Online Appendix for
A Market Based Solution for Fire Sales and Other Pecuniary Externalities
(Not For Publication)

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This appendix contains omitted formulations, proofs, and results for the paper titled "A Market Based Solution for Fire Sales and Other Pecuniary Externalities" by Weerachart T. Kilenthong and Robert M. Townsend.

A Derivations of the Necessary Optimal Conditions for the Market Maker Problem

Consider the following market maker problem (the same as the programming problem (20) in the main text)

$$\max_{\left[p_2,\delta^h(p_2),\mathbf{x}^h(p_2)\right]_h} \sum_h \sum_{p_2} \lambda^h \alpha^h \delta^h(p_2) \left[u^h\left(c^h_{w1}(p_2), c^h_{z1}(p_2)\right) + V^h\left(k^h(p_2), p_2\right)\right] \tag{A.1}$$

subject to

$$\delta^h(p_2)\, k^h(p_2) \geq 0, \forall h; p_2, \tag{A.2}$$

$$\sum_{h,p_2} \alpha^h \delta^h(p_2)\, c^h_{w1}(p_2) = \sum_h \alpha^h e^h_{w1}, \tag{A.3}$$

$$\sum_{h,p_2} \alpha^h \delta^h(p_2)\left[c_{z1}^h(p_2)+k^h(p_2)\right]=\sum_h \alpha^h e_{z1}^h, \tag{A.4}$$

$$\sum_h \delta^h(p_2)\,\alpha^h \Delta^h\left(k^h(p_2),p_2\right)=0, \forall p_2. \tag{A.5}$$

The Lagrangian for this problem is

$$\begin{aligned}\mathscr{L}=\sum_h \alpha^h \sum_{p_2} \delta^h(p_2)\Bigg\{&\lambda^h\left[u^h\left(c_{w1}^h(p_2),c_{z1}^h(p_2)\right)+V^h\left(k^h(p_2),p_2\right)\right]\\ &+\frac{\mu^h(p_2)}{\alpha^h}k^h(p_2)+\mu_{w1}\left[e_{w1}^h-c_{w1}^h(p_2)\right]\\ &+\mu_{z1}\left[e_{z1}^h-c_{z1}^h(p_2)-k^h(p_2)\right]-\mu_\Delta(p_2)\,\Delta^h\left(k^h(p_2),p_2\right)\Bigg\}\end{aligned}$$

The FOCs with respect to $c_{w1}^h(p_2), c_{z1}^h(p_2), k^h(p_2)$, respectively, are

$$\delta^h(p_2)\left[\lambda^h\alpha^h u_{w1}^h-\mu_{w1}\alpha^h\right]=0, \forall h; p_2, \tag{A.6}$$

$$\delta^h(p_2)\left[\lambda^h\alpha^h u_{z1}^h-\mu_{z1}\alpha^h\right]=0, \forall h; p_2, \tag{A.7}$$

$$\delta^h(p_2)\left[\lambda^h\alpha^h V_k^h+\mu^h(p_2)-\mu_\Delta(p_2)\,\alpha^h\Delta_k^h\left(k^h(p_2),p_2\right)-\mu_{z1}\alpha^h\right]=0, \forall h; p_2, \tag{A.8}$$

where $u_{w1}^h \equiv \frac{\partial u^h\left(c_{w1}^h(p_2),c_{z1}^h(p_2)\right)}{\partial c_{w1}^h(p_2)}$, $u_{z1}^h \equiv \frac{\partial u^h\left(c_{w1}^h(p_2),c_{z1}^h(p_2)\right)}{\partial c_{z1}^h(p_2)}$, $V_k^h \equiv \frac{\partial V^h\left(k^h(p_2),p_2\right)}{\partial k^h(p_2)}$ and $\Delta_k^h\left(k^h(p_2),p_2\right) \equiv \frac{\partial \Delta^h\left(k^h(p_2),p_2\right)}{\partial k^h(p_2)}$. Considering only the case with $\delta^h(p_2)>0$, we can show that the

$$\frac{\mu_{z1}}{\mu_{w1}}=\frac{u_{z1}^h}{u_{w1}^h}, \tag{A.9}$$

$$\frac{\mu_{z1}}{\mu_{w1}}=\frac{V_k^h}{u_{w1}^h}+\frac{\mu^h(p_2)}{\lambda^h\alpha^h u_{w1}^h}-\frac{\mu_\Delta(p_2)}{\mu_{w1}}\Delta_k^h\left(k^h(p_2),p_2\right) \tag{A.10}$$

Using the envelope theorem for the value function of the second period utility maximization problem, we can show that, for any p_2 with $\delta^h(p_2)>0$, $V_k^h\left(k^h(p_2),p_2\right)=u_{z2}^h\left(c_{w2}^h(p_2),c_{z2}^h(p_2)\right)R$. Therefore,

$$\frac{u_{z1}^h}{u_{w1}^h}=\frac{u_{z2}^h}{u_{w1}^h}R+\frac{\mu^h(p_2)}{\lambda^h\alpha^h u_{w1}^h}-\frac{\mu_\Delta(p_2)}{\mu_{w1}}\Delta_k^h\left(k^h(p_2),p_2\right), \forall h=a,b. \tag{A.11}$$

