IMPLEMENTATION VIA INFORMATION DESIGN USING GLOBAL GAMES

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Abstract. For binary-action supermodular games with a continuum of symmetric players, we show that simple global game information structures can be used to implement an optimal outcome under adversarial equilibrium selection.

1. Introduction

In Morris et al. (2020), we studied implementation by information design in binary-action supermodular (BAS) games. An outcome is smallest equilibrium implementable if there exists an information structure such that that outcome is induced by the smallest equilibrium of the game with that information structure. We characterized smallest equilibrium implementable outcomes in general finite BAS games. In particular, we provided a canonical implementing information structure that works for all implementable outcomes and all BAS games.

In this companion note, we provide a simple, alternative smallest equilibrium implementation of a particular "target outcome" under the additional restrictions that payoffs are symmetric and higher states give a higher incentive to choose the high action. The target outcome is an (approximate) solution to an information design problem—to be described in detail below—where the information designer prefers the high action to be chosen. The target outcome is one where all players choose the high action if and only if the state is above some threshold state, where the threshold state is the lowest state

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Date: July 1, 2022.
Keywords. Information design, supermodular game, smallest equilibrium implementation, global game, potential game.
We are grateful for the discussions with Fei Li. Stephen Morris gratefully acknowledges financial support from NSF Grant SES-2049744 and SES-1824137. Daisuke Oyama gratefully acknowledges financial support from JSPS KAKENHI Grants 18KK0359 and 19K01556. Part of this research was conducted while Daisuke Oyama was visiting the Department of Economics, Massachusetts Institute of Technology, whose hospitality is gratefully acknowledged.

with the property that, conditional on the state being higher than the threshold, the high action is in expectation "Laplacian", i.e., the best response to a uniform belief about the proportion of the players choosing the high action. The target outcome is shown to be smallest equilibrium implementable by first publicly announcing whether or not the state lies above the threshold state, and then creating a latent state that garbles the true state and further having the players observe a noisy signal about the latent state, as in the standard global games.

We carry out our analysis with a continuum of symmetric players and a continuous state space. The class of environments we consider encompasses the regime change games studied by Inostroza and Pavan (2020) and Li et al. (2019) and other coordination games widely studied in the literature. Among many possible constructions that implement the target outcome, ours has the attractive feature that the implementing information structure depends only on the threshold defining the target outcome, but otherwise the same construction works for all symmetric BAS games. Li et al. (2019) present an alternative, arguably more complicated, implementation of the same outcome, which is tailored to regime change games. But a finite signal version of their information structure is an (essentially unique) optimal information structure when the designer is constrained to only use finite information structures with up to K signals, and their unconstrained optimal information structure is characterized as the limit of the optimal finite information structures as $K \to \infty$.

We now describe in more detail (i) the setting, (ii) the target outcome, (iii) the optimal information design problem for which the target outcome is a solution, and (iv) the implementing information structure.

The setting is as follows. A continuum of players choose between a low action, 0, and a high action, 1. A player's "payoff gain" is the difference in payoffs from choosing action 1 over action 0 as a function of the proportion of players choosing action 1 and the state θ unknown to the players. We assume action monotonicity (the payoff gain is nondecreasing in the proportion of players who choose 1), state monotonicity (the payoff gain is nondecreasing in the state θ), and upper dominance region (the payoff gain is positive for sufficiently high θ, even if all other players choose action 0).

We now describe the target outcome. The Laplacian payoff gain at state θ is a player's expected payoff gain if he has a uniform belief over the proportion of other players choosing action 1. Let θ^* be the unique state such that the expected Laplacian payoff gain conditional on $\theta > \theta^*$ is zero. Our "target outcome" is the outcome where all players choose action 1 if and only if $\theta > \theta^*$. The contribution of this note is to show how standard "global game information structures" implement the target outcome.

Arguments from Morris et al. (2020) establish that the target outcome is an (approximate) optimal outcome of an information design problem with adversarial equilibrium selection. Consider an information designer who benefits from more players choosing action 1, with the benefit increasing in the state. The designer can commit to any information structure which sends signals privately or publicly to the players depending on the state. Suppose that the designer chooses an information structure to maximize her objective function anticipating that the worst equilibrium for her will be played in the resulting Bayesian supermodular game. By the action monotonicity of the objective, this will be the smallest equilibrium. Under an additional restriction on the designer's objective function, a discrete analogue of the target outcome is an optimal outcome in an N-player, N-state analogue of the setting in this note, and those optimal outcomes in the discrete games converge to the target outcome as $N \to \infty$.[1] Our conditions cover, in particular, the case of a regime change game where the designer's objective is to maximize the probability that the status quo is maintained (the problem studied by Li et al. (2019)).

We now describe the implementing information structure in more detail. Recall that the target outcome generates "success" (action 1 played by all players) if and only if the state θ is above the critical threshold θ^*. The implementing information structure is constructed as follows. First, we partition the state space into two regions, a "failure region" $(-\infty, \theta^* + \varepsilon)$ and a "success region"for $[\theta^* + \varepsilon, \infty)$, and publicly announce which region the state θ belongs to. Then, if the state θ is in the success region, we garble θ to produce a latent state ω that is distributed uniformly over a certain interval and only reveals which element of two sub-regions, $[\theta^* + \varepsilon, \overline{\theta}')$ and $[\overline{\theta}', \infty)$, θ belongs to, where $[\overline{\theta}', \infty)$ is a "dominance region" $[\overline{\theta}', \infty)$ in which action 1 is a dominant action. Finally,

[1] In Appendix B, we provide a formal treatment of this convergence result. In Appendix C, we present an alternative formulation of the information design problem directly dealing with a continuum of symmetric players and a continuous state space and show that the target outcome is an optimal outcome of this version.

players observe a noisy signal of the latent state ω as in global games. We can choose $\varepsilon > 0$ arbitrarily small and $\overline{\theta}'$ arbitrarily large, so the success region is arbitrarily close to the set of states where there is success under the target outcome. Now if it is publicly announced that the state is in the failure region, the smallest equilibrium will have all players choosing action 0. But if the state is in the success region, the game is equivalent to a global game (with state ω) with a one-sided dominance region and possibly discontinuous payoffs. By a standard iterative dominance argument for global games, the smallest equilibrium will have all players choosing action 1, since, by construction, the Laplacian payoff gain is always positive within this sub-game (Morris and Shin (2003)).[2] Thus the target outcome is approximately smallest equilibrium implementable, where the approximation can be made arbitrarily tight as $\varepsilon \to 0$. Note that the same implementing information structure works for any payoff function satisfying our maintained assumptions. While the details of the implementing information structure are delicate, the main ingredients are very simple: make a public announcement about whether the state is in the success region, and, if the state is in the success region, introduce noisy global game signals about its exact value.

2. Setting

We consider a setting with a continuum of players and symmetric payoffs as in Morris and Shin (2003, Section 2.2). Each player i chooses an action $a_i \in A_i = \{0, 1\}$. A state $\theta \in \Theta$ is drawn according to a distribution function P with support Θ, where $\Theta \subset \mathbb{R}$ is a closed interval (with nonempty interior $\operatorname{int}\Theta$). A player's payoff gain function is given by $d\colon [0,1] \times \Theta \to \mathbb{R}$, where $d(\ell, \theta)$ is the gain from choosing action 1 over action 0 when proportion ℓ of players choose action 1 and the state is θ.

We impose the following assumptions:

A1. Action Monotonicity: For each θ, $d(\ell, \theta)$ is nondecreasing in ℓ.

A2. State Monotonicity: For each ℓ, $d(\ell, \theta)$ is nondecreasing in θ.

A3. Upper Dominance Region: There exists $\overline{\theta} \in \operatorname{int}\Theta$ such that $d(0, \theta) > 0$ for all $\theta \geq \overline{\theta}$.

Under A1, A3 implies that $d(\ell, \theta) > 0$ for all $\ell \in [0, 1]$ and $\theta \geq \overline{\theta}$.

[2] In Appendix A, we provide a formal proof that accommodates payoff discontinuity, under a restriction on the noise distribution.

We write

$$\bar{\Phi}(\theta) = \int_0^1 d(\ell, \theta) d\ell$$

for the *Laplacian payoff gain* at state θ, namely, the expected payoff gain when a player has a Laplacian (i.e., uniform) belief over the proportion of players choosing action 1 when the state is θ. By A2, $\bar{\Phi}$ is nondecreasing, and by A3, $\bar{\Phi}(\theta) > 0$ for all $\theta \geq \overline{\theta}$. The *Laplacian threshold* is the critical state $\theta^\sharp \in \Theta$ such that the Laplacian payoff gain becomes nonpositive:

$$\theta^\sharp = \sup\{\theta \in \Theta \mid \bar{\Phi}(\theta) \leq 0\}$$

with convention $\sup \emptyset = \inf \Theta$.

In a *global game* (Carlsson and van Damme (1993), Morris and Shin (2003)), players observe the true state with noise. Specifically, each player i observes a private signal $x_i = \theta + \kappa\zeta_i$, $\kappa > 0$, where the noise terms ζ_i are distributed independently of θ and across players according to a density f. Morris and Shin (2003) consider the setting described above that embeds many applications in the literature. Under additional technical assumptions, one can show that action 1 is played whenever $\theta > \theta^\sharp$, if the noise is sufficiently small, by any strategy that survives iterative dominance. For completeness, in Appendix A, we state and prove this result under the assumptions that the distribution of θ admits a continuous and strictly positive density (Assumption A10) and that the conditional distribution of θ given the observation of x_i is increasing in x_i in the first-order stochastic dominance order (Assumption A11), to accommodate possible discontinuity of payoffs in θ in our construction below.

3. Target Outcome

In the following, we assume that the payoffs (and hence $\bar{\Phi}$) are integrable with respect to P:

A4. Integrability: $\int_\Theta \max_{\ell \in \{0,1\}} |d(\ell, \theta)| dP(\theta) < \infty$.

Note that, under A1 and A2, this condition is automatically satisfied if Θ is bounded. To ease the exposition, we also assume that $\int_\Theta \bar{\Phi}(\theta) dP(\theta) < 0$ and that P is continuous (i.e., the probability measure it induces is atomless).

