g(r)q'(s'-r)}{q(s'-r)}\left[\int_s^{s'} q(\tilde{s}-r)d\tilde{s}\int_{s'}^{s''} q(\tilde{s}'-r)d\tilde{s}' - \int_{s'}^{s''} q(\tilde{s}'-r)d\tilde{s}'\int_s^{s'} q(\tilde{s}-r)d\tilde{s}\right] = 0,$$

where the first equality is by (5)–(7), and the inequality is by strict log-concavity of $q$, which implies that the derivative of $\ln q$ is strictly decreasing, yielding

$$\frac{q'(\tilde{s}-r)}{q(\tilde{s}-r)} > \frac{q'(s'-r)}{q(s'-r)} > \frac{q'(\tilde{s}'-r)}{q(\tilde{s}'-r)}, \quad \text{for } \tilde{s} < s' < \tilde{s}'. \qquad \textit{Q.E.D.}$$

### *A.3. Characterization of Segregate-Pair Districting*

LEMMA 5: *For any segregate-pair districting plan $\mathcal{H}$, there exists a bifurcation point $r^b \in (\underline{s}, \overline{s}]$, a decreasing function $s_1 : (r^b, \overline{s}) \to [\underline{s}, r^b)$, and an increasing function $s_2 : (r^b, \overline{s}) \to (r^b, \overline{s}]$ satisfying $s_1(r) < r < s_2(r)$, such that for each $P \in \mathrm{supp}(\mathcal{H})$, we have $P = \delta_{r^*(P)}$ if $r^*(P) \le r^b$ and $\mathrm{supp}(P) = \{s_1(r^*(P)), s_2(r^*(P))\}$ if $r^*(P) > r^b$.*

PROOF OF LEMMA 5: Let $\mathcal{H}$ be a segregate-pair districting plan. Since $\mathcal{H}$ is strictly single-dipped, the support of each $P \in \mathrm{supp}(\mathcal{H})$ has at most two elements and thus can be represented as $\{s_1(r^*(P)), s_2(r^*(P))\}$ with $s_1(r^*(P)) \le r^*(P) \le s_2(r^*(P))$. Moreover, for each $P, P' \in \mathrm{supp}(\mathcal{H})$ with $r^*(P) < r^*(P')$, we have $s_2(r^*(P)) \le s_2(r^*(P'))$, as otherwise we would have $s_2(r^*(P')) \in (s_1(r^*(P)), s_2(r^*(P)))$, contradicting strict single-dippedness of $\mathcal{H}$.

Assume that there exists $P \in \mathrm{supp}(\mathcal{H})$ such that $s_1(r^*(P)) < s_2(r^*(P))$, as otherwise the lemma obviously holds with $r^b = \overline{s}$. Define $r^b = \inf\{r^*(\tilde{P}) : \tilde{P} \in \mathrm{supp}(\mathcal{H}),\ s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P}))\}$, so that, for each $P \in \mathrm{supp}(\mathcal{H})$ with $r^*(P) < r^b$, we have $\mathrm{supp}(P) = \{r^*(P)\}$. Since $\mathrm{supp}(\mathcal{H})$ is compact, there exists $P^b \in \mathrm{supp}(\mathcal{H})$ with $r^*(P^b) = r^b$. It follows that $\mathrm{supp}(P^b) = \{r^b\}$, as otherwise (i.e., if $s_1(r^*(P^b)) < r^b < s_2(r^*(P^b))$ voter types in $(r^b, s_2(r^*(P^b)))$ (which have strictly positive mass since $f$ is strictly positive on $[\underline{s}, \overline{s}]$) cannot be segregated, as this would contradict strict single-dippedness of $\mathcal{H}$, and also cannot be paired with other types, as this would contradict either strict single-dippedness of $\mathcal{H}$ or the definition of $r^b$.

Next, we show that, for each $P, P' \in \mathrm{supp}(\mathcal{H})$ with $r^b < r^*(P) < r^*(P')$, we have $s_1(r^*(P)) \ge s_1(r^*(P'))$. Suppose by contradiction that $s_1(r^*(P)) < s_1(r^*(P'))$. Since $\mathcal{H}$ is a strictly single-dipped segregate-pair districting plan, by the definition of $r^b$, we have $s_1(r^*(P)) < r^*(P) < s_2(r^*(P)) \le s_1(r^*(P')) < r^*(P') < s_2(r^*(P'))$. Define $r^\dagger = \inf\{r^*(\tilde{P}) : \tilde{P} \in \mathrm{supp}(\mathcal{H}),\ s_1(r^*(P')) \le s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P})) \le s_2(r^*(P'))\} \ge$

$s_1(r^*(P'))$. By the same argument as in the previous paragraph, we have $\delta_{r^\dagger} \in \text{supp}(\mathcal{H})$, contradicting that $\mathcal{H}$ is segregate-pair.

Finally, we have (i) $r^b > \underline{s}$, as otherwise all voter types above $\underline{s}$ are paired with $\underline{s}$, contradicting that $F$ has no atom at $\underline{s}$, and (ii) $\sup\{r^*(\tilde{P}) : \tilde{P} \in \text{supp}(\mathcal{H}),\ s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P}))\} < \overline{s}$ when $r^b < \overline{s}$, as otherwise $\delta_{\overline{s}} \in \text{supp}(\mathcal{H})$ by the same argument as in the second paragraph, contradicting that $\mathcal{H}$ is segregate-pair. Then we can extend functions $s_1$ and $s_2$ from the set $\tilde{R} = \{r^*(\tilde{P}) : \tilde{P} \in \text{supp}(\mathcal{H}),\ s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P}))\} \subset (r^b, \underline{s})$ to the interval $(r^b, \underline{s})$ by setting $s_1(r) = \inf\{s_1(\tilde{r}) : \tilde{r} \in \tilde{R},\ \tilde{r} < r\}$ and $s_2(r) = \sup\{s_2(\tilde{r}) : \tilde{r} \in \tilde{R},\ \tilde{r} < r\}$ for all $r \in (r^b, \overline{s}) \setminus \tilde{R}$. By construction, extended functions $s_1$ and $s_2$ are as required. *Q.E.D.*

### A.4. *Auxiliary Lemmas*

Lemmas 6–9 are used to prove Theorems 1 and 2.

LEMMA 6: *If for all $s < r < s'$ such that*

$$G(r) + \lambda(r)\left(Q(s-r) - Q(0)\right) \geq G(s), \tag{8}$$

*where*

$$\lambda(r) = \frac{g(r)(Q(s'-r)) - Q(s-r))}{(Q(s'-r) - Q(0))q(s-r) - (Q(s-r) - Q(0))q(s'-r)}, \tag{9}$$

*we have, for all $s'' \geq s'$,*

$$G(r) + \lambda(r)(Q(s-r) - Q(0)) < G(s'') + \frac{g(s'')}{q(0)}(Q(s-s'') - Q(0)), \tag{10}$$

*then there is a unique optimal districting plan, which is segregate-pair.*

PROOF OF LEMMA 6: Suppose by contradiction that there exists an optimal non-segregate-pair plan $\mathcal{H}$. By Lemma 1, $\mathcal{H}$ is strictly single-dipped. Consequently, since $\mathcal{H}$ is not segregate-pair, there exist $s < r < s' \leq s''$ and $P, P' \in \text{supp}(\mathcal{H})$ such that $r^*(P) = r$, $\text{supp}(P) = \{s, s'\}$, and $\text{supp}(P') = \{s''\}$. By Lemma 2, condition (8) holds and condition (10) fails, yielding a contradiction.[52] Finally, for uniqueness, by Theorem 7 in Kolotilin, Corrao, and Wolitzky (2024), it suffices to show that $\mathcal{H}$ is *regular*, in that for each $P \in \text{supp}(\mathcal{H})$, there exists $\varepsilon > 0$ such that either (i) $|\text{supp}(\tilde{P})| = 1$ for all $\tilde{P} \in \text{supp}(\mathcal{H})$ satisfying $r^*(\tilde{P}) \in (r^*(P) - \varepsilon, r^*(P))$, or (ii) $|\text{supp}(\tilde{P})| = 2$ for all $\tilde{P} \in \text{supp}(\mathcal{H})$ satisfying

[52]Intuitively, (8) says that the designer prefers not to move a few type-$s$ voters from district $P$ to district $\delta_s$, and (10) says that the designer strictly prefers to move a few type-$s$ voters from district $P$ to district $\delta_{s''}$.