Thus far, we have been treating p_2 as taking discrete values in a finite grid, as we have finite sum over such values. Here, we ensure our grid is judiciously fine enough, i.e., contains the best possible p_2, by treating p_2 as a continuous parameter. We then can optimize over p_2 using the envelope condition, taking the derivative of the Lagrangian with respect to p_2, and for the optimization condition, set that derivative to zero. As a result, for each agent type h with $\delta^h(p_2)>0$,

$$\lambda^h V_p^h\left(k^h(p_2),p_2\right)=\mu_\Delta(p_2)\,\Delta_p^h\left(k^h(p_2),p_2\right). \tag{A.12}$$

B Derivations of the Necessary Optimal Conditions for Utility Maximization Problem in the Decentralization with Individually Chosen Rights to Trade

Consider the following utility maximization problem (the same as the maximization problem (27) in the main text)

$$\max_{p_2} \max_{p_2, \delta^h(p_2), \mathbf{x}^h(p_2)} \sum_{p_2} \delta^h(p_2) \left[u^h \left(c_{w1}^h(p_2), c_{z1}^h(p_2) \right) + V^h \left(k^h(p_2), p_2 \right) \right] \tag{A.13}$$

subject to

$$\delta^h(p_2) k^h(p_2) \geq 0, \forall h; p_2, \tag{A.14}$$

$$\sum_{p_2} \delta^h(p_2) \left[c_{w1}^h(p_2) + p_1 c_{z1}^h(p_2) + p_1 k^h(p_2) + P_\Delta(p_2) \Delta^h \left(k^h(p_2), p_2 \right) \right] \leq e_{w1}^h + p_1 e_{z1}^h, \tag{A.15}$$

The Lagrangian for this problem is

$$\begin{aligned} \mathscr{L} = \sum_{p_2} \delta^h(p_2) \Bigg\{ & u^h \left(c_{w1}^h(p_2), c_{z1}^h(p_2) \right) + V^h \left(k^h(p_2), p_2 \right) \\ & + \eta^h(p_2) k^h(p_2) + \eta_{bc,1}^h \left[e_{w1}^h + p_1 e_{z1}^h - c_{w1}^h(p_2) - p_1 c_{z1}^h(p_2) - p_1 k^h(p_2) - P_\Delta(p_2) \Delta^h \left(k^h(p_2), p_2 \right) \right] \Bigg\} \end{aligned}$$

The FOCs with respect to $c_{w1}^h(p_2), c_{z1}^h(p_2), k^h(p_2)$, respectively, are

$$\delta^h(p_2) \left[u_{w1}^h - \eta_{bc,1}^h \right] = 0, \forall h; p_2, \tag{A.16}$$

$$\delta^h(p_2) \left[u_{z1}^h - \eta_{bc,1}^h p_1 \right] = 0, \forall h; p_2, \tag{A.17}$$

$$\delta^h(p_2) \left[V_k^h + \eta^h(p_2) - \eta_{bc,1}^h p_1 - \eta_{bc,1}^h P_\Delta(p_2) \Delta_k^h \left(k^h(p_2), p_2 \right) \right] = 0, \forall h; p_2, \tag{A.18}$$

Using a similar process as in online Appendix A, we can show that

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h} R + \frac{\eta^h(p_2)}{u_{w1}^h} - P_\Delta(p_2) \Delta_k^h \left(k^h(p_2), p_2 \right), \forall h = a, b. \tag{A.19}$$

Again, we here ensure our grid is judiciously fine enough, i.e., contains the best possible p_2, by treating p_2 as a continuous parameter. We then can optimize over p_2 using the envelope condition, taking the derivative of the Lagrangian with respect to p_2, and for the optimization condition, set that derivative to zero. As a result, for each agent type h with $\delta^h(p_2) > 0$,

$$\left(\frac{1}{\eta_{bc,1}^h} \right) V_p^h \left(k^h(p_2), p_2 \right) = P_\Delta(p_2) \Delta_p^h \left(k^h(p_2), p_2 \right), \tag{A.20}$$

With the matching conditions: $P_\Delta(p_2) = \frac{\mu_\Delta(p_2)}{\mu_{w1}}$ and $\lambda^h = \frac{\mu_{w1}}{\eta_{bc,1}^h}$, this condition is equivalent to condition (A.12).

C Proofs of the First Welfare Theorem and the Existence Theorem for the General Economy in Section of 3 the Main Text

Proof of the First Welfare Theorem 1. This proof follows Prescott and Townsend (1984a). Let allocations $\left(\mathbf{x}^h, \mathbf{b}\right)$, and prices $(p_1, P(\mathbf{w}))$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e., there is an attainable allocation $\tilde{\mathbf{x}}$ such that $\sum_{\mathbf{w}} \tilde{x}^h(\mathbf{w}) U^h(\mathbf{w}) \geq \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{w})$ for all h and $\sum_{\mathbf{w}} \tilde{x}^{\hat{h}}(\mathbf{w}) U^{\hat{h}}(\mathbf{w}) > \sum_{\mathbf{w}} x^{\hat{h}}(\mathbf{w}) U^{\hat{h}}(\mathbf{w})$ for some $\hat{h}$. With local non-satiation of preferences, $\sum_{\mathbf{w}} P(\mathbf{w}) x^h(\mathbf{w}) \leq \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{x}^h(\mathbf{w})$ for all h, and $\sum_{\mathbf{w}} P(\mathbf{w}) x^{\hat{h}}(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{x}^{\hat{h}}(\mathbf{w})$ for some $\hat{h}$. Summing over all agents with weights α^h, we have

$$\sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h x^h(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h \tilde{x}^h(\mathbf{w