An outcome is a mapping $\nu \colon \Theta \to \Delta([0,1])$, where $\nu_\theta \in \Delta([0,1])$ is a probability distribution on $[0,1]$ which represents a distribution of the proportion of the players playing action 1 at state θ. We now define the target outcome. The *expected Laplacian*

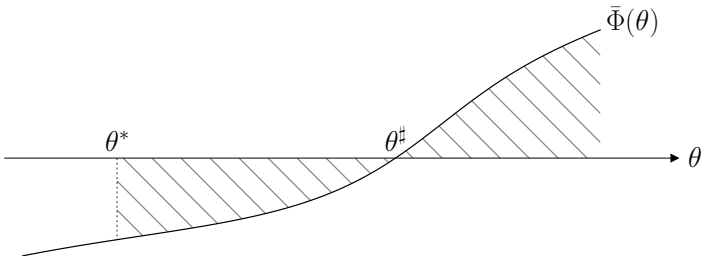

FIGURE 1. Laplacian thresholds

threshold is the critical state $\theta^* \in \operatorname{int}\Theta$ such that the expected Laplacian payoff gain conditional on $\theta > \theta^*$ becomes zero, i.e., the unique $\theta^* \in \operatorname{int}\Theta$ that solves

$$\int_{\theta>\theta^*} \bar{\Phi}(\theta)dP(\theta) = 0. \tag{1}$$

This is well defined by the assumptions of A3, $\int_\Theta \bar{\Phi}(\theta)dP(\theta) < 0$, and the continuity of P. Note that $\theta^* < \theta^\sharp$ by construction (Figure 1). Now the definition of our target outcome ν^* is

$$\nu^*_\theta = \begin{cases} \delta_1 & \text{if } \theta > \theta^*, \\ \delta_0 & \text{if } \theta \leq \theta^*, \end{cases} \tag{2}$$

where $\delta_\ell \in \Delta([0,1])$ is the Dirac measure on $\ell \in [0,1]$.

As an illustration, let us consider two leading examples, satisfying Assumptions A1–A3, from the global games literature. Morris and Shin (2003) studied global games with the linear payoff gain function

$$d(\ell, \theta) = \ell + \theta - 1$$

with $\{\theta \in \Theta \mid \theta > 1\} \neq \emptyset$ (to satisfy A3). In this case, we have $\bar{\Phi}(\theta) = \theta - \frac{1}{2}$, and thus the Laplacian threshold is $\theta^\sharp = \frac{1}{2}$. Hence,

$$\int_{\theta'>\theta} \bar{\Phi}(\theta')dP(\theta') = (1 - P(\theta)) \left(\mathbb{E}[\theta'|\theta' > \theta] - \theta^\sharp\right),$$

and therefore, the expected Laplacian threshold θ^* is the unique solution to

$$\mathbb{E}[\theta|\theta > \theta^*] = \theta^\sharp.$$

Morris and Shin (1998, 2004) studied regime change games with the payoff gain function

$$d(\ell, \theta) = \begin{cases} c & \text{if } \ell > 1 - \theta, \\ c - 1 & \text{if } \ell \leq 1 - \theta \end{cases} \tag{3}$$

with $\{\theta \in \Theta \mid \theta > 1\} \neq \emptyset$ (to satisfy A3), where action 0 corresponds to attacking the regime (with cost $c \in (0,1)$), while action 1 to abstaining from attacking. In this case,

we have $\bar{\Phi}(\theta) = k(\theta) - (1-c)$, where

$$k(\theta) = \begin{cases} 0 & \text{if } \theta < 0, \\ \theta & \text{if } 0 \le \theta < 1, \\ 1 & \text{if } \theta \ge 1, \end{cases}$$

and thus the Laplacian threshold is $\theta^\sharp = 1 - c$. Hence,

$$\int_{\theta' > \theta} \bar{\Phi}(\theta') dP(\theta') = (1 - P(\theta)) \left(\mathbb{E}[k(\theta') | \theta' > \theta] - \theta^\sharp \right),$$

and therefore, the expected Laplacian threshold θ^* is the unique solution to

$$\mathbb{E}[k(\theta) | \theta > \theta^*] = \theta^\sharp.$$

4. Smallest Equilibrium Implementation of the Target Outcome

We now show that the target outcome ν^* is arbitrarily approximately smallest equilibrium implementable. For $\varepsilon > 0$, define the outcome ν^ε by

$$\nu^\varepsilon(\theta) = \begin{cases} \delta_1 & \text{if } \theta \ge \theta^* + \varepsilon, \\ \delta_0 & \text{if } \theta < \theta^* + \varepsilon. \end{cases}$$

Proposition 1. *Assume A1–A4. For any sufficiently small $\varepsilon > 0$, there exists an information structure whose smallest equilibrium induces the outcome ν^ε.*

Proof. Let $\varepsilon > 0$, where $\int_{\theta \ge \theta^* + \varepsilon} \bar{\Phi}(\theta) dP(\theta) > 0$. We assume that ε is sufficiently small that

$$d(0, \theta) \le 0 \text{ for all } \theta < \theta^* + \varepsilon.^3 \tag{4}$$

Let $\overline{\theta}' \in \operatorname{int} \Theta$ be sufficiently large that

$$\int_{\theta^* + \varepsilon}^{\overline{\theta}'} \bar{\Phi}(\theta) dP(\theta) > 0 \tag{5}$$

and

$$d(0, \theta) > 0 \text{ for all } \theta \ge \overline{\theta}'. \tag{6}$$

We denote

$$\underline{P} = P(\theta^* + \varepsilon), \quad \overline{P} = P(\overline{\theta}'),$$

where $0 < \underline{P} < \overline{P} < 1$.

We construct a desired information structure as follows. Conditional on the realization of θ, a signal x_i is sent to each player i according to the following law:

- If $\theta < \theta^* + \varepsilon$, then $x_i = -\infty$ for all players i.

[3] Since $\bar{\Phi}(\theta) > 0$ for large θ by A3, we must have $\bar{\Phi}(\theta^* + \varepsilon) < 0$, and hence $d(0, \theta^* + \varepsilon) < 0$, for small $\varepsilon > 0$.

- If $\theta \geq \theta^* + \varepsilon$, then $x_i = \omega + \kappa\zeta_i$ for each player i, where
  - ω is drawn from the uniform distribution on $[\underline{P}, \overline{P}]$ if $\theta^* + \varepsilon \leq \theta < \overline{\theta}'$ and from the uniform distribution on $[\overline{P}, 1]$ if $\theta \geq \overline{\theta}'$, and
  - ζ_i are distributed independently of θ and ω and across players according to a distribution F on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with a log-concave density, say the uniform distribution, and $\kappa > 0$ is a scaler parameter with $\kappa \leq 1 - \overline{P}$.

Thus, first, there is a public announcement whether $\theta < \theta^* + \varepsilon$ (in which case all players observe a signal $-\infty$) or $\theta \geq \theta^* + \varepsilon$ (in which case all players observe a signal of finite value). Then in the latter event, unconditionally on whether $\theta^* + \varepsilon \leq \theta < \overline{\theta}'$ or $\theta \geq \overline{\theta}'$, the "latent state" ω is uniformly distributed over $\Omega = [\underline{P}, 1]$, which is observed with idiosyncratic noise ξ_i by each player i. A strategy of a player is a measurable function $s\colon \{-\infty\} \cup (\Omega + \kappa\left[-\frac{1}{2}, \frac{1}{2}\right]) \to \{0, 1\}$, where $s(x)$ is the action that the player plays when observing signal x. Denote this game by $\hat{G}$.

Now, if $\theta < \theta^* + \varepsilon$, the expected payoff gain when the others play action 0, $\frac{1}{\underline{P}} \int_{\theta < \theta^* + \varepsilon} d(0, \theta) dP(\theta)$, is nonpositive by (4), and hence, all players play action 0 in the smallest equilibrium of $\hat{G}$. If $\theta \geq \theta^* + \varepsilon$, the game is equivalent to the global game with uniform prior where the base payoff gain function $\hat{d}(\ell, \omega)$, $\omega \in \Omega$, is given by

$$\hat{d}(\ell, \omega) = \begin{cases} \dfrac{1}{\overline{P} - \underline{P}} \displaystyle\int_{\theta^* + \varepsilon}^{\overline{\theta}'} d(\ell, \theta) dP(\theta) & \text{if } \omega < \overline{P}, \\ \dfrac{1}{1 - \overline{P}} \displaystyle\int_{\theta \geq \overline{\theta}'} d(\ell, \theta) dP(\theta) & \text{if } \omega \geq \overline{P}, \end{cases}$$

which is nondecreasing in ℓ and ω, where $\hat{d}(0, \omega) > 0$ for all $\omega \geq \overline{P}$ by (6). The Laplacian payoff gain $\hat{\bar{\Phi}}(\omega)$ in this game is thus given by

$$\hat{\bar{\Phi}}(\omega) = \int_0^1 \hat{d}(\ell, \omega) d\ell = \begin{cases} \dfrac{1}{\overline{P} - \underline{P}} \displaystyle\int_{\theta^* + \varepsilon}^{\overline{\theta}'} \bar{\Phi}(\theta) dP(\theta) & \text{if } \omega < \overline{P}, \\ \dfrac{1}{1 - \overline{P}} \displaystyle\int_{\theta \geq \overline{\theta}'} \bar{\Phi}(\theta) dP(\theta) & \text{if } \omega \geq \overline{P}, \end{cases}$$

and hence $\hat{\bar{\Phi}}(\omega) > 0$ for all $\omega \in \Omega$. By the assumption $\kappa \leq 1 - \overline{P}$, it follows from Proposition A.2(ii) in Appendix A that the global game has a unique equilibrium, in which all players play action 1. Therefore, in all states $\theta \geq \theta^* + \varepsilon$, all players play action 1 in any (hence in the smallest) equilibrium of $\hat{G}$. $\square$

5. Optimality of the Target Outcome

The target outcome ν^* studied in this note is, under player symmetry, a continuous analog of the optimal outcome derived in Morris et al. (2020, Theorem 2) for the finite-player, finite-state case, under certain assumptions to be described below. In this section, we briefly review the analysis there. In Appendix B, we formally construct a sequence of games with N symmetric players and N states along which the optimal outcome converges as $N \to \infty$ to ν^*. In Appendix C, we present an alternative framework of the information design problem maintaining continuum players and continuous states (with a heuristic "law of large numbers" assumption) and prove by the same arguments as in Morris et al. (2020) (modulo technical differences) that ν^* is an optimal outcome of the continuous version of the problem.