$r^*(\tilde{P}) \in (r^*(P) - \varepsilon, r^*(P))$. But each segregate-pair plan $\mathcal{H}$ is clearly regular, with any $\varepsilon > 0$ for $r^*(P) \leq r^b$ and with any $\varepsilon \in (0, r^*(P) - r^b)$ for $r^*(P) > r^b$. *Q.E.D.*

LEMMA 7: *If $s < r < s'$, then $\lambda(r)$ given by (9) satisfies $\lambda(r) > g(r)/q(0)$.*

PROOF OF LEMMA 7: It follows from (9) and $q$ being uniquely maximized at 0. *Q.E.D.*

LEMMA 8: *If $\eta \geq 1$ and $s < r < s'$ satisfy (8), with $\lambda(r)$ given by (9), then $r > 0$.*

PROOF OF LEMMA 8: If $r \leq 0$, then (8) fails, because

$$G(r) - G(s) = \int_s^r g(x)dx \leq \frac{g(r)}{g(0)} \int_s^r g(x-r)dx = \frac{g(r)}{g(0)}(G(0) - G(s-r))$$
$$= \frac{g(r)}{\eta q(0)}(Q(\eta 0) - Q(\eta(s-r)) \leq \frac{g(r)}{q(0)}(Q(0) - Q(s-r)) < \lambda(r)\,(Q(0) - Q(s-r))\,,$$

where the first inequality is by strict log-concavity of $g$ on $[s, 0]$, the second inequality is by $\eta \geq 1$ and strict convexity of $Q$ on $[s-r, 0]$, and the last inequality is by Lemma 7. *Q.E.D.*

LEMMA 9: *If $\mathcal{H}$ is optimal and $\delta_s, \delta_{s'} \in \mathrm{supp}(\mathcal{H})$ with $s < s'$, then $s < 0$.*

PROOF OF LEMMA 9: Suppose by contradiction that $s \geq 0$. By Lemma 2, for each $r \in (s, s')$ there exists $\lambda(r)$ such that

$$G(s) \geq G(r) + \lambda(r)(Q(s-r) - Q(0)) \quad \text{and} \tag{11}$$
$$G(s') \geq G(r) + \lambda(r)(Q(s'-r) - Q(0)). \tag{12}$$

Summing (11) multiplied by $Q(s'-r) - Q(0)$ and (12) multiplied by $Q(0) - Q(s-r)$ yields

$$(G(s') - G(r))\,(Q(0) - Q(s-r)) \geq (G(r) - G(s))\,(Q(s'-r) - Q(0))\,.$$

But this inequality cannot hold for $r$ sufficiently close to $s'$, because

$$g(s')\,(Q(0) - Q(s-s')) < (G(s') - G(s))q(0),$$

since $G$ is strictly concave on $[s, s']$ and $Q$ is strictly convex on $[s-s', 0]$. *Q.E.D.*

### A.5. *Proof of Theorem 1*

If $s<r<s'$ satisfy (8), with $\lambda(r)$ given by (9), then $r>0$, by Lemma 8. Theorem 1 then follows from Lemma 6, as (10) holds for all $s''\geq s'$, because

$$
\begin{aligned}
&G(s'')+\frac{g(s'')}{q(0)}\left(Q(s-s'')-Q(0)\right)-G(r)-\lambda(r)\left(Q(s-r)-Q(0)\right)\\
&>\frac{g(r)}{q(0)}(Q(0)-Q(s-r))+G(s'')-G(r)-\frac{g(s'')}{q(0)}(Q(0)-Q(s-s''))\\
&>\frac{g(s'')}{q(0)}\left[Q(0)-Q(s-r)+q(0)(s''-r)-(Q(0)-Q(s-s''))\right]>0,
\end{aligned}
$$

where the first inequality is by Lemma 7, the second inequality is by strict concavity of $G$ on $[r,s'']$, and the third inequality is by strict convexity of $Q$ on $[s-s'',0]$.

### A.6. *Proof of Theorem 2*

*Part 1.* Let $\mathcal{H}$ be an optimal strictly single-dipped plan. By Lemma 9, there do not exist $s<s'$ in $[\underline{s},\overline{s}]$ such that $\delta_s,\delta_{s'}\in\operatorname{supp}(\mathcal{H})$. Then, by Theorem 6 in Kolotilin, Corrao, and Wolitzky (2024), $\mathcal{H}$ is NAD.

*Part 2.* Suppose by contradiction that there exist an optimal strictly single-dipped plan $\mathcal{H}$ and $P\in\operatorname{supp}(\mathcal{H})$ such that $r^*(P)=r$ and $\operatorname{supp}(P)=\{s,s'\}$ with $s<r<s'$. By Lemma 2, (8) holds with $\lambda(r)$ given by (9). So, by Lemma 8, $r^*(P)>0$, contradicting that $r<s'\leq\overline{s}\leq 0$.

*Part 3.* Since $f$ is strictly positive on $[\underline{s},\overline{s}]$ and $\underline{s}<\overline{s}$, we have $\underline{s}<r^*(F)=0<\overline{s}$, so segregation is suboptimal by Lemma 9.

Suppose by contradiction that there exists an optimal NAD plan $\mathcal{H}$. By Lemma 5, for each $P\in\operatorname{supp}(\mathcal{H})$ except for $\delta_{r^b}$, we have $s_1(r^*(P))<r^*(P)<s_2(r^*(P))$, where $s_1$ is decreasing and $s_2$ is increasing. Note that $r^b<r^*(F)=0$, because

$$
\begin{aligned}
\int Q(s-r^*(F))dF(s)=Q(0)=\iint Q(s-r^*(P))dP(s)d\mathcal{H}(P)\\
<\iint Q(s-r^b)dP(s)d\mathcal{H}(P)=\int Q(s-r^b)dF(s),
\end{aligned}
$$

where the first two equalities hold by the definition of $r^*(F)$ and $r^*(P)$, the inequality holds by $r^*(P)>r^b$ for all $P\in\operatorname{supp}(\mathcal{H})$ except for $P=\delta_{r^b}$, and the last equality holds by $\int Pd\mathcal{H}(P)=F$. Since $f$ is strictly positive on $[\underline{s},\overline{s}]$, we get $\lim_{r\downarrow r^b}s_1(r)=\lim_{r\downarrow r^b}s_2(r)=r^b$, as otherwise voter types in $(\lim_{r\downarrow r^b}s_1(r),\lim_{r\downarrow r^b}s_2(r))$ are not assigned to any district. Thus, for any $\varepsilon>0$, there exists $P\in\operatorname{supp}(\mathcal{H})$ such that, for $r=r^*(P)$, $s=s_1(r)$, and $s'=s_2(r)$, we have $r^b-\varepsilon\leq s_1(r)<r<s_2(r)\leq r^b+\varepsilon$, and

$$
G(r)+\frac{g(r)}{q(0)}(Q(s-r)-Q(0))>G(r)+\lambda(r)(Q(s-r)-Q(0))\geq G(s),
$$

where the first inequality is by Lemma 7, and the second inequality is by Lemma 2. But this yields a contradiction because there exists a small enough $\varepsilon\in(0,-r^b)$ such that, for all $s < r$ in $[r^b - \varepsilon, r^b + \varepsilon]$, we have

$$\frac{G(r) - G(s)}{g(r)} < \frac{Q(0) - Q(s-r)}{q(0)},$$

because $g'(r) > 0 = q'(0)$, and $g$ and $q$ are strictly positive and three-times differentiable.

