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A Short Proof of the No-Trade Theorem

Hendren (2013), *Private Information and Insurance Rejections*

Let P have an arbitrary distribution F with support $\Psi \subseteq [0,1]$. Type p has endowment $(w-l, w)$ in the loss and no-loss states and expected utility $pu(c_L) + (1-p)u(c_{NL})$, where u is twice continuously differentiable, $u' > 0$, and $u'' < 0$. An allocation is *implementable* if it is incentive compatible (IC), individually rational relative to the endowment (IR), and satisfies

$$\mathbb{E}\big[Pc_L(P) + (1-P)c_{NL}(P)\big] \le w - l\mathbb{E}[P].$$

Theorem 1 (No trade). *The endowment is the only implementable allocation if and only if*

$$\frac{p}{1-p}\frac{u'(w-l)}{u'(w)} \le \frac{\mathbb{E}[P \mid P \ge p]}{1-\mathbb{E}[P \mid P \ge p]} \qquad \textit{for every } p \in \Psi \setminus \{1\}. \tag{1}$$

If this condition fails, there is an implementable allocation with strictly positive aggregate resource slack and a strict utility improvement for a positive mass of types.

Uniqueness is understood up to F-null types and consumption in states of probability zero. The integral formulation below avoids undefined ratios when a tail consists entirely of type 1.

Proof. Define

$$a = \frac{1}{u'(w-l)}, \qquad b = \frac{1}{u'(w)}, \qquad 0 < a < b,$$

and, for any cutoff $t \in [0,1]$, let

$$K(t) = \mathbb{E}\left[\big(aP(1-t) - b(1-P)t\big)\mathbf{1}_{\{P \ge t\}}\right]. \tag{2}$$

Condition (1) is equivalent to $K(t) \ge 0$ on $\Psi \setminus \{1\}$. It implies this inequality on all of $[0,1]$: $K(1) = 0$; in any gap in the support, moving the cutoff down from the next support point leaves its pool unchanged and increases the integrand. If the tail is empty, $K(t) = 0$.

1. Express IC and IR through a convex utility gain. For any IC, IR allocation, write its state-specific utility gains as

$$x(p) = u(c_L(p)) - u(w-l), \qquad y(p) = u(c_{NL}(p)) - u(w).$$

Include the endowment in the menu and define, for every $p \in [0,1]$,

$$V(p) = \max\left\{0,\ \sup_{q\in\Psi}\big[px(q) + (1-p)y(q)\big]\right\}. \tag{3}$$

The function V is convex and nonnegative.[1] At actual types, IC and IR imply

$$V(p) = px(p) + (1-p)y(p), \qquad s(p) := x(p) - y(p) \in \partial V(p). \tag{4}$$

[1]For a fixed contract q, $px(q) + (1-p)y(q) = y(q) + p[x(q) - y(q)]$ is affine in p. The pointwise supremum of affine functions, including the constant zero, is convex: at $p_\lambda = \lambda p_1 + (1-\lambda)p_2$, each contract's value is at most $\lambda V(p_1) + (1-\lambda)V(p_2)$, and taking the supremum preserves this bound.

Thus $x(p) = V(p) + (1-p)s(p)$ and $y(p) = V(p) - ps(p)$. All functions used below are bounded: IC implies that x is nondecreasing and y is nonincreasing on the compact support, so their endpoint values bound the menu. In particular, V is Lipschitz.

2. Bound resource cost by a linear functional of V. The inverse utility function is strictly convex. Its tangent inequalities at the endowment therefore give

$$c_L(p) - (w-l) \geq ax(p), \qquad c_{NL}(p) - w \geq by(p).$$

Let C denote aggregate resource cost relative to the endowment. Substituting (4) yields

$$\begin{aligned} C &:= \mathbb{E}\big[P(c_L(P) - (w-l)) + (1-P)(c_{NL}(P) - w)\big] \\ &\geq \mathbb{E}\big[aPx(P) + b(1-P)y(P)\big] \\ &= \mathbb{E}\big[(aP + b(1-P))V(P) - (b-a)P(1-P)s(P)\big] \\ &\geq \mathcal{L}(V), \end{aligned} \tag{5}$$

where

$$\mathcal{L}(V) = \mathbb{E}\big[(aP + b(1-P))V(P) - (b-a)P(1-P)V'_+(P)\big]. \tag{6}$$

The last inequality follows from $s(p) \leq V'_+(p)$ for interior p. The derivative's coefficient is zero at the endpoints, so its value there is irrelevant. The first inequality in (5) is strict for any allocation that changes consumption on a set of positive probability.