Within this section, we assume the following technical conditions to guarantee the convergence as $N \to \infty$:

A5. State Space Compactness: Θ is bounded (and hence compact).

A6. Laplacian State Continuity: $\bar{\Phi}(\theta)$ is continuous in θ.

A6 is satisfied in the two examples discussed in Section 3.

Suppose that an information designer has an objective function $V\colon [0,1] \times \Theta \to \mathbb{R}$ with $V(0,\theta) = 0$ for all $\theta \in \Theta$ by normalization, where $V(\ell,\theta)$ is the value that the designer receives when proportion ℓ of players play 1 and the state is θ. We impose:

A7. Objective Action Monotonicity: For each θ, $V(\ell,\theta)$ is nondecreasing in ℓ.

Suppose that the designer chooses an information structure to maximize V anticipating that the worst—hence smallest by A7—equilibrium of the resulting Bayesian supermodular game. Morris et al. (2020) identified sufficient conditions for an outcome to be an optimal solution of this information design problem for general finite BAS games. A *perfect coordination outcome* is one where either all players choose action 1 or all players choose action 0: in our setting, this requires $\nu_\theta(\{0,1\}) = 1$ for all θ. Morris et al. (2020) show that a perfect coordination outcome will always be optimal if (i) the game has a convex potential, and (ii) the designer's objective satisfies restricted convexity with respect to the potential. In our current setting with symmetric players, (i) the game has a potential

$$\Phi(\ell,\theta) = \int_0^\ell d(\ell',\theta)d\ell',$$

which, by A1, satisfies convexity, $\Phi(\ell,\theta) \leq \ell\Phi(1,\theta)$ for all ℓ and θ, while (ii) requires:

A8. Restricted Convexity: $V(\ell, \theta) \leq \ell V(1, \theta)$ whenever $\Phi(\ell, \theta) > \Phi(1, \theta)$.

In order to guarantee the optimal outcome to be monotone in the state, we in addition require:

A9. Objective State Monotonicity: For each ℓ, $V(\ell, \theta)$ is nondecreasing in θ.

We also assume that $V(1, \theta) > V(0, \theta) = 0$ for all $\theta \in \Theta$ by removing irrelevant states.

Since $\bar{\Phi}(\theta) = \Phi(1, \theta)$ is nondecreasing by A2, this guarantees that the "cost benefit ratio" $\frac{\Phi(1,\theta)}{V(1,\theta)}$ is nondecreasing in θ whenever $\Phi(1, \theta) < 0$. Intuitively, $V(1, \theta)$ represents the benefit of having all play action 1 at state θ, and, when $\Phi(1, \theta) < 0$, $-\Phi(1, \theta)$ measures the cost of inducing players to do so. Thus, by Theorem 2 of Morris et al. (2020), it is optimal to have action 1 played at as many higher states as possible consistent with the smallest equilibrium implementability, which is equivalent to the expectation of the potential being positive. Thus, we have a perfect coordination optimal outcome. In Appendix B, this optimal outcome for the N-player N-state game is shown to converge to the target outcome ν^* as $N \to \infty$ (Proposition B.1).

Assumptions A7–A9 are satisfied in the following examples. An easy case is where, for some $\alpha \geq 1$, $V(\ell, \theta) = \ell^\alpha$ for all θ, where $V(1, \theta) \leq \ell V(1, \theta)$ for all ℓ and θ. This designer objective V encompasses the case where the designer wants to maximize the expected proportion of players playing action 1 (when $\alpha = 1$) and the case where she wants to maximize the probability that all players play action 1 (when $\alpha \to \infty$).

A more subtle case arises for the regime change game (described in Section 3) when the designer's objective is to maximize the probability of regime change:

$$V(\ell, \theta) = \begin{cases} 1 & \text{if } \ell > 1 - \theta, \\ 0 & \text{if } \ell \leq 1 - \theta, \end{cases}$$

which is nondecreasing in ℓ and θ (A7, A9). This is the designer objective maximized in Li et al. (2019). In this game, the potential is given by

$$\Phi(\ell, \theta) = \begin{cases} c\ell - (1 - k(\theta)) & \text{if } \ell > 1 - \theta, \\ -(1 - c)\ell & \text{if } \ell \leq 1 - \theta, \end{cases}$$

where $k(\theta) = \theta$ if $0 < \theta \leq 1$ and $k(\theta) = 1$ if $\theta > 1$. This function is increasing in ℓ if $\ell > 1 - \theta$, and therefore we have $\Phi(\ell, \theta) > \Phi(1, \theta)$ only if $\ell \leq 1 - \theta$. But if $\ell \leq 1 - \theta$, we have $V(\ell, \theta) = 0 \leq \ell V(1, \theta)$, establishing the restricted convexity of V with respect to Φ (A8).

6. Discussion: Finite-Player BAS Potential Games

We could extend the argument in this note to general BAS potential games, allowing asymmetric payoffs across players. We would analogously let θ^* be the state such that the expected potential of action profile $\mathbf{1}$ (where all players play action 1) conditional on $\theta > \theta^*$ exceeds the expected potential of all other action profiles, and define the target outcome to be one where all play action 1 above θ^* and all play action 0 below. The construction in this note could be combined with arguments in Frankel et al. (2003) to implement the target outcome.

Appendix A. Binary-Action Global Games with Continuum Players

In this section, we prove an equilibrium uniqueness result for the global game with a continuum of players and binary actions as described in Section 2, imposing, along with the standard monotonicity properties on the payoffs, a restriction on the noise distribution but dispensing with the continuity of the payoffs in the state (while the monotonicity entails the continuity of a certain key object).[4]

Let $d\colon [0,1] \times \Theta \to \mathbb{R}$ be the payoff gain function, where $d(\ell, \theta)$ is the gain from action 1 over action 0 when proportion ℓ of players play action 1 and the state is $\theta \in \Theta = [\theta_0, \theta_1] \subset \mathbb{R}$, $-\infty \leq \theta_0 < \theta_1 \leq \infty$.[5] We maintain Assumptions A1–A3 in Section 2 on the function d. The state θ is drawn according to a distribution function P with support Θ, which we assume to admit a continuous and strictly positive density:

A10. Continuous Prior Density: P admits a density p that is continuous and strictly positive on Θ.

Each player i observes a private signal $x_i = \theta + \kappa\zeta_i$, $\kappa > 0$, where the noise terms ζ_i are distributed independently of θ and across players according to density f (with distribution function F) with support $\left[-\frac{1}{2}, \frac{1}{2}\right]$.[6] A strategy of a player is a measurable function $s\colon X_\kappa \to \{0,1\}$, $X_\kappa = \Theta + \kappa\left[-\frac{1}{2}, \frac{1}{2}\right]$, where $s(x)$ is the action that the player plays when observing signal x. Denote this game by $G(\kappa)$.

[4] Proposition 2.2 of Morris and Shin (2003) report a version of this result under different assumptions (in particular, with their continuity assumption A5 but without a restriction on the noise distribution). However, there appear to be gaps in the reported proof and we have not been able to verify or disprove Proposition 2.2 as stated.

[5] $[\theta_0, \theta_1]$ is understood as $(-\infty, \theta_1]$ if $\theta_0 = -\infty$ and $\theta_1 < \infty$, and so on.

[6] As long as the support of f is bounded, the results will hold only with slight notational changes.

Let $F_\kappa(\theta|x)$ denote the distribution function of θ conditional on the observation of $x_i = x$:

- If $\theta_0 > -\infty$,

$$F_\kappa\left(\theta\middle|\theta_0 - \frac{1}{2}\kappa\right) = \begin{cases} 0 & \text{if } \theta < \theta_0, \\ 1 & \text{if } \theta \geq \theta_0. \end{cases}$$

- For $x \in \left(\theta_0 - \frac{1}{2}\kappa, \theta_1 + \frac{1}{2}\kappa\right)$,

$$F_\kappa(\theta|x) = \frac{\int_{-\infty}^{\theta} f\left(\frac{x-\theta'}{\kappa}\right) p(\theta')d\theta'}{\int_{-\infty}^{\infty} f\left(\frac{x-\theta'}{\kappa}\right) p(\theta')d\theta'}.$$

- If $\theta_1 < \infty$,

$$F_\kappa\left(\theta\middle|\theta_1 + \frac{1}{2}\kappa\right) = \begin{cases} 0 & \text{if } \theta < \theta_1, \\ 1 & \text{if } \theta \geq \theta_1. \end{cases}$$

We assume the following:

A11. First-Order Stochastic Dominance: For all κ, $F_\kappa(\theta|x)$ is nonincreasing in x.

A11 is satisfied in particular when f is log-concave, or equivalently, it satisfies the monotone likelihood ratio property: $\frac{f(x'-\theta)}{f(x-\theta)} \leq \frac{f(x'-\theta')}{f(x-\theta')}$ whenever $x < x'$ and $\theta < \theta'$.

As in Section 2, define the function $\bar{\Phi}(\theta)$ by

$$\bar{\Phi}(\theta) = \int_0^1 d(\ell, \theta)d\ell,$$

and let

$$\theta^\sharp = \sup\{\theta \in \Theta \mid \bar{\Phi}(\theta) \leq 0\}$$

with convention $\sup \emptyset = \theta_0$.