### A.7. *Proof of Theorem 3*

Theorem 3 follows from Lemmas 10–14.

For each $r$, let $\mathcal{R}(r)$ be the set of all plans $\mathcal{H}$ that maximize the designer's seat share when the aggregate shock is $r$. Lemma 10 characterizes $\mathcal{R}(r)$. If $r^*(F) \geq r$, then $\mathcal{H} \in \mathcal{R}(r)$ assigns all voters to districts that the designer wins. If $r^*(F) < r$, then $\mathcal{H} \in \mathcal{R}(r)$ assigns all voter types above $s^*(r)$ to cracked districts that the designer wins with exactly 50% of the vote and packs the remaining voters arbitrarily.

Lemma 10: *The following hold.*

1. *Let $r^*(F) \geq r$. Then $\mathcal{H} \in \mathcal{R}(r)$ iff, for each $P \in \operatorname{supp}(\mathcal{H})$, we have $r^*(P) \geq r$.*
2. *Let $r^*(F) < r$. Then $\mathcal{H} \in \mathcal{R}(r)$ iff, for each $P \in \operatorname{supp}(\mathcal{H})$, we have either $\operatorname{supp}(P) \subset [\underline{s}, s^*(r)]$ or $\operatorname{supp}(P) \subset [s^*(r), \overline{s}]$ and $r^*(P) = r$.*

Proof of Lemma 10: *Part 1.* Since $r^*(F) \geq r$, $s^*(r) = \underline{s}$ and $\delta_F \in \mathcal{R}(r)$ is optimal. Hence, $\mathcal{H} \in \mathcal{R}(r)$ iff $\int \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}(P) = 1$, which is equivalent to $r^*(P) \geq r$ for all $P \in \operatorname{supp}(\mathcal{H})$, because the set $\{P \in \Delta[\underline{s}, \overline{s}] : r^*(P) \geq r\}$ is closed by the continuity of $r^*$, which follows from the continuity and strict monotonicity of $Q$.

*Part 2.* Assume that $r^*(F) < r < \overline{s}$, as if $r \geq \overline{s}$ then $s^*(r) = \overline{s}$, so part 2 holds trivially. For each plan $\mathcal{H}$, we have

$$\begin{aligned}
\int \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}(P) &= \int \mathbf{1}\left\{\mathbb{E}_P[Q(s-r) - Q(0)] \geq 0\right\} d\mathcal{H}(P) \\
&\leq \int \max\left\{0, \frac{\mathbb{E}_P[Q(s-r)] - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} d\mathcal{H}(P) \\
&\leq \iint \max\left\{0, \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} dP(s) d\mathcal{H}(P) \qquad (13) \\
&= \int \max\left\{0, \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} dF(s) \\
&= \int_{s^*(r)}^{\overline{s}} \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)} dF(s) = 1 - F(s^*(r)),
\end{aligned}$$

where the first equality is by the definition of $r^*(P)$, the first inequality is by pointwise dominace of the integrands, the second inequality is by Jensen's inequality, the second equality is by $\int P d\mathcal{H}(P) = F$, the third equality is by strict monotonicity of $Q$, and the last equality is by the definition of $s^*(r)$. Hence, $\mathcal{H} \in \mathcal{R}(r)$ iff, for a measure-1 set of districts $P$ under $\mathcal{H}$, we have (a) $\mathbb{E}_P[Q(s-r)] \leq Q(s^*(r)-r)$ or $\mathbb{E}_P[Q(s-r)] = Q(0)$ (as otherwise the first inequality in (13) is strict) and (b) $\mathrm{supp}(P) \subset [\underline{s}, s^*(r)]$ or $\mathrm{supp}(P) \subset [s^*(r), \overline{s}]$ (as otherwise the second inequality in (13) is strict), or equivalently, either (i) $\mathrm{supp}(P) \subset [\underline{s}, s^*(r)]$ (which implies that $\mathbb{E}_P[Q(s-r)] \leq Q(s^*(r)-r)$) or (ii) $\mathrm{supp}(P) \subset [\underline{s}, s^*(r)]$ and $r^*(P) = r$ (which is equivalent to $\mathbb{E}_P[Q(s-r)] = Q(0)$). Finally, as in the proof of part 1, continuity implies that properties (i) or (ii) hold for all $P \in \mathrm{supp}(\mathcal{H})$, rather than just for a measure-1 set. *Q.E.D.*

Lemma 11 shows that pack-and-crack districting is approximately optimal. An upper bound on the designer's optimal expected seat share $V_\eta$ can be obtained by allowing the designer to choose $\mathcal{H}_r \in \mathcal{R}(r)$ after observing each realization $r$,

$$\overline{V}_\eta = \int (1 - F(s^*(r)))dG_\eta(r).$$

A lower bound on $V_\eta$ can be obtained by restricting attention to $\mathcal{H}_{\tilde{r}} \in \mathcal{R}(\tilde{r})$ for some $\tilde{r}$,

$$\underline{V}_\eta(\tilde{r}) = \int (1 - F(s^*(\tilde{r})))\mathbf{1}\{r \leq \tilde{r}\}dG_\eta(r).$$

LEMMA 11: *For all $\eta$ and all $\tilde{r}$, we have $\underline{V}_\eta(\tilde{r}) \leq V_\eta \leq \overline{V}_\eta$. Moreover, if $\eta \to \infty$, then $\overline{V}_\eta \to 1 - F(s^*(0))$, $\underline{V}_\eta(\tilde{r}) \to 1 - F(s^*(\tilde{r}))$ for all $\tilde{r} > 0$, and $V_\eta \to 1 - F(s^*(0))$.*

PROOF OF LEMMA 11: Let $\mathcal{H}_\eta$ be the optimal plan and let $\mathcal{H}_r$ be any districting plan in $\mathcal{R}(r)$. On the one hand, we have

$$V_\eta = \iint \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_\eta(P)dG_\eta(r)$$
$$< \iint \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_r(P)dG_\eta(r) = \int (1 - F(s^*(r)))dG_\eta(r) = \overline{V}_\eta,$$

where the inequality holds because $\int \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_\eta(P) \leq \int \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_r(P)$ for all $r$ by the definition of $\mathcal{H}_r$.

On the other hand, for any $\tilde{r}$, we have

$$V_\eta = \iint \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_\eta(P)dG_\eta(r)$$
$$\geq \iint \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}_{\tilde{r}}(P)dG_\eta(r) \geq \int (1 - F(s^*(\tilde{r})))\mathbf{1}\{r \leq \tilde{r}\}dG_\eta(r) = \underline{V}_\eta(\tilde{r}),$$

where the first inequality holds because $\mathcal{H}_\eta$ is optimal and $\mathcal{H}_{\tilde{r}}$ is feasible, and the second inequality holds by Lemma 10.

Suppose now that $\eta \to \infty$, which implies that $G_\eta \to \delta_0$. By the implicit function theorem, $F(s^*(r))$ is continuous in $r$, so $\overline{V}_\eta \to 1 - F(s^*(0))$. For $\tilde{r} > 0$, $\underline{V}_\eta(\tilde{r}) \to 1 - F(s^*(\tilde{r}))$, which converges to $1 - F(s^*(0))$ as $\tilde{r} \downarrow 0$, implying that $V_\eta \to 1 - F(s^*(0))$. *Q.E.D.*

Lemma 12 shows that limit points of optimal plans $H_n = H_{\eta_n}$, for $\eta_n \to \infty$, belong to $\mathcal{P}(0)$.