3. Check the linear functional on elementary convex functions. Every nonnegative convex Lipschitz function on $[0,1]$ admits a representation

$$V(p) = m + \int_{[0,1]} (p-t)_+ \, d\mu_+(t) + \int_{[0,1]} (t-p)_+ \, d\mu_-(t), \tag{7}$$

where $m \geq 0$ and μ_+, μ_- are finite nonnegative measures. To obtain this representation, choose a minimizer r of V and set $m = V(r)$. To the right of r, integrate the nonnegative, nondecreasing slope; to the left, integrate the nonpositive slope backward. Changes in slope supply the measures, with any initial slopes represented by mass at r. If the minimum is at an endpoint, only the corresponding side is needed.

Because $\mathcal{L}$ is linear, it suffices to check the three components in (7). First,

$$\mathcal{L}(1) = \mathbb{E}[aP + b(1-P)] > 0.$$

For an increasing hinge, its right derivative is $\mathbf{1}_{\{p \geq t\}}$, giving

$$\mathcal{L}((p-t)_+) = \mathbb{E}\left[(aP(1-t) - b(1-P)t)\mathbf{1}_{\{P \geq t\}}\right] = K(t) \geq 0. \tag{8}$$

For a decreasing hinge, its right derivative is $-\mathbf{1}_{\{p<t\}}$, giving

$$\mathcal{L}((t-p)_+) = \mathbb{E}\left[(bt(1-P) - a(1-t)P)\mathbf{1}_{\{P<t\}}\right] \geq 0, \tag{9}$$

since, when $P < t$,

$$bt(1-P) - a(1-t)P = a(t-P) + (b-a)t(1-P) \geq 0.$$

The weak and strict cutoff conventions in (8) and (9) are deliberate; they account for atoms in F. Finiteness of the measures in (7) justifies interchanging expectations and integrals.

Consequently, $\mathcal{L}(V) \geq 0$. Equation (5) then implies $C > 0$ for any nontrivial allocation, whereas resource feasibility requires $C \leq 0$. This proves sufficiency, including when the no-trade condition holds with equality.

4. Construct profitable trade when the condition fails. Suppose $K(p_0) < 0$ at some $p_0 \in \Psi \setminus \{1\}$. Necessarily $p_0 > 0$. The function K is left-continuous by dominated convergence, including at atoms. Hence choose $t < p_0$ sufficiently close to p_0 that $K(t) < 0$.

Offer the endowment and a single contract with utility gains

$$(x_\varepsilon, y_\varepsilon) = (\varepsilon(1-t), -\varepsilon t), \qquad \varepsilon > 0.$$

Its expected utility gain for type p is exactly

$$px_\varepsilon + (1-p)y_\varepsilon = \varepsilon(p-t).$$

Assign the contract to types $p \geq t$ and the endowment to types $p < t$. This allocation is IC and IR. A positive mass of types strictly benefits because p_0 is in the support and $t < p_0$.

The contract's consumption levels are

$$c_L^\varepsilon = u^{-1}\big(u(w-l) + \varepsilon(1-t)\big), \qquad c_{NL}^\varepsilon = u^{-1}\big(u(w) - \varepsilon t\big).$$

For sufficiently small ε these are well-defined. Their aggregate resource cost satisfies

$$C(0) = 0, \qquad C'(0) = \mathbb{E}\left[\big(aP(1-t) - b(1-P)t\big)\mathbf{1}_{\{P \geq t\}}\right] = K(t) < 0.$$

Thus sufficiently small $\varepsilon > 0$ gives $C(\varepsilon) < 0$: the allocation is implementable and leaves strictly positive resource slack. □

Interpretation. IC makes the utility gain convex, so it decomposes into elementary contracts serving upper or lower tails. The no-trade condition makes the linearized resource cost of each upper-tail component nonnegative; risk aversion does the same for the lower-tail components. Strict convexity of consumption cost then rules out every nontrivial menu.

Source. Nathaniel Hendren (2013), "Private Information and Insurance Rejections," *Econometrica* 81(5), 1713–1762, Theorem 1, p. 1719. Published paper; Supplemental Appendix A.1. The argument above is an alternative proof.