Proposition A.1. *Assume A1–A3, A10, and A11.* (i) *For any $\theta^+ > \theta^\sharp$, there exists $\bar{\kappa} > 0$ such that for any $\kappa \leq \bar{\kappa}$, if strategy s survives iterated deletion of strictly dominated strategies in $G(\kappa)$, then $s(x) = 1$ for all $x \geq \theta^+ - \frac{1}{2}\kappa$, and therefore, in any equilibrium of $G(\kappa)$, all players play action 1 whenever $\theta \geq \theta^+$.* (ii) *If $\theta_0 > -\infty$ and $\bar{\Phi}(\theta_0) > 0$, then there exists $\bar{\kappa} > 0$ such that for any $\kappa \leq \bar{\kappa}$, if strategy s survives iterated deletion of strictly dominated strategies in $G(\kappa)$, then $s(x) = 1$ for all $x \in X_\kappa$, and therefore, $G(\kappa)$ has a unique equilibrium, in which all players play action 1 for any $\theta \in \Theta$.*

Proof. Write $D_\kappa(x, k)$ for the expected payoff gain when the player observes signal x and others play the k-threshold strategy (i.e., the strategy s such that $s(x) = 1$ if $x \geq k$ and

$s(x) = 0$ if $x < k$):

$$D_\kappa(x,k) = \int_\Theta d\left(1 - F\left(\frac{k-\theta}{\kappa}\right), \theta\right) dF_\kappa(\theta|x),$$

which is nondecreasing in x by A1, A2, and A11 and nonincreasing in k by A1. $D_\kappa(x,k)$ is also continuous in (x,k) by A1, A2, and A10, as shown in Lemma 1 below.

For part (i), let $\theta^+ \in (\theta^\sharp, \overline{\theta}]$ (for $\theta^+ > \overline{\theta}$, the conclusion clearly holds for sufficiently small κ by A3); for part (ii), let $\theta^+ = \theta_0$. Let $\kappa_0 > 0$ be small enough that $\overline{\theta} + \frac{1}{2}\kappa_0 \leq \theta_1 - \frac{1}{2}\kappa_0$, and for part (i), $\theta^+ - \kappa_0 > \theta^\sharp$. In the following, we assume that $\kappa \leq \kappa_0$. Define ξ_κ^n inductively by $\xi_\kappa^0 = \infty$, and

$$\xi_\kappa^n = \sup\left\{x \geq \theta^+ - \frac{1}{2}\kappa \;\middle|\; D_\kappa(x, \xi_\kappa^{n-1}) \leq 0\right\}$$

with convention $\sup\emptyset = \theta^+ - \frac{1}{2}\kappa$, where by A3 and the monotonicity of D_κ, $\theta^+ - \frac{1}{2}\kappa \leq \cdots \leq \xi_\kappa^2 \leq \xi_\kappa^1 \leq \overline{\theta} + \frac{1}{2}\kappa$, and by the continuity of $D_\kappa(x,k)$ in x, we have $D_\kappa(\xi_\kappa^n, \xi_\kappa^{n-1}) = 0$ if $\xi_\kappa^n > \theta^+ - \frac{1}{2}\kappa$. Thus, if a strategy s survives n rounds of iterated deletion of strictly dominated strategies, then $s(x) = 1$ for all $x > \xi_\kappa^n$. Denote the limit of ξ_κ^n by ξ_κ^∞. If $\xi_\kappa^\infty > \theta^+ - \frac{1}{2}\kappa$, then we must have $D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) = 0$ by the continuity of $D_\kappa(x,k)$ in (x,k). We want to show that $\xi_\kappa^\infty = \theta^+ - \frac{1}{2}\kappa$ for all sufficiently small κ. Assume the contrary, that there is a subsequence, again denoted ξ_κ^∞, such that $\xi_\kappa^\infty > \theta^+ - \frac{1}{2}\kappa$, and hence $D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) = 0$, for all κ. Let $\kappa_1(\cdot)$ be as in Lemma 2 below.

For part (i), let $\phi_1 > 0$ be such that $\phi_1 < \bar{\Phi}(\theta^+ - \kappa_0)$, and let $\hat{\Theta}_1 = [\theta^+ - \kappa_0, \overline{\theta} + \kappa_0]$. Let $d_1 = \max_{\ell\in[0,1]} |d(\ell, \theta^+ - \kappa_0)|$. Then if $\kappa \leq \kappa_1\big(\frac{\phi_1}{d_1}, \hat{\Theta}_1\big)$, we have

$$\begin{aligned}
D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) &\geq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} d(1-F(z), \theta^+ - \kappa_0) f(z) p(\xi_\kappa^\infty - \kappa z) dz}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z) p(\xi_\kappa^\infty - \kappa z) dz} \\
&\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} d(1-F(z), \theta^+ - \kappa_0) f(z) dz - \int_{-\frac{1}{2}}^{\frac{1}{2}} |d(1-F(z), \theta^+ - \kappa_0)| f(z) \frac{\phi_1}{d_1} dz \\
&\geq \bar{\Phi}(\theta^+ - \kappa_0) - \phi_1 > 0,
\end{aligned}$$

contradicting $D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) = 0$.

For (ii), let $\phi_2 > 0$ be such that $\phi_2 < \Phi^*(\theta_0)$, and let $\hat{\Theta}_2 = [\theta_0, \overline{\theta} + \kappa_0]$. Let $d_2 = \max_{\ell\in[0,1]} |d(\ell, \theta_0)|$. Then let $\kappa \leq \kappa_1\big(\frac{\phi_2}{d_2}, \hat{\Theta}_2\big)$. If $\xi_\kappa^\infty \geq \theta_0 + \frac{1}{2}\kappa$, we have

$$D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) \geq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} d(1-F(z), \theta_0) f(z) p(\xi_\kappa^\infty - \kappa z) dz}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z) p(\xi_\kappa^\infty - \kappa z) dz} \geq \Phi^*(\theta_0) - \phi_2 > 0$$

as in part (i), while if $\xi_\kappa^\infty < \theta_0 + \frac{1}{2}\kappa$, we have

$$\begin{aligned}
D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) &\geq \frac{\int_{-\frac{1}{2}}^{\frac{\xi_\kappa^\infty - \theta_0}{\kappa}} d(1-F(z),\theta_0) f(z) p(\xi_\kappa^\infty - \kappa z) dz}{\int_{-\frac{1}{2}}^{\frac{\xi_\kappa^\infty - \theta_0}{\kappa}} f(z) p(\xi_\kappa^\infty - \kappa z) dz} \\
&\geq \frac{1}{F\left(\frac{\xi_\kappa^\infty - \theta_0}{\kappa}\right)} \int_{-\frac{1}{2}}^{\frac{\xi_\kappa^\infty - \theta_0}{\kappa}} d(1-F(z),\theta_0) f(z) dz \\
&\quad - \frac{1}{F\left(\frac{\xi_\kappa^\infty - \theta_0}{\kappa}\right)} \int_{-\frac{1}{2}}^{\frac{\xi_\kappa^\infty - \theta_0}{\kappa}} |d(1-F(z),\theta_0)| f(z) \frac{\phi_2}{d_2} dz \\
&\geq \frac{1}{F\left(\frac{\xi_\kappa^\infty - \theta_0}{\kappa}\right)} \int_{1-F\left(\frac{\xi_\kappa^\infty - \theta_0}{\kappa}\right)}^{1} d(\ell,\theta_0) d\ell - \phi_2 \geq \bar{\Phi}(\theta_0) - \phi_2 > 0,
\end{aligned}$$

contradicting $D_\kappa(\xi_\kappa^\infty, \xi_\kappa^\infty) = 0$.

This shows that $\xi_\kappa^\infty = \theta^+ - \frac{1}{2}\kappa$ for all sufficiently small κ. Finally, for part (i), $D_\kappa\left(\theta^+ - \frac{1}{2}\kappa, \theta^+ - \frac{1}{2}\kappa\right) \geq \bar{\Phi}(\theta^+ - \kappa_0) - \phi_1 > 0$ for sufficiently small κ as shown above, and for part (ii), $D_\kappa\left(\theta_0 - \frac{1}{2}\kappa, \theta_0 - \frac{1}{2}\kappa\right) = d(1,\theta_0) \geq \bar{\Phi}(\theta_0) > 0$. Thus, we have shown that for all sufficiently small κ, any strategy $s(x)$ that survives iterated deletion of strictly dominated strategies plays action 1 for all $x \geq \theta^+ - \frac{1}{2}\kappa$. □

If p is constant (i.e., P is a uniform distribution), the inequality in Lemma 2 holds as an equality with $r = 0$ for any $\kappa > 0$. Thus the proof of Proposition A.1 in fact proves the following proposition:

A12. Uniform Prior: Θ is bounded, and P is the uniform distribution on Θ (with p denoting its density).[7]

Proposition A.2. *Assume A1–A3, A11, and A12.* (i) *For any $\theta^+ > \theta^\sharp$ and any $\kappa < \min\{\theta^+ - \theta^\sharp, \theta_1 - \overline{\theta}\}$, if strategy s survives iterated deletion of strictly dominated strategies in $G(\kappa)$, then $s(x) = 1$ for all $x \geq \theta^+ - \frac{1}{2}\kappa$, and therefore, in any equilibrium of $G(\kappa)$, all players play action 1 whenever $\theta \geq \theta^+$.* (ii) *If $\theta_0 > -\infty$ and $\bar{\Phi}(\theta_0) > 0$, then for any $\kappa \leq \theta_1 - \overline{\theta}$, if strategy s survives iterated deletion of strictly dominated strategies in $G(\kappa)$, then $s(x) = 1$ for all $x \in X_\kappa$, and therefore, $G(\kappa)$ has a unique equilibrium, in which all players play action 1 for any $\theta \in \Theta$.*

The essential uniqueness of equilibrium is obtained under the following additional assumptions:

[7] In fact, Proposition A.2 holds also when $\Theta = \mathbb{R}$, and P is the improper uniform distribution on $\mathbb{R}$, where the conditional probabilities are well defined. In this case, A11 is automatically satisfied.

A13. Lower Dominance Region: There exists $\underline{\theta} \in (\theta_0, \theta_1)$ such that $d(1, \theta) < 0$ for all $\theta \leq \underline{\theta}$.

A14. Strict Laplacian State Monotonicity: $\inf\{\theta \in \Theta \mid \bar{\Phi}(\theta) \geq 0\} = \sup\{\theta \in \Theta \mid \bar{\Phi}(\theta) \leq 0\}$.