LEMMA 12: *Let $\mathcal{H}_n \to \mathcal{H}$ as $\eta_n \to \infty$. Then $\mathcal{H} \in \mathcal{R}(0)$.*

PROOF OF LEMMA 12: Suppose by contradiction that there exists a sequence $\eta_n \to \infty$ such that an optimal plan $\mathcal{H}_n$ converges weakly to $\mathcal{H} \notin \mathcal{R}(0)$. Then we have

$$1 - F(s^*(0)) = \lim_{n\to\infty} \int Q(\eta_n r^*(P)) d\mathcal{H}_n(P) \leq \int \mathbf{1}\{r^*(P) \geq 0\} d\mathcal{H}(P) < 1 - F(s^*(0)),$$

where the equality is by Lemma 11, the first inequality is by the Portmanteau theorem, and the second inequality is by $\mathcal{H} \notin \mathcal{R}(0)$ and Lemma 10. *Q.E.D.*

Lemma 13 shows that, in the limit, all districts are equally strong when $r^*(F) \geq 0$.

LEMMA 13: *Let $r^*(F) \geq 0$ and $\mathcal{H}_n \to \mathcal{H}$ as $\eta_n \to \infty$. Then, for each $P \in \text{supp}(\mathcal{H})$, we have $r^*(P) = r^*(F)$.*

PROOF OF LEMMA 13: If $r^*(F) = 0$, then, for each $P \in \text{supp}(\mathcal{H})$, we have $r^*(P) \geq 0$ by Lemma 10, so $r^*(P) = 0$ by $\int P d\mathcal{H}(P) = F$. So suppose that $r^*(F) > 0$. Moreover, suppose by contradiction that there exists $\varepsilon \in (0, r^*(F))$, $\delta \in (0,1)$, and a sequence $\eta_n \to \infty$ such that $\int \mathbf{1}\{r^*(P) \leq r^*(F) - \varepsilon\} d\mathcal{H}_n(P) \geq \delta$ for all $n$. We obtain a contradiction for sufficiently large $n$, because

$$\int Q(\eta_n r^*(P)) d\mathcal{H}_n(P) \leq \delta Q(\eta_n(r^*(F) - \varepsilon)) + (1 - \delta) < Q(\eta_n r^*(F)),$$

where the first inequality is by the supposition and the second inequality is by

$$\frac{1 - Q(\eta r^*(F))}{1 - Q(\eta(r^*(F) - \varepsilon))} \to 0, \quad \text{as } \eta \to \infty,$$

which we prove below. Denote $c = q'(r^*(F) - \varepsilon)/q(r^*(F) - \varepsilon)$. Since $q'(0) = 0$ and $q$ is strictly log-concave, for all $\eta > 1$, we have

$$0 = \frac{q'(0)}{q(0)} > c = \frac{q'(r^*(F) - \varepsilon)}{q(r^*(F) - \varepsilon)} > \frac{q'(\eta(r^*(F) - \varepsilon))}{q(\eta(r^*(F) - \varepsilon))} > \frac{q'(x)}{q(x)}, \quad \text{for all } x > \eta(r^*(F) - \varepsilon).$$

Hence Gronwall's inequality gives $\lim_{\eta\to\infty} q(\eta r^*(F))/q(\eta(r^*(F)-\varepsilon)) \leq \lim_{\eta\to\infty} e^{c\varepsilon\eta} = 0$, so, by L'Hopital's rule, we have

$$\lim_{\eta\to\infty} \frac{1-Q(\eta r^*(F))}{1-Q(\eta(r^*(F)-\varepsilon))} = \lim_{\eta\to\infty} \frac{q(\eta r^*(F))r^*(F)}{q(\eta(r^*(F)-\varepsilon))(r^*(F)-\varepsilon)} = 0. \qquad Q.E.D.$$

Lemma 14 shows that, in the limit, types below $s^*(0)$ are segregated and types above $s^*(0)$ are paired in a negatively assortative manner.

LEMMA 14: *Let $\mathcal{H}_n \to \mathcal{H}$ as $\eta_n \to \infty$.*

1. *For any $P \in \text{supp}(\mathcal{H})$ with $r^*(P) \leq s^*(0)$, we have $|\text{supp}(P)| = 1$.*
2. *For any $P, P' \in \text{supp}(\mathcal{H})$ with $r^*(P) = r^*(P') \geq 0$, we have $\text{supp}(P) = \{s_1(P), s_2(P)\}$ and $\text{supp}(P') = \{s_1(P'), s_2(P')\}$ with $s_1(P) \leq s_2(P)$, $s_1(P') \leq s_2(P')$, and $(s_2(P') - s_2(P))(s_1(P) - s_1(P')) \geq 0$.*

PROOF OF LEMMA 14: Denote $\Lambda_n = \text{supp}(\mathcal{H}_n)$. Since the set of compact subsets of a compact set is compact (in the Hausdorff topology), taking a subsequence if necessary, $\Lambda_n$ converges to some compact set $\Lambda$. By Box 1.13 in Santambrogio (2015), we have $\text{supp}(\mathcal{H}) \subset \Lambda$. Since $\mathcal{H}_n$ is strictly single-dipped by Lemma 1, we have $|\text{supp}(P_n)| \leq 2$ for all $P_n \in \Lambda_n$, and thus $|\text{supp}(P)| \leq 2$ for all $P \in \Lambda$.

Suppose part 2 fails. Then, by Lemmas 10, 12, and 13, there exist $P, P' \in \text{supp}(\mathcal{H})$ such that $s_1(P') < s_1(P) < r^*_+(F) < s_2(P') < s_2(P)$. But then since $\Lambda_n \to \Lambda$, there exist $n$ and $P_n, P'_n \in \Lambda_n$ such that $\text{supp}(P_n) = \{s_1(P_n), s_2(P_n)\}$, $\text{supp}(P'_n) = \{s_1(P'_n), s_2(P'_n)\}$, and $s_1(P'_n) < s_1(P_n) < r^*_+(F) < s_2(P'_n) < s_2(P_n)$, contradicting that $\mathcal{H}_n$ is strictly single-dipped.

Suppose part 1 fails. Then, by Lemmas 10 and 12, there exists $P \in \text{supp}(\mathcal{H})$ such that $\text{supp}(P) = \{s, s'\}$ with $\underline{s} \leq s < s' \leq s^*(0)$. Moreover, by Lemmas 10 and 12 and part 2, there exists $P' \in \text{supp}(\mathcal{H})$ such that $\text{supp}(P') = \{s^*(0), \overline{s}\}$. But then since $\Lambda_n \to \Lambda$, there exist $n$ and $P_n, P'_n \in \Lambda_n$ with $(s_1(P_n), s_2(P_n), s_1(P'_n), s_2(P'_n))$ close to $(s, s', s^*(0), \overline{s})$. Then, by Lemma 5, $\mathcal{H}_n$ cannot be segregate-pair, contradicting Theorem 1. *Q.E.D.*

To complete the proof of Theorem 3, note that Lemmas 10 (for $r = 0$), 12, 13, and 14 show that if a sequence of optimal plans $\mathcal{H}_\eta$ converges to $\mathcal{H}$, then $\mathcal{H}$ must segregate types below $s^*(0)$ and pair types above $s^*(0)$ in a negatively assortative manner in equally strong districts. The unique such plan is $\mathcal{H} = \mathcal{H}^*$. Finally, since every convergent sequence $\mathcal{H}_n$ converges to $\mathcal{H}^*$, compactness of $\Delta\Delta[\underline{s}, \overline{s}]$ implies that $\mathcal{H}_\eta$ also converges to $\mathcal{H}^*$.