Corollary A.3. *Assume A1–A3, A10, A11, A13, and A14, and let $\theta^\sharp$ be the value of the equality in A14. For any $\varepsilon > 0$, there exists $\bar{\kappa} > 0$ such that for any $\kappa \leq \bar{\kappa}$, if strategy s survives iterated deletion of strictly dominated strategies in $G(\kappa)$, then $s(x) = 0$ for all $x \leq \theta^\sharp - \varepsilon$ and $s(x) = 1$ for all $x \geq \theta^\sharp + \varepsilon$.*

In the following, we state and prove the lemmas used in the proof of Proposition A.1.

Lemma 1. *For $x \in \left(\theta_0 - \frac{1}{2}\kappa, \theta_1 + \frac{1}{2}\kappa\right)$, $D_\kappa(x, k)$ is continuous in (x, k).*

Proof. For $x \in \left(\theta_0 - \frac{1}{2}\kappa, \theta_1 + \frac{1}{2}\kappa\right)$, $D_\kappa(x, k)$ is written as

$$D_\kappa(x, k) = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} E(x, k, z) f(z) p(x - \kappa z) dz}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z) p(x - \kappa z) dz} \tag{A.1}$$

with

$$E(x, k, z) = d\left(1 - F\left(z + \frac{k - x}{\kappa}\right), x - \kappa z\right)$$

(where $d(\ell, \cdot)$ is extended outside Θ so that $d(\ell, \theta)$ is still nondecreasing in ℓ and θ). Fix any (x_0, k_0) with $x_0 \in \left(\theta_0 - \frac{1}{2}\kappa, \theta_1 + \frac{1}{2}\kappa\right)$. We want to show that for almost all $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $E(x, k, z)$ is continuous in (x, k) at $(x, k) = (x_0, k_0)$.

The function $z \mapsto E(x_0, k_0, z)$ is nonincreasing by A1 and A2 and hence continuous except possibly on some null set $N_0 \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. Take any $z_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus N_0$, and take any $\varepsilon > 0$. Let $\eta_0 > 0$ be such that $E(x_0, k_0, z_0 - \eta_0) - \varepsilon \leq E(x_0, k_0, z_0) \leq E(x_0, k_0, z_0 + \eta_0) + \varepsilon$, and let $\eta = \frac{\kappa}{2}\eta_0$. Let $|x - x_0| \leq \eta$ and $|k - k_0| \leq \eta$. Then we have

$$\begin{aligned} E(x, k, z_0) &= d\left(1 - F\left(z_0 + \frac{k - x}{\kappa}\right), x - \kappa z_0\right) \\ &\leq d\left(1 - F\left(z_0 - \eta_0 + \frac{k_0 - x_0}{\kappa}\right), x_0 - \kappa(z - \eta_0)\right) \\ &= E(x_0, k_0, z_0 - \eta_0) \leq E(x_0, k_0, z_0) + \varepsilon \end{aligned}$$

by A1 and A2. Similarly, we have $E(x, k, z_0) \geq E(x_0, k_0, z_0 + \eta_0) \geq E(x_0, k_0, z_0) - \varepsilon$. Hence, $|E(x, k, z_0) - E(x_0, k_0, z_0)| \leq \varepsilon$. This shows that $E(x, k, z_0)$ is continuous in (x, k) at (x_0, k_0).

Now, by A10, $z \mapsto p(x_0 - \kappa z)$ is continuous except possibly at $z = \frac{x_0-\theta_0}{\kappa}, \frac{x_0-\theta_1}{\kappa}$. Thus, for all $z \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \left(N_0 \cup \left\{\frac{x_0-\theta_0}{\kappa}, \frac{x_0-\theta_1}{\kappa}\right\}\right)$, the integrands $E(x,k,z)$ and $p(x-\kappa z)$ in (A.1) converge to $E(x_0,k_0,z)$ and $p(x_0-\kappa z)$, respectively, as $(x,k) \to (x_0,k_0)$. Together with the boundedness of d and p on the integral domain, the continuity of D_κ therefore follows by the dominated convergence theorem. □

Lemma 2. *For any $r > 0$ and for any compact $\hat{\Theta} \subset \Theta$, there exists $\kappa_1(r, \hat{\Theta}) > 0$ such that for any $\kappa \leq \kappa_1(r, \hat{\Theta})$, if $\bar{z} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $\left[x - \kappa\bar{z}, x + \frac{1}{2}\kappa\right] \subset \hat{\Theta}$, then*

$$\left| \frac{F(\bar{z})p(x-\kappa z)}{\int_{-\frac{1}{2}}^{\bar{z}} f(z')p(x-\kappa z')dz'} - 1 \right| \leq r.$$

for all $z \in \left[-\frac{1}{2}, \bar{z}\right]$.

Proof. By A10, for any $r > 0$ and any compact $\hat{\Theta} \subset \Theta$, there exists $\eta > 0$ such that if $\theta, \theta' \in \hat{\Theta}$ and $|\theta - \theta'| \leq \eta$, then $|p(\theta) - p(\theta')| \leq \underline{p}r$ where $\underline{p} = \min_{\theta \in \hat{\Theta}} p(\theta) > 0$. Set $\kappa_1(r, \hat{\Theta})$ to be any such η, and let $\kappa \leq \kappa_1(r, \hat{\Theta})$. Then if $\bar{z} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, $\left[x - \kappa\bar{z}, x + \frac{1}{2}\kappa\right] \subset \hat{\Theta}$, and $z \in \left[-\frac{1}{2}, \bar{z}\right]$, we have

$$\begin{aligned} \left| \frac{F(\bar{z})p(x-\kappa z)}{\int_{-\frac{1}{2}}^{\bar{z}} f(z')p(x-\kappa z')dz'} - 1 \right| &\leq \frac{\int_{-\frac{1}{2}}^{\bar{z}} f(z')|p(x-\kappa z) - p(x-\kappa z')|dz'}{\int_{-\frac{1}{2}}^{\bar{z}} f(z')p(x-\kappa z')dz'} \\ &\leq \frac{\int_{-\frac{1}{2}}^{\bar{z}} f(z')\underline{p}rdz'}{\int_{-\frac{1}{2}}^{\bar{z}} f(z')\underline{p}dz'} = r, \end{aligned}$$

since $x - \kappa z, x - \kappa z' \in \hat{\Theta}$ and $|(x-\kappa z) - (x-\kappa z')| \leq \kappa \leq \kappa_1(r, \hat{\Theta})$ for all $z' \in \left[-\frac{1}{2}, \bar{z}\right]$. □

Appendix B. Optimal Information Design with Finite Approximations

In this section, by applying the results of Morris et al. (2020), we demonstrate that the target outcome in the continuous game studied in the main text is obtained as the limit of optimal outcomes in approximating finite games.

Within this section, assume A5 (State Space Compactness), and denote $\Theta = [\theta_0, \theta_1]$, $-\infty < \theta_0 < \theta_1 < \infty$. Let the payoff gain function $d\colon [0,1] \times \Theta \to \mathbb{R}$ be given, where we assume A1–A3. The potential of the game is given by

$$\Phi(\ell, \theta) = \int_0^\ell d(\ell', \theta)d\ell',$$

and the Laplacian payoff gain is $\bar{\Phi}(\theta) = \Phi(1, \theta)$, where we assume A6 (Laplacian State Continuity). As in Section 3, assume that $\int_\Theta \bar{\Phi}(\theta)dP(\theta) < 0$ and that P is continuous

(ease of exposition). The expected Laplacian threshold is the unique $\theta^* \in (\theta_0, \theta_1)$ that solves $\Psi(\theta^*) = 0$, where

$$\Psi(\theta) = \int_{\theta' > \theta} \bar{\Phi}(\theta') dP(\theta').$$

For expositional purpose, we identify the target outcome (2) with the probability distribution ν^* on $\{0,1\} \times \Theta$ defined by

$$\nu^*(\{\ell\} \times E) = \begin{cases} \mu(E \cap (\theta^*, \theta_1]) & \text{if } \ell = 1, \\ \mu(E \cap [\theta_0, \theta^*]) & \text{if } \ell = 0, \end{cases} \tag{B.1}$$

where μ is the probability distribution on Θ induced by the distribution function P.

The objective function is given by $V\colon [0,1] \times \Theta \to \mathbb{R}$ with $V(0,\theta) = 0$ for all $\theta \in \Theta$. We assume A7 (Objective Action Monotonicity), A8 (Restricted Convexity), and A9 (Objective State Monotonicity), and also assume that $V(1,\theta) > 0$ for all $\theta \in \Theta$ (relevancy).

B.1. **Finite Games.** For each natural number $N \geq 2$, we construct a finite game as follows. There are N players, $I^N = \{1, \ldots, N\}$, and each player $i \in I^N$ has binary actions $A_i = \{0,1\}$, where we denote $A^N = \prod_{i \in I^N} A_i$ and $A^N_{-i} = \prod_{j \neq i} A_j$. There are N possible states: the state space is

$$\Theta^N = \left\{ \theta_0 + \frac{\theta_1 - \theta_0}{N} m \;\middle|\; m = 1, \ldots, N \right\}.$$

The prior distribution on Θ^N is given by the probability distribution μ^N on Θ with support Θ^N defined by

$$\mu^N\left(\left\{\theta_0 + \frac{\theta_1 - \theta_0}{N} m\right\}\right) = \mu\left(\left(\theta_0 + \frac{\theta_1 - \theta_0}{N}(m-1), \theta_0 + \frac{\theta_1 - \theta_0}{N} m\right]\right).$$

The payoff gain for each player $i \in I^N$ is given by the function d_i^N on $\left\{\frac{n}{N} \;\middle|\; n = 0, \ldots, N-1\right\} \times \Theta$ defined by

$$d_i^N\left(\frac{n}{N}, \theta\right) = \frac{1}{N} d\left(\frac{n+1}{N}, \theta\right),$$

where $d_i^N\left(\frac{n(a_{-i})}{N}, \theta\right)$ is the payoff gain for opponents' action profile $a_{-i} \in A^N_{-i}$ (with $n(a_{-i})$ denoting the number of players $j \neq i$ such that $a_j = 1$) and state θ. For all $i \in I^N$, $d_i^N\left(\frac{n(a_{-i})}{N}, \theta\right)$ is nondecreasing in a_{-i} by the monotonicity of $d(\ell, \theta)$ in ℓ (Assumption A1), and the dominance state assumption is satisfied at $\theta = \theta_1$, $d_i^N(0, \theta_1) > 0$, by $d(0, \theta_1) > 0$ (Assumption A3). Define the function Φ^N on $\left\{\frac{n}{N} \;\middle|\; n = 0, \ldots, N\right\} \times \Theta$ by

$$\Phi^N\left(\frac{n}{N}, \theta\right) = \sum_{k=1}^{n} \frac{1}{N} d\left(\frac{k}{N}, \theta\right).$$

Then the game is a potential game with potential function $\Phi^N\left(\frac{n(a)}{N},\theta\right)$, $a \in A^N$ (where $n(a)$ denotes the number of players $i \in I^N$ such that $a_i = 1$).

The objective function of the information designer is given by the function V^N on $\left\{\frac{n}{N} \mid n = 0,\ldots,N\right\} \times \Theta$ defined by

$$V^N\left(\frac{n}{N},\theta\right) = V\left(\frac{n}{N},\theta\right),$$

where $V^N\left(\frac{n(a)}{N},\theta\right)$ is the value for the designer for action profile $a \in A^N$ and state θ, which is nondecreasing in a by the monotonicity of $V(\ell,\theta)$ in ℓ (Assumption A7). As in Morris et al. (2020), the designer chooses an information structure to maximize the expected value of V^N under adversarial equilibrium selection, i.e., the assumption that the players will play the worst—hence smallest—equilibrium of the resulting Bayesian supermodular game. An information structure is represented by countable type spaces T_i, $i \in I^N$, and a common prior π over $T \times \Theta^N$ consistent with μ^N (i.e., $\pi(T \times \{\theta\}) = \mu^N(\{\theta\})$ for all $\theta \in \Theta^N$), where we write $T = \prod_{i \in I^N} T_i$ and $T_{-i} = \prod_{j \neq i} T_j$. In the Bayesian game that an information structure defines, a (pure) strategy of player $i \in I^N$ is a function $\sigma_i \colon T_i \to A_i$, and the expected payoff gain for type $t_i \in T_i$ against opponents' strategy profile $\sigma_{-i} = (\sigma_j)_{j \neq i}$ is

$$D_i^N(\sigma_{-i}|t_i) = \sum_{t_{-i} \in T_{-i}, \theta \in \Theta^N} d_i^N\left(\frac{n(\sigma_{-i}(t_{-i}))}{N},\theta\right)\pi(t_{-i},\theta|t_i),$$

where $\pi(t_{-i},\theta|t_i) = \frac{\pi(t,\theta)}{\sum_{t'_{-i},\theta'} \pi((t_i,t'_{-i}),\theta')}$. A strategy profile $\sigma = (\sigma_i)_{i \in I^N}$ is an equilibrium if for all $i \in I^N$, $D_i^N(\sigma_{-i}|t_i) \geq 0$ whenever $\sigma_i(t_i) = 1$ and $D_i^N(\sigma_{-i}|t_i) \leq 0$ whenever $\sigma_i(t_i) = 0$. Here, we represent an outcome by a probability distribution on $A^N \times \Theta^N$. An outcome $\nu \in \Delta(A^N \times \Theta^N)$ is smallest equilibrium implementable (S-implementable) if there exists an information structure $((T_i)_{i \in I}, \pi)$ whose smallest equilibrium $\underline{\sigma}$ induces that outcome, i.e., $\nu(a,\theta) = \pi(\{t \in T \mid \underline{\sigma}(t) = a\} \times \{\theta\})$ for all $a \in A^N$ and $\theta \in \Theta^N$. Let $SI^N \subset \Delta(A^N \times \Theta^N)$ be the set of S-implementable outcomes. Thus, the adversarial information design problem reduces to

$$\sup_{\nu \in SI^N} \sum_{a \in A^N, \theta \in \Theta^N} \nu(a,\theta) V^N\left(\frac{n(a)}{N},\theta\right) = \max_{\nu \in \overline{SI}^N} \sum_{a \in A^N, \theta \in \Theta^N} \nu(a,\theta) V^N\left(\frac{n(a)}{N},\theta\right),$$

where $\overline{SI}^N$ denotes the closure of SI^N. An optimal outcome is an element $\overline{SI}^N$ that attains the optimal value.

Morris et al. (2020) identify a set of sufficient conditions (for general finite BAS games) under which an optimal outcome is a perfect coordination outcome, an outcome that

assigns positive probability only on the action profiles $\mathbf{0}$ (all playing action 0) and $\mathbf{1}$ (all playing action 1), and is monotone in the state. The conditions are the following, which we verify to be satisfied under the current assumptions:

- The potential Φ^N satisfies convexity: By A1, for each $\theta \in \Theta^N$,
$$\Phi^N\left(\frac{n}{N},\theta\right) \leq \frac{n}{N}\sum_{k=1}^{N}\frac{1}{N}d\left(\frac{k}{N},\theta\right) = \frac{n}{N}\Phi^N(1,\theta)$$
for all $n = 0,\ldots,N$.
- V^N satisfies the restricted convexity with respect to Φ^N: Suppose that $\Phi^N(1,\theta) - \Phi^N\left(\frac{n}{N},\theta\right) < 0$. Then we have
$$\begin{aligned}\Phi(1,\theta) - \Phi\left(\frac{n}{N},\theta\right) &= \int_{\frac{n}{N}}^{1} d(\ell,\theta)d\ell \\ &\leq \sum_{k=n+1}^{N}\frac{1}{N}d\left(\frac{k}{N},\theta\right) = \Phi^N(1,\theta) - \Phi^N\left(\frac{n}{N},\theta\right) < 0,\end{aligned}$$
where the weak inequality follows from A1. Thus, by the restricted convexity of V with respect to Φ (Assumption A8), we have
$$V^N\left(\frac{n}{N},\theta\right) = V\left(\frac{n}{N},\theta\right) \leq \frac{n}{N}V(1,\theta) = \frac{n}{N}V^N(1,\theta).$$
- $\frac{\Phi^N(1,\theta)}{V^N(1,\theta)}$ is nondecreasing in θ on the set $\{\theta \in \Theta^N \mid \Phi^N(1,\theta) < 0\}$: This follows from the monotonicity of $d(\ell,\theta)$ and $V(\ell,\theta)$ in θ (Assumptions A2, A9).

Hence, Theorem 2 of Morris et al. (2020) applies to our finite game. Define the function
$$\Psi^N(\theta) = \int_{(\theta,\theta_1]} \Phi^N(1,\theta')d\mu^N(\theta').$$
Let $\theta^N \in \Theta^N$ be the unique element in Θ^N such that $\Psi^N(\theta) \geq 0$ if and only if $\theta \geq \theta^N$, and let $p^N = \frac{\Psi(\theta^N)}{-\Phi^N(1,\theta^N)}$. Then let ν^N be the perfect coordination outcome as defined in the expression (4.4) in Morris et al. (2020), where we view ν^N as a probability distribution on $\{0,1\} \times \Theta$ with support $\{0,1\} \times \Theta^N$:
$$\nu^N(\ell,\theta) = \begin{cases} \mu^N(\{\theta\}) & \text{if } \ell = 1 \text{ and } \theta > \theta^N, \\ p^N & \text{if } \ell = 1 \text{ and } \theta = \theta^N, \\ \mu^N(\{\theta\}) - p^N & \text{if } \ell = 0 \text{ and } \theta = \theta^N, \\ \mu^N(\{\theta\}) & \text{if } \ell = 0 \text{ and } \theta < \theta^N. \\ 0 & \text{otherwise,} \end{cases} \tag{B.2}$$
which by construction satisfies
$$\int_{\{0,1\}\times\Theta} \Phi^N(\ell,\theta)d\nu^N(\ell,\theta) = 0.$$

By Theorem 2 of Morris et al. (2020), ν^N is an optimal outcome of the adversarial information design problem for the finite game given by $(d_i^N)_{i\in I^N}$ and μ^N.

B.2. **Convergence.** We now show that ν^N as defined in (B.2) converges weakly to the target outcome ν^* as defined in (B.1) as $N \to \infty$.

Proposition B.1. *Assume A1–A3 and A5–A9. Then $\nu^N \to \nu^*$ weakly as $N \to \infty$.*

Proof. Note first that, by construction, as $N \to \infty$, μ^N converges to μ weakly and $\Phi^N(1,\theta)$ converges to $\Phi(1,\theta)$ uniformly over $\theta \in \Theta$.

Take any subsequence of $\{\nu^N\}$, again denoted by $\{\nu^N\}$. The corresponding sequence of $\{\theta^N\}$, passing to a subsequence, converges to some $\theta^\infty \in \Theta$. Let ν^∞ be the probability distribution on $\{0,1\} \times \Theta$ defined by

$$\nu^\infty(\{\ell\} \times E) = \begin{cases} \mu(E \cap (\theta^\infty, \theta_1]) & \text{if } \ell = 1, \\ \mu(E \cap [\theta_0, \theta^\infty]) & \text{if } \ell = 0. \end{cases}$$

By construction, ν^N converges to ν^∞ weakly. By $\int_{\{0,1\}\times\Theta} \Phi^N(\ell,\theta) d\nu^N(\ell,\theta) = 0$, we have

$$\begin{aligned} |\Psi(\theta^\infty)| &= \left| \int_{\{0,1\}\times\Theta} \Phi(\ell,\theta) d\nu^\infty(\ell,\theta) \right| \\ &= \left| \int_{\{0,1\}\times\Theta} \Phi(\ell,\theta) d\nu^\infty(\ell,\theta) - \int_{\{0,1\}\times\Theta} \Phi^N(\ell,\theta) d\nu^N(\ell,\theta) \right| \\ &\le \left| \int_{\{0,1\}\times\Theta} \Phi(\ell,\theta) d\nu^\infty(\ell,\theta) - \int_{\{0,1\}\times\Theta} \Phi(\ell,\theta) d\nu^N(\ell,\theta) \right| \\ &\quad + \sup_{\theta\in\Theta} \left| \Phi(1,\theta) - \Phi^N(1,\theta) \right| \to 0 \end{aligned}$$

as $N \to \infty$ by the weak convergence $\nu^N \to \nu^\infty$ (where Φ is continuous on $\{0,1\} \times \Theta$ by Assumption A6) and the uniform convergence $\Phi^N \to \Phi$. Thus we have $\Psi(\theta^\infty) = 0$, but since θ^* is a unique solution to $\Psi(\theta) = 0$, this implies that $\theta^\infty = \theta^*$, and hence $\nu^\infty = \nu^*$. This shows that the original sequence ν^N converges to ν^*. □

Appendix C. Optimal Information Design with Continuum Players

In this section, we formulate a framework of smallest equilibrium implementation with a continuum of symmetric players and a possibly uncountable signal space, and then apply the arguments of Morris et al. (2020) to obtain a necessary condition for smallest equilibrium implementability (Proposition C.1) and show that the target outcome studied in the main text is an optimal outcome to the adversarial information design problem (Proposition C.2).