### A.8. *Proof of Theorem 4*

Theorem 4 follows from Lemmas 15–18.

Let $\mathcal{T}$ be the set of all plans $\mathcal{H}$ that maximize the designer's seat share when each voter's idiosyncratic shock is 0. Lemma 15 shows that $\mathcal{H} \in \mathcal{T}$ iff each district $P \in$

supp$(H)$ contains 50% voters with some type $s^P \geq s^m$ and 50% voters with types $s \leq s^m$ (so the designer wins district $P$ iff $r \leq s^P$).

For $P \in \Delta[\underline{s}, \overline{s}]$, define $\overline{P}(r) = \int \mathbf{1}\{s \geq r\} dP(s) = 1 - P(r_-)$ for all $r$. The designer wins district $P$ iff the aggregate shock $r$ satisfies $r \leq r_0^*(P) = \{\max \tilde{r} : \overline{P}(\tilde{r}) \geq 1/2\}$. For $\mathcal{H} \in \Delta\Delta[\underline{s}, \overline{s}]$, define $\overline{H}(r) = \int \mathbf{1}\{r_0^*(P) \geq r\} d\mathcal{H}(P)$ for all $r$.

LEMMA 15: *$\mathcal{H} \in \mathcal{T}$ iff, for each $P \in \text{supp}(\mathcal{H})$, there exists $s^P \geq s^m$ such that $P(s) = 1$ for all $s \geq s^P$, $P(s) = 1/2$ for all $s \in [s^m, s^P)$, and $P(s) \leq 1/2$ for all $s < s^m$.*

PROOF OF LEMMA 15: For each $r \geq s^m$, we have

$$\begin{aligned}\overline{F}(r) &= \int \overline{P}(r) d\mathcal{H}(P) = \int \mathbf{1}\{\overline{P}(r) \geq \tfrac{1}{2}\} \overline{P}(r) d\mathcal{H}(P) + \int \mathbf{1}\{\overline{P}(r) < \tfrac{1}{2}\} \overline{P}(r) d\mathcal{H}(P) \\ &\geq \int \mathbf{1}\{\overline{P}(r) \geq \tfrac{1}{2}\} \tfrac{1}{2} d\mathcal{H}(P) = \int \mathbf{1}\{r_0^*(P) \geq r\} \tfrac{1}{2} d\mathcal{H}(P) = \tfrac{1}{2} \overline{H}(r).\end{aligned} \tag{14}$$

So, any feasible $\mathcal{H}$ satisfies $\overline{H}(r) \leq \overline{H}^*(r)$ for all $r$, where

$$\overline{H}^*(r) = \begin{cases} 1, & \text{if } r \leq s^m, \\ 2\overline{F}(r), & \text{if } r > s^m. \end{cases}$$

Thus, the designer's expected seat share for any feasible plan is $\int \overline{H}(r) dG(r) \leq \int \overline{H}^*(r) dG(r)$, with strict inequality if $\overline{H}(r) < \overline{H}^*(r)$ for some $r$ (and thus on some interval $(r', r)$ with $r' < r$, by continuity of $\overline{H}^*$ and monotonicity and left-continuity of $\overline{H}$), because $G(r)$ is strictly increasing in $r$. Hence, a districting plan $\mathcal{H}$ is optimal iff it induces $\overline{H} = \overline{H}^*$. In turn, $\overline{H} = \overline{H}^*$ iff, for each $r \geq s^m$, the inequality in (14) holds with equality, or equivalently, $\int \mathbf{1}\{\overline{P}(r) = 1/2\} d\mathcal{H}(P) = 2\overline{F}(r)$ and $\int \mathbf{1}\{\overline{P}(r) = 0\} d\mathcal{H}(P) = 1 - 2\overline{F}(r)$. Finally, this holds for all $r \geq s^m$ iff, for each $P \in \text{supp}(\mathcal{H})$, there exists $s^P \geq s^m$ such that $\overline{P}(s) = 0$ for all $s > s^P$, $\overline{P}(s) = 1/2$ for all $s \in (s^m, s^P]$, and $\overline{P}(s) \geq 1/2$ for all $s \leq s^m$. *Q.E.D.*

LEMMA 16: *If $\eta \to 0$, then $V_\eta \to 2\int_{s^m}^{\overline{s}} G(r) dF(r)$.*

PROOF OF LEMMA 16: Let $\mathcal{H}_\eta$ be the optimal plan and let $\mathcal{H}_r$ be any districting plan in $\mathcal{R}_\eta(r)$. We have

$$\begin{aligned}V_\eta &= \iint \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_\eta(P) dG(r) \\ &\leq \iint \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_r(P) dG(r) = \int (1 - F(s_\eta^*(r))) dG(r) = \overline{V}_\eta,\end{aligned}$$

where the inequality holds because $\int \mathbf{1}\{r \le r^*_\eta(P)\} d\mathcal{H}_\eta(P) \le \int \mathbf{1}\{r \le r^*_\eta(P)\} dH_r(P)$ for all $r$ by the definition of $\mathcal{H}_r$.

Let $\mathcal{H}^*_q$, with $q \in (0,1/2)$, be NAD with a $q$-$1-q$ split in each district. Formally, $\mathcal{H}^*_q$ is the unique plan $\mathcal{H}$ such that, for any $P \in \operatorname{supp}(\mathcal{H})$, we have either (a) $\operatorname{supp}(P) = \{s^q\}$ with $s^q = F^{-1}(q)$ or (b) $\operatorname{supp}(P) = \{s_1(P), s_2(P)\}$ such that $\underline{s} \le s_1(P) < s^q < s_2(P) \le \overline{s}$, and $(1-q)F(s_1(P)) = q(1-F(s_2(P)))$. We have

$$V_\eta = \int G(r^*_\eta(P)) d\mathcal{H}_\eta(P) \ge \int G(r^*_\eta(P)) d\mathcal{H}^*_q(P) = \underline{V}_\eta(q),$$

where the inequality holds because $\mathcal{H}_\eta$ is optimal and $\mathcal{H}^*_q$ is feasible.

Suppose now that $\eta \to 0$, which implies that $Q_\eta \to \delta_0$. For each $r$, $1 - F(s^*_\eta(r)) \to \overline{H}^*(r)$, so, by the dominated convergence theorem and integration by parts, $\overline{V}_\eta \to \int \overline{H}^*(r) dG(r) = 2\int_{s^m}^{\overline{s}} G(r) dF(r)$. For $q < 1/2$ and $s < s'$, $r^*_\eta(q\delta_s + (1-q)\delta_{s'}) \to s'$, so, by the dominated convergence theorem, $\underline{V}_\eta(q) \to \int_{s^q}^{\overline{s}} G(r) dF(r)/(1-q)$, which converges to $2\int_{s^m}^{\overline{s}} G(r) dF(r)$ as $q \uparrow 1/2$. *Q.E.D.*

Lemma 17 shows that limit points of optimal plans $H_n = H_{\eta_n}$, for $\eta_n \to 0$, belong to $\mathcal{T}$.

LEMMA 17: *Let $\mathcal{H}_n \to \mathcal{H}$ as $\eta_n \to \infty$. Then $\mathcal{H} \in \mathcal{T}$.*

PROOF OF LEMMA 17: Suppose by contradiction that there exists a sequence $\eta_n \to 0$ such that $\mathcal{H}_n$ converges weakly to $\mathcal{H} \notin \mathcal{T}$. Then we have

$$2\int_{s^m}^{\overline{s}} G(r) dF(r) = \lim_{n\to\infty} \int G(r^*_{\eta_n}(P)) d\mathcal{H}_n(P) \le \int \overline{H}(r) dG(r) < 2\int_{s^m}^{\overline{s}} G(r) dF(r),$$

where the equality is by Lemma 16, the first inequality is by the Portmanteau theorem and integration by parts, and the second inequality is by $\mathcal{H} \notin \mathcal{T}$ and Lemma 15. *Q.E.D.*

Lemma 18 shows that, in the limit, all types are paired in a negatively assortative manner.