The following notions will be used. A partially ordered set $(X, \leq)$ is upward sequentially complete if any nondecreasing sequence $\{x^n\}_{n=0}^{\infty}$ in X has a supremum, $\sup\{x^n\}_{n=0}^{\infty}$, in X. For partially ordered sets $(X, \leq)$ and $(Y, \leq)$ that are each upward sequentially complete, a nondecreasing function $f\colon X \to Y$ is upward sequentially continuous if for any nondecreasing sequence $\{x^n\}_{n=0}^{\infty}$ in X, we have $\sup\{f(x^n)\}_{n=0}^{\infty} = f(\sup\{x^n\}_{n=0}^{\infty})$.

As in Section 2, let Θ be the state space with a distribution function P, and let the payoff gain function be given by the measurable function $d\colon [0,1] \times \Theta \to \mathbb{R}$, where we assume A1 (Action Monotonicity) and A4 (Integrability).[8] To guarantee, for any information structure, the smallest equilibrium to be reached by sequential best responses from the smallest strategy, we also assume the following:

A15. Action Continuity: For each θ, $d(\ell, \theta)$ is lower semi-continuous in ℓ.

Under A1, A15 implies that $d(\ell, \theta)$ is upward sequentially continuous in ℓ (with respect to the usual order on $[0,1]$). Note that A15 is satisfied by the regime change game with our choice of tie breaking in (3) in Section 3.

We focus on the following class of information structures. For any Polish (i.e., separable and completely metrizable) space Z, let $\Delta(Z)$ denote the set of probability measures on Z with respect to its Borel sigma-algebra $\mathcal{B}(Z)$, and endow $\Delta(Z)$ with the weak topology, so that it is again a Polish space. An information structure is a pair $(X, (\pi_\theta)_{\theta\in\Theta})$ such that

- X is a Polish space of signals; and
- for each $\theta \in \Theta$, $\pi_\theta \in \Delta(\Delta(X))$, and for each $Q \in \mathcal{B}(\Delta(X))$, $\pi_\theta(Q)$ is measurable in θ.

For an information structure $(X, (\pi_\theta)_{\theta\in\Theta})$, the probability measure π on $X \times \Delta(X) \times \Theta$ is defined by $\pi(S \times Q \times R) = \int_R \int_Q q(S) d\pi_\theta(q) dP(\theta)$ for $S \in \mathcal{B}(X)$, $Q \in \mathcal{B}(\Delta(X))$, and $R \in \mathcal{B}(\Theta)$, and a regular conditional probability $\pi(\cdot|x)$ on $\Delta(X) \times \Theta$ conditional on $x \in X$ is well defined (where, in particular, for any $Q \in \mathcal{B}(\Delta(X))$ and $R \in \mathcal{B}(\Theta)$, $\pi(Q \times R|x)$ is measurable in x) and is unique up to π-a.s.; we fix any member of the equivalent class in such a way that $\int_{\Delta(X)\times\Theta} \max_{\ell\in\{0,1\}} |d(\ell, \theta)| d\pi(q, \theta|x) < \infty$ for all $x \in X$ (by A4). The marginal probability measure of π on X is denoted π_X.

An interpretation based on a heuristic "law of large numbers" is as follows.[9] Suppose that the information designer commits to an information structure $(X, (\pi_\theta)_{\theta\in\Theta})$. For

[8] Proposition C.1 in fact holds for any Polish state space Θ.

[9] See Sun (2006) for a formal measure-theoretic modelling with a continuum of players with independent randomness.

a state realization $\theta \in \Theta$, a probability distribution $q \in \Delta(X)$ is drawn according to π_θ. Then signals are drawn identically and independently across players according to q and sent privately to each player. By the "law of large numbers", q also represents the empirical distribution of signal realizations. The interim belief of a player receiving a signal $x \in X$ about the signal distribution and the state is given by $\pi(\cdot|x) \in \Delta(\Delta(X)\times\Theta)$.

The global game information structures, say those described in Appendix A, and the one constructed in the proof of Proposition 1 fall within this class of information structures. Indeed, for the former, $X = \Theta + \kappa\left[-\frac{1}{2}, \frac{1}{2}\right]$, and for $\theta \in \Theta$, $\pi_\theta \in \Delta(\Delta(X))$ is the Dirac measure on the distribution of $x = \theta + \kappa\zeta$ where ζ follows the density f. For the latter, $X = \{-\infty\} \cup \left(\Omega + \kappa\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$, and $\pi_\theta \in \Delta(\Delta(X))$ is the Dirac measure on the distribution $\bar{q}_\theta \in \Delta(X)$ defined as follows:

- If $\theta < \theta^* + \varepsilon$, then $\bar{q}_\theta = \delta_{-\infty}$.
- If $\theta^* + \varepsilon \leq \theta < \bar{\theta}'$ (resp. $\theta \geq \bar{\theta}'$), then $\bar{q}_\theta$ is the distribution of $x = \omega + \kappa\zeta$ where ω follows the uniform distribution over $[\underline{P}, \overline{P}]$ (resp. $[\overline{P}, 1]$) and ζ follows the uniform distribution over $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

An information structure $(X, (\pi_\theta)_{\theta\in\Theta})$ defines a Bayesian game with a continuum of symmetric players and binary actions, where we only consider symmetric strategy profiles. A strategy is a measurable function $s\colon X \to \{0, 1\}$, which also represents a symmetric strategy profile. To simplify the notation, we identify a strategy s with the set $S = \{x \in X \mid s(x) = 1\}$, and thus identify the set Σ of strategies with the sigma-algebra $\mathcal{B}(X)$. Endowed with set inclusion as the partial order, Σ is upward sequentially complete (while not complete in general). An outcome is a mapping $\nu\colon \Theta \to \Delta([0, 1])$ such that for any $E \in \mathcal{B}([0, 1])$, $\nu_\theta(E)$ is measurable in θ, where ν_θ is the probability distribution of the proportion of players who plays action 1. Denote the set of outcomes by O. A strategy $S \in \Sigma$ induces an outcome $\nu \in O$ by $\nu_\theta(E) = \pi_\theta(\{q \in \Delta(X) \mid q(S) \in E\})$ for $E \in \mathcal{B}([0, 1])$.

The expected payoff gain against strategy $S \in \Sigma$ conditional on $x \in X$ is

$$D(S|x) = \int_{\Delta(X)\times\Theta} d(q(S), \theta)d\pi(q, \theta|x),$$

which is nondecreasing and upward sequentially continuous in S by A1 and A15 and is measurable in x. A strategy $S \in \Sigma$ is an equilibrium if $D(S|x) \geq 0$ for all $x \in S$ and $D(S|x) \leq 0$ for all $x \in X \setminus S$. For $S \in \Sigma$ and $x \in X$, define

$$\underline{\beta}(S) = \{x \in X \mid D(S|x) > 0\} \in \Sigma,$$

which is nondecreasing and upward sequentially continuous in S. By the Tarski-Kantorovich Fixed Point Theorem (Van Zandt (2010, Theorem 10)), $\underline{\beta}\colon \Sigma \to \Sigma$ has a smallest fixed point $\underline{S} \in \Sigma$, which is the smallest equilibrium, where the sequential application of $\underline{\beta}$ to the smallest element $\emptyset \in \Sigma$ converges to $\underline{S}$. An outcome $\nu \in O$ is smallest equilibrium implementable (S-implementable) if there exists an information structure whose smallest equilibrium induces ν. Let $SI \subset O$ be the set of S-implementable outcomes. The following proposition provides a necessary condition for S-implementability, where $\Phi(\ell, \theta) = \int_0^\ell d(\ell', \theta) d\ell'$.

Proposition C.1. *Assume A1, A4, and A15. If an outcome ν is S-implementable, then it satisfies*

$$\int_\Theta \int_{[0,1]} \Phi(\ell, \theta) d\nu_\theta(\ell) dP(\theta) \geq 0.$$

Proof. The proof is analogous to that of Theorem 1(1) of Morris et al. (2020), but exploits the symmetry of the players assumed here.

Suppose that ν is S-implementable, and let $(X, (\pi_\theta)_{\theta \in \Theta})$ be an information structure whose smallest equilibrium $\underline{S}$ induces ν. Define the sequence $\{S^n\}_{n=0}^\infty$ by $S^0 = \emptyset$ and $S^n = \underline{\beta}(S^{n-1})$ for $n = 1, 2, \ldots$, where $S^0 \subset S^1 \subset S^2 \subset \cdots$ and $\bigcup_{n=0}^\infty S^n = \underline{S}$. By construction, we have $D(S^{n-1}|x) > 0$ for all $x \in S^n$. Therefore, we have

$$\begin{aligned}
0 &\leq \sum_{n=1}^\infty \int_{S^n \setminus S^{n-1}} \int_\Theta d(q(S^{n-1}), \theta) d\pi(q, \theta|x) d\pi_X(x) \\
&= \int_\Theta \sum_{n=1}^\infty d(q(S^{n-1}), \theta) q(S^n \setminus S^{n-1}) d\pi_\theta(q) dP(\theta) \\
&\leq \int_\Theta \sum_{n=1}^\infty \int_{q(S^{n-1})}^{q(S^n)} d(\ell, \theta) d\ell d\pi_\theta(q) dP(\theta) \\
&= \int_\Theta \Phi(q(\underline{S}), \theta) d\pi_\theta(q) dP(\theta) = \int_\Theta \int_{[0,1]} \Phi(\ell, \theta) d\nu_\theta(\ell) dP(\theta),
\end{aligned}$$

as desired, where the second weak inequality follows from A1. □

The first weak inequality in the proof holds with strict inequality, and so does the conclusion, if $\pi_X(\underline{S}) > 0$.