LEMMA 18: *Let $\mathcal{H}_n \to \mathcal{H}$ as $\eta_n \to 0$. For any $P, P' \in \operatorname{supp}(\mathcal{H})$, we have $\operatorname{supp}(P) = \{s_1(P), s_2(P)\}$ and $\operatorname{supp}(P') = \{s_1(P'), s_2(P')\}$ with $s_1(P) \le s_2(P)$, $s_1(P') \le s_2(P')$, and $(s_2(P') - s_2(P))(s_1(P) - s_1(P')) \ge 0$.*

PROOF OF LEMMA 18: Denoting $\Lambda_n = \operatorname{supp}(\mathcal{H}_n)$, the same argument as in the proof of Lemma 14 implies that there exists $\Lambda$ such that, up to a subsequence, $\Lambda_n \to \Lambda$, $\operatorname{supp}(\mathcal{H}) \subset \Lambda$, and $|\operatorname{supp}(P)| \le 2$ for all $P \in \Lambda$.

By Lemmas 15 and 17, if the conclusion of the lemma fails, there must exist $P, P' \in \operatorname{supp}(\mathcal{H})$ such that $\operatorname{supp}(P) = \{s_1(P), s_2(P)\}$ and $\operatorname{supp}(P') = \{s_1(P'), s_2(P')\}$

with $s_1(P') < s_1(P) < s^m < s_2(P') < s_2(P)$. Then, since $\Lambda_n \to \Lambda$, there exist $n$ and $P_n, P'_n \in \Lambda_n$ such that $\mathrm{supp}(P_n) = \{s_1(P_n), s_2(P_n)\}$, $\mathrm{supp}(P'_n) = \{s_1(P'_n), s_2(P'_n)\}$, and $s_1(P'_n) < s_1(P_n) < s^m < s_2(P'_n) < s_2(P_n)$, contradicting that $\mathcal{H}_n$ is strictly single-dipped. *Q.E.D.*

To complete the proof of Theorem 4, note that Lemmas 15, 17, and 18 show that if a sequence of optimal plans $\mathcal{H}_\eta$ converges to $\mathcal{H}$, then $\mathcal{H}$ must pair all types in a negatively assortative manner, with 50% mass on the higher type. Clearly, the unique such plan is $\mathcal{H} = \mathcal{H}^{**}$. Since every convergent sequence $\mathcal{H}_n$ converges to $\mathcal{H}^{**}$, compactness of $\Delta\Delta[\underline{s}, \overline{s}]$ implies that $\mathcal{H}_\eta$ also converges to $\mathcal{H}^{**}$.

### *A.9. Proof of Theorem 5*

By Lemma 2, $\lambda(r)$ has a derivative $\lambda'(r)$ at each $r \in (r^b, r^b + \varepsilon]$ satisfying

$$g(r) - \lambda(r) q(s_2(r) - r) + \lambda'(r) \left( Q(s_2(r) - r) - Q(0) \right) = 0,$$
$$g(r) - \lambda(r) q(s_1(r) - r) + \lambda'(r) \left( Q(s_1(r) - r) - Q(0) \right) = 0.$$

Solving for $\lambda(r)$ and $\lambda'(r)$ yields, for all $r \in (r^b, r^b + \varepsilon]$,

$$\lambda(r) = \frac{g(r)[Q(s_2(r) - r) - Q(s_1(r) - r)]}{\left(Q(s_2(r) - r) - Q(0)\right) q(s_1(r) - r) - \left(Q(s_1(r) - r) - Q(0)\right) q(s_2(r) - r)},$$
$$\lambda'(r) = \frac{g(r)[q(s_2(r) - r) - q(s_1(r) - r)]}{\left(Q(s_2(r) - r) - Q(0)\right) q(s_1(r) - r) - \left(Q(s_1(r) - r) - Q(0)\right) q(s_2(r) - r)}.$$

Since $\lambda'$ is the derivative of $\lambda$, we have $d\lambda(r)/dr = \lambda'(r)$ for all $r \in (r^b, r^b + \varepsilon]$. Since $s_1$ and $s_2$ are twice differentiable and satisfy $\lim_{r \downarrow r^b} s_1(r) = \lim_{r \downarrow r^b} s_2(r) = r^b$, we can apply L'Hopital's rule to evaluate $d\lambda(r)/dr = \lambda'(r)$ in the limit $r \downarrow r^b$ to obtain

$$\frac{g'(r^b) q(0)}{(q(0))^2} = \frac{g(r^b) q'(0)}{(q(0))^2},$$

which implies that $r^b = 0$, because $G(r) = Q(\eta r)$ for all $r$ and $q'(r) = 0$ iff $r = 0$. Denote $\lim_{r \downarrow r^b} s'_1(r) = 1 - \beta_1$ and $\lim_{r \downarrow r^b} s'_2(r) = 1 + \beta_2$, where $\beta_1 \geq 1$ (because $s_1$ is decreasing) and $\beta_2 \geq 0$ (because $s_2(r) > r$). Differentiating $d\lambda(r)/dr = \lambda'(r)$ with respect to $r$ and taking the limit $r \downarrow 0$, we get

$$\frac{\eta q''(0)(\eta^2 - \beta_2 \beta_1)}{q(0)} = \frac{\eta q''(0)(\beta_2 - \beta_1)}{2q(0)},$$

and hence

$$2\eta^2 = 2\beta_2 \beta_1 + \beta_2 - \beta_1. \tag{15}$$

Since, for small enough $r>0$, type $s_1(r)$ is assigned to both district $\delta_{s_1(r)}$ and district $P$ with $r^*(P)=r$ and $\text{supp}(P)=\{s_1(r),s_2(r)\}$, we must have, by Lemma 2,

$$Q(\eta s_1(r)) = Q(\eta r) + \lambda(r)\left(Q(s_1(r)-r) - Q(0)\right).$$

In the limit $r \downarrow 0$, the values and the derivatives up to order 2 of both sides always coincide, while the third derivatives coincide iff

$$q''(0)\eta^3(-\beta_1+1)^3 = q''(0)\eta^3 - 3q''(0)\eta^3\beta_1 + 3q''(0)\eta\beta_2\beta_1^2 - q''(0)\eta\beta_1^3,$$

which simplifies to

$$-\eta^2\beta_1 + 3\eta^2 = 3\beta_2 - \beta_1. \tag{16}$$

Since, for small enough $r>0$, type $s_1(r)$ is assigned to both district $\delta_{s_1(r)}$ and district $P$ with $r^*(P)=r$, while type $s_2(r)$ is assigned only to district $P$, we have

$$f(s_1(r))s_1'(r)\left(Q(s_1(r)-r)-Q(0)\right) \geq f(s_2(r))s_2'(r)\left(Q(s_2(r)-r)-Q(0)\right).$$

In the limit $r \downarrow 0$, both sides are equal, and hence their derivatives must satisfy

$$-f(0)q(0)\beta_1(1-\beta_1) \geq f(0)q(0)\beta_2(\beta_2+1),$$

which, given that $\beta_1+\beta_2>0$, simplifies to

$$\beta_1 \geq \beta_2 + 1. \tag{17}$$

Recalling that $\gamma = \eta^2/(1+\eta^2)$, equations (15) and (16) have two solutions

$$(\beta_1,\beta_2) = \left(\tfrac{3\eta^2}{(2(\eta^2-1))}, \tfrac{\eta^2}{2}\right) = \left(\tfrac{3\gamma}{2(2\gamma-1)}, \tfrac{\gamma}{2(1-\gamma)}\right) \quad \text{and} \quad (\beta_1',\beta_2') = \left(1, \tfrac{(2\eta^2+1)}{3}\right) = \left(1, \tfrac{\gamma+1}{3(1-\gamma)}\right),$$

unless $\gamma = 1/2$, in which case (15) and (16) have only one solution $(\beta_1,\beta_2)=(1,1)$. The solution $(\beta_1',\beta_2')$ never satisfies (17) and thus is discarded. Moreover, for the solution $(\beta_1,\beta_2)$, condition $\beta_1 \geq 1$ yields $\gamma > 1/2$, and condition (17) yields $\gamma \leq \sqrt{3}-1$. Thus, for Y-districting to be optimal, we must have $\gamma \in (1/2, \sqrt{3}-1]$. Finally, the statement in Footnote 34 holds because

$$\lim_{r\downarrow 0} s_1'(r) = 1-\beta_1 = -\frac{2-\gamma}{2(2\gamma-1)} < 0 \quad \text{and} \quad \lim_{r\downarrow 0} s_2'(r) = 1+\beta_2 = \frac{2-\gamma}{2(1-\gamma)} > 0$$

are both strictly increasing in $\gamma$.