Now we consider the optimal information design problem within the class of information structures as described above. The objective of the information designer is given by a measurable function $V\colon [0,1] \times \Theta \to \mathbb{R}$ with $V(0, \theta) = 0$ (normalization), where we assume A7 (Objective Action Monotonicity) and:

A16. Objective Integrability: $\int_\Theta V(1,\theta)dP(\theta) < \infty$.

Suppose that the designer anticipates that the worst—hence smallest—equilibrium will be played once an information structure is given. Thus our adversarial information design problem is:

$$V^* = \sup_{\nu \in SI} V(\nu),$$

where, by abuse of notation,

$$V(\nu) = \int_\Theta \int_{[0,1]} V(\ell,\theta) d\nu_\theta(\ell) dP(\theta).$$

An outcome $\nu \in O$ is an optimal outcome of the adversarial information design problem if there exists a sequence of S-implementable outcomes $\nu^k \in SI$ such that $\nu^k_\theta \to \nu_\theta$ as $k \to \infty$ for each $\theta \in \Theta$ and $V^* = \sup_k V(\nu^k)$. In light of Proposition C.1, we consider the problem

$$\max_{\nu \in O} V(\nu) \tag{C.1a}$$

subject to

$$\int_\Theta \int_{[0,1]} \Phi(\ell,\theta) d\nu_\theta(\ell) dP(\theta) \geq 0. \tag{C.1b}$$

We now want to show the optimality of the target outcome studied in the main text. Assume A2 (State Monotonicity) and A3 (Upper Dominance Region). As in Section 3, also assume that $\int_\Theta \bar{\Phi}(\theta) dP(\theta) < 0$ and that P is continuous (ease of exposition). The expected Laplacian threshold is the unique $\theta^* \in \Theta$ such that

$$\int_{\theta > \theta^*} \Phi(1,\theta) dP(\theta) = 0,$$

and the target outcome $\nu^* \in O$ is

$$\nu^*_\theta = \begin{cases} \delta_1 & \text{if } \theta > \theta^*, \\ \delta_0 & \text{if } \theta \leq \theta^*, \end{cases}$$

where $\delta_\ell \in \Delta([0,1])$ is the Dirac measure on ℓ. For the objective function V, assume A8 (Restricted Convexity) and A9 (Objective State Monotonicity), and also assume that $V(1,\theta) > 0$ for all $\theta \in \Theta$ (relevancy).

Proposition C.2. *Assume A1–A4, A7–A9, A15, and A16. Then ν^* is an optimal outcome of the adversarial information design problem.*

Proof. Since by Proposition 1 in Section 4, there exists a sequence of S-implementable outcomes ν^ε such that $\nu^\varepsilon_\theta \to \nu^*_\theta$ and $V(\nu^\varepsilon) \to V(\nu^*)$ as $\varepsilon \to 0$, it suffices to prove that ν^*

is an optimal solution to the problem (C.1). The proof is analogous to that of Theorem 2 of Morris et al. (2020).

An outcome $\nu \in O$ is a perfect coordination outcome if $\nu_\theta(\{0,1\}) = 1$ for all $\theta \in \Theta$. The outcome ν^* is a perfect coordination outcome.

Claim 1. *For any outcome $\nu \in O$ satisfying* (C.1b)*, there exists a perfect coordination outcome ν' satisfying* (C.1b) *such that $V(\nu') \geq V(\nu)$.*

Proof. Let

$$\alpha(\ell,\theta) = \begin{cases} 1 & \text{if } \Phi(\ell,\theta) \leq \Phi(1,\theta), \\ \ell & \text{if } \Phi(\ell,\theta) > \Phi(1,\theta). \end{cases}$$

Then for all (ℓ,θ), we have $\Phi(\ell,\theta) \leq \alpha(\ell,\theta)\Phi(1,\theta)$ (by A1) and $V(\ell,\theta) \leq \alpha(\ell,\theta)V(1,\theta)$ (by A7 and A8). Let ν satisfy (C.1b). Let

$$\alpha(\theta) = \int_{[0,1]} \alpha(\ell,\theta) d\nu_\theta(\ell),$$

and define the perfect coordination outcome ν' by

$$\nu'_\theta = \alpha(\theta)\delta_1 + (1-\alpha(\theta))\delta_0.$$

Then we have

$$\begin{aligned} \int_\Theta \int_{[0,1]} \Phi(\ell,\theta) d\nu'_\theta(\ell) dP(\theta) &= \int_\Theta \int_{[0,1]} \alpha(\ell,\theta)\Phi(1,\theta) d\nu_\theta(\ell) dP(\theta) \\ &\geq \int_\Theta \int_{[0,1]} \Phi(\ell,\theta) d\nu_\theta(\ell) dP(\theta) \geq 0. \end{aligned}$$

Therefore, ν' satisfies (C.1b). We also have

$$\begin{aligned} V(\nu') &= \int_\Theta \int_{[0,1]} \alpha(\ell,\theta) V(1,\theta) d\nu_\theta(\ell) dP(\theta) \\ &\geq \int_\Theta \int_{[0,1]} V(\ell,\theta) d\nu_\theta(\ell) dP(\theta) = V(\nu), \end{aligned}$$

as claimed. □

Let $O_1 \subset O$ be the set of perfect coordination outcomes. In light of Claim 1, the problem (C.1) reduces to

$$\max_{\nu \in O_1} \int_\Theta \nu_\theta(\{1\}) V(1,\theta) dP(\theta) \tag{C.2a}$$

subject to

$$\int_\Theta \nu_\theta(\{1\}) \Phi(1,\theta) dP(\theta) \geq 0. \tag{C.2b}$$

Claim 2. *ν^* is an optimal solution to* (C.2)*.*

Proof. Let $\theta_\sharp = \inf\{\theta \in \Theta \mid \Phi(1,\theta) \geq 0\}$, where $\theta^* < \theta_\sharp$.

First, by construction, ν^* satisfies the constraint (C.2b) with equality. Then, let $\nu \in O_1$ be such that $V(\nu) > V(\nu^*)$. Define the functions $w, w^*, w^{**} \colon \Theta \to \mathbb{R}$ by

$$w(\theta) = \begin{cases} \nu_\theta(\{1\})V(1,\theta) & \text{if } \theta \leq \theta_\sharp, \\ V(1,\theta) & \text{if } \theta > \theta_\sharp, \end{cases}$$

$$w^*(\theta) = \nu^*_\theta(\{1\})V(1,\theta) = \begin{cases} 0 & \text{if } \theta \leq \theta^*, \\ V(1,\theta) & \text{if } \theta > \theta^*, \end{cases}$$

$$w^{**}(\theta) = \begin{cases} w^*(\theta) + \frac{V' - V(\nu^*)}{P(\theta_\sharp) - P(\theta^*)} & \text{if } \theta^* < \theta \leq \theta_\sharp, \\ w^*(\theta) & \text{otherwise,} \end{cases}$$

where $V' = \int_\Theta w(\theta)dP(\theta) \geq V(\nu) > V(\nu^*)$. By construction, $w(\theta) = w^{**}(\theta)$ for all $\theta > \theta_\sharp$, $\int_\theta^{\theta_\sharp} w(\theta')dP(\theta') \leq \int_\theta^{\theta_\sharp} w^{**}(\theta')dP(\theta')$ for all $\theta \leq \theta_\sharp$, and $\int_{\theta \leq \theta_\sharp} w(\theta)dP(\theta) = \int_{\theta \leq \theta_\sharp} w^{**}(\theta)dP(\theta)$. Since $\frac{\Phi(1,\theta)}{V(1,\theta)}$ (< 0) is nondecreasing for $\theta < \theta_\sharp$ (by A2 and A9), we thus have $\int_{\theta \leq \theta_\sharp} \frac{\Phi(1,\theta)}{V(1,\theta)} w(\theta)dP(\theta) \leq \int_{\theta \leq \theta_\sharp} \frac{\Phi(1,\theta)}{V(1,\theta)} w^{**}(\theta)dP(\theta)$, and therefore $\int_\Theta \frac{\Phi(1,\theta)}{V(1,\theta)} w(\theta)dP(\theta) \leq \int_\Theta \frac{\Phi(1,\theta)}{V(1,\theta)} w^{**}(\theta)dP(\theta)$. Hence, we have

$$\begin{aligned} \int_\Theta \nu_\theta(\{1\})\Phi(1,\theta)dP(\theta) &\leq \int_\Theta \frac{\Phi(1,\theta)}{V(1,\theta)} w(\theta)dP(\theta) \\ &\leq \int_\Theta \frac{\Phi(1,\theta)}{V(1,\theta)} w^{**}(\theta)dP(\theta) \\ &= \int_\Theta \frac{\Phi(1,\theta)}{V(1,\theta)} w^*(\theta)dP(\theta) + \frac{V' - V(\nu^*)}{P(\theta_\sharp) - P(\theta^*)} \int_{\theta^*}^{\theta_\sharp} \frac{\Phi(1,\theta)}{V(1,\theta)} dP(\theta) \\ &< \int_\Theta \nu^*_\theta(\{1\})\Phi(1,\theta)dP(\theta) = 0, \end{aligned}$$

which means that ν does not satisfy the constraint (C.2b). This implies that ν^* is an optimal solution to the problem (C.2). □

The proof of Proposition C.2 is thus completed. □

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