## APPENDIX B: Estimators

In this section, we formally define our estimators and show that they satisfy standard statistical properties. Fix a US state. We assume throughout that there is a large number of voters, so that the vote share in a precinct $n$ with type $s_n$ in district $d$ and election $y$ with aggregate shock $r_{dy}$ is given by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$. Let $\mu_s$ and $\sigma_s^2$ be the mean and variance of the distribution of precinct types, defined by $\mu_s = \mathbb{E}_F[s]$ and $\sigma_s^2 = Var_F[s]$. For convenience, we repeat some definitions from the main text. Let $w_{ny} = \Phi^{-1}(v_{ny})$, $T$ denote the number of elections, $D$ the number of districts, and $\mathcal{N}_{dy}$ the set of precincts in district $d$ and election $y$. Define

$$w_{dy} = \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} w_{ny}}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}}, \quad w_{d\bullet} = \frac{\sum_y w_{dy}}{T}, \quad w_{\bullet y} = \frac{\sum_d w_{dy}}{D}, \quad w_{\bullet\bullet} = \frac{\sum_{d,y} w_{dy}}{DT},$$

$$e_n^2 = \frac{1}{DT} \sum_{d,y} \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} (w_{ny} - w_{\bullet y})^2}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}},$$

$$e_d^2 = \frac{\sum_{d,y} (w_{dy} - w_{d\bullet})^2}{D(T-1)}, \quad e^2 = \frac{\sum_y (w_{\bullet y} - w_{\bullet\bullet})^2}{T-1},$$

$$cov = \frac{\sum_{y,d,d'>d} (w_{dy} - w_{d\bullet})(w_{d'y} - w_{d'\bullet})}{\frac{D(D-1)}{2}(T-1)} = \frac{De^2 - e_d^2}{D-1},$$

where the last equality follows from

$$e^2 = \frac{\sum_y \left( \sum_d \frac{1}{D} (w_{dy} - w_{d\bullet}) \right)^2}{(T-1)}$$

$$= \frac{1}{D} \frac{\sum_{d,y} (w_{dy} - w_{d\bullet})^2}{D(T-1)} + \frac{D-1}{D} \frac{\sum_{y,d,d'>d} (w_{dy} - w_{d\bullet})(w_{d'y} - w_{d'\bullet})}{\frac{D(D-1)}{2}(T-1)}$$

$$= \frac{1}{D} e_d^2 + \frac{D-1}{D} cov.$$

To construct our estimators, we use the following proposition.

Proposition 1: *In our empirical model,*

$$\mathbb{E}e_d^2 = \frac{1-\gamma}{\gamma}, \quad \mathbb{E}cov = \rho \frac{1-\gamma}{\gamma}, \quad \mathbb{E}w_{\bullet\bullet} = \frac{\mu_s}{\sqrt{\gamma}}, \quad \textit{and} \quad \mathbb{E}e_n^2 = \frac{\sigma_s^2}{\gamma} + (1-\rho) \frac{D-1}{D} \frac{1-\gamma}{\gamma},$$

*and*

$$e_d^2 \stackrel{d}{=} \frac{1-\gamma}{D(T-1)\gamma} \left[ (1-\rho) \chi^2_{(D-1)(T-1)} + (1 + (D-1)\rho) \chi^2_{T-1} \right],$$

*where* $\stackrel{d}{=}$ *denotes equality in distribution, and* $\chi^2_{(D-1)(T-1)}$ *and* $\chi^2_{T-1}$ *denote independent* $\chi^2$ *random variables with* $(D-1)(T-1)$ *and* $T-1$ *degrees of freedom, respectively.*

Consider the following point estimators of $\gamma$, $\rho$, $\mu_s$, and $\sigma_s$:

$$\widehat{\gamma}=\frac{1}{1+e_d^2},\quad \widehat{\rho}=\frac{cov}{e_d^2},\quad \widehat{\mu}_s=\frac{w_{\bullet\bullet}}{\sqrt{1+e_d^2}},\quad \text{and}\quad \widehat{\sigma}_s=\sqrt{\frac{e_n^2-\frac{D-1}{D}(e_d^2-cov)}{1+e_d^2}}.$$

By Proposition 1, $1/\widehat{\gamma}$, $\widehat{\rho}/\widehat{\gamma}-\widehat{\rho}$, $\widehat{\mu}_s/\sqrt{\widehat{\gamma}}$, and $\widehat{\sigma}_s^2/\widehat{\gamma}$ are unbiased estimators of $1/\gamma$, $\rho/\gamma-\rho$, $\mu_s/\sqrt{\gamma}$, and $\sigma_s^2/\gamma$. Moreover, by the law of large numbers for $D(T-1)\to\infty$, we have that $\widehat{\gamma}$, $\widehat{\rho}$, $\widehat{\mu}_s$, and $\widehat{\sigma}_s$ are consistent estimators of $\gamma$, $\rho$, $\mu_s$, and $\sigma_s$.

Proposition 1 also gives us a confidence interval for $\gamma$. Specifically, for any $\alpha\in(0,1)$, let $q_\alpha$ be the $\alpha$-quantile for $(1-\widehat{\rho})\chi^2_{(D-1)(T-1)}+(1+(D-1)\widehat{\rho})\chi^2_{T-1}$. Then, a one-sided $1-\alpha$ confidence interval for $\gamma$ is $(\widehat{\gamma}_\alpha,1)$ where

$$\widehat{\gamma}_\alpha=\frac{1}{1+\frac{D(T-1)}{q(\alpha)}e_d^2}.$$

Proof of Proposition 1: Denote

$$r_{d\bullet}=\frac{\sum_y r_{dy}}{T},\quad r_{\bullet y}=\frac{\sum_d r_{dy}}{D},\quad s_{dy}=\frac{\sum_{n\in\mathcal{N}_{dy}}k_{ny}s_n}{\sum_{n\in\mathcal{N}_{dy}}k_{ny}},\quad s_{\bullet y}=\frac{\sum_d s_{dy}}{D}.$$

First, we have

$$\mathbb{E}w_{\bullet\bullet}=\mathbb{E}\frac{1}{DT}\sum_{d,y}\frac{\sum_{n\in N_{dy}}k_{ny}(s_n-r_{dy})}{\sqrt{\gamma}\sum_{n\in N_{dy}}k_{ny}}=\mathbb{E}\frac{\sum_{n\in N_{dy}}k_{ny}s_n}{\sqrt{\gamma}\sum_{n\in N_{dy}}k_{ny}}=\frac{\mu_s}{\sqrt{\gamma}},$$

where the first equality is by $v_{ny}=\Phi((s_n-r_{dy})/\sqrt{\gamma})$ and the definition of $v_{ny}$ and $w_{\bullet\bullet}$, the second is by $\mathbb{E}[r_{dy}]=0$ and district equipopulation, and the fourth is by the definition of $\mu_s$. Second, we have

$$\mathbb{E}e_d^2=\mathbb{E}\frac{\sum_{d,y}\left(\frac{T-1}{T}r_{dy}-\frac{1}{T}\sum_{y'\neq y}r_{dy'}\right)^2}{D(T-1)\gamma}=\frac{DT\left[\left(\frac{T-1}{T}\right)^2+\frac{T-1}{T^2}\right](1-\gamma)}{D(T-1)\gamma}=\frac{1-\gamma}{\gamma},$$

where the first equality is by $v_{ny}=\Phi((s_n-r_{dy})/\sqrt{\gamma})$, the definition of $w_{dy}$ and $w_{d\bullet}$, and rearrangement, the second is by $Var[r_{dy}]=1-\gamma$ and $Cov[r_{dy},r_{dy'}]=0$ for $y\neq y'$, and the third is by rearrangement. Third, we have

$$\mathbb{E}cov=\mathbb{E}\frac{\sum_{y,d,d'>d}(r_{dy}-r_{d\bullet})(r_{d'y}-r_{d'\bullet})}{\frac{D(D-1)}{2}(T-1)\gamma}=\rho\frac{1-\gamma}{\gamma},$$

where the first equality is again by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$ and the definition of $w_{dy}$ and $w_{d\bullet}$, and the second is by $Cov[r_{dy}, r_{d'y}] = \rho(1-\gamma)$ for $d \neq d'$, $Cov[r_{dy}, r_{d'y'}] = 0$ for $y \neq y'$, and rearrangement. Fourth, we have

$$\mathbb{E}e_n^2 = \mathbb{E}\frac{1}{DT}\sum_{d,y}\frac{\sum_{n\in\mathcal{N}_{dy}} k_{ny}(s_n - s_{\bullet y} + r_{dy} - r_{\bullet y})^2}{\gamma\sum_{n\in\mathcal{N}_{dy}} k_{ny}} = \mathbb{E}\frac{\sum_{n\in\mathcal{N}_{dy}} k_{ny}(s_n - s_{\bullet y})^2}{\gamma\sum_{n\in\mathcal{N}_{dy}} k_{ny}}$$

$$+\mathbb{E}\frac{\sum_d (r_{dy} - r_{\bullet y})^2}{\gamma D} = \frac{\sigma_s^2}{\gamma} + \mathbb{E}\frac{\sum_d \left(\frac{D-1}{D} r_{dy} - \frac{1}{D}\sum_{d'\neq d} r_{d'y}\right)^2}{\gamma D}$$

$$= \frac{\sigma_s^2}{\gamma} + \mathbb{E}\frac{\sum_d \left[\left(\frac{D-1}{D}\right)^2 r_{dy}^2 + \frac{1}{D^2}\sum_{d'\neq d} r_{d'y}^2 - \frac{2(D-1)}{D^2} r_{dy} r_{d'y} + \frac{2}{D^2}\sum_{d'\neq d, d''>d'} r_{d'y} r_{d''y}\right]}{\gamma D}$$

$$= \frac{\sigma_s^2}{\gamma} + \left[\left(\frac{D-1}{D}\right)^2 + \frac{D-1}{D^2} - \rho\frac{2(D-1)}{D^2} + \rho\frac{(D-1)(D-2)}{D^2}\right]\frac{1-\gamma}{\gamma}$$

$$= \sigma_s^2 + (1-\rho)\frac{D-1}{D}\frac{1-\gamma}{\gamma},$$

where the first equality is by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$, the definition of $w_{ny}$ and $w_{\bullet y}$, and rearrangement, the second is by independence across elections and district equipopulation, the third is by the large number of voters and rearrangement of the second term, the fourth is by quadratic expansion, the fifth is by $\mathbb{E}[r_{dy}^2] = 1-\gamma$ and $\mathbb{E}[r_{dy} r_{d'y}] = \rho(1-\gamma)$ for $d' \neq d$, and the sixth is by rearrangement.

Finally, let $r = (r_{11}, \ldots, r_{1T}, \ldots, r_{D1}, \ldots, r_{DT})'$. Then we can write

$$\sum_{d,y}(r_{dy} - r_{d\bullet})^2 = r' A r$$

where

$$A = \begin{pmatrix} \frac{T-1}{T} & \ldots & -\frac{1}{T} & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & \ldots & \frac{T-1}{T} & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & \frac{T-1}{T} & \ldots & -\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & -\frac{1}{T} & \ldots & \frac{T-1}{T} \end{pmatrix}.$$

Note that

$$\frac{\mathbb{E}[rr']}{1-\gamma} = \Sigma = \begin{pmatrix} 1 \dots 0 \dots \rho \dots 0 \\ \vdots \ddots \vdots \ddots \vdots \ddots \vdots \\ 0 \dots 1 \dots 0 \dots \rho \\ \vdots \ddots \vdots \ddots \vdots \ddots \vdots \\ \rho \dots 0 \dots 1 \dots 0 \\ \vdots \ddots \vdots \ddots \vdots \ddots \vdots \\ 0 \dots \rho \dots 0 \dots 1 \end{pmatrix}.$$

By the spectral theorem, there is an orthogonal matrix $P$ (so that $P'P = P'P = I$) and a diagonal matrix $\Lambda$ with positive diagonal elements $\lambda_1, \dots, \lambda_{DT}$ such that $\Sigma^{1/2} A \Sigma^{1/2} = P' \Lambda P$. Define $u = P \Sigma^{-1/2} r / \sqrt{1-\gamma}$ (so that $r = \Sigma^{1/2} P' u \sqrt{1-\gamma}$). Then

$$\frac{r'Ar}{1-\gamma} = u'P\Sigma^{1/2}A\Sigma^{1/2}P'u = u'PP'\Lambda PP'u = u'\Lambda u = \sum_{i=1}^{DT} \lambda_i u_i^2$$

where $u \sim N(0, I)$, and $\lambda_1, \dots \lambda_{DT}$ are the roots of the characteristic equation

$$|\Sigma^{1/2} A \Sigma^{1/2} - \lambda I| = 0 \iff |A\Sigma - \lambda I| = 0.$$

Note that

$$A\Sigma = \begin{pmatrix} \frac{T-1}{T} & \dots & -\frac{1}{T} & \dots & \rho\frac{T-1}{T} & \dots & -\rho\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & \dots & \frac{T-1}{T} & \dots & -\rho\frac{1}{T} & \dots & \rho\frac{T-1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho\frac{T-1}{T} & \dots & -\rho\frac{1}{T} & \dots & \frac{T-1}{T} & \dots & -\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\rho\frac{1}{T} & \dots & \rho\frac{T-1}{T} & \dots & -\frac{1}{T} & \dots & \frac{T-1}{T} \end{pmatrix}.$$

After some algebra, we obtain

$$|A\Sigma - \lambda I| = (-1)^{DT} \lambda^D (\lambda - 1 + \rho)^{(D-1)(T-1)} (\lambda - 1 - (D-1)\rho)^{T-1},$$

showing that $r'Ar/(1-\gamma) \stackrel{d}{=} (1-\rho)\chi^2_{(D-1)(T-1)} + (1+(D-1)\rho)\chi^2_{T-1}$, and hence

$$e_d^2 = \frac{r'Ar}{D(T-1)\gamma} \stackrel{d}{=} \frac{1-\gamma}{D(T-1)\gamma}\left[(1-\rho)\chi^2_{(D-1)(T-1)} + (1+(D-1)\rho)\chi^2_{T-1}\right]. \qquad \textit{Q.E.D.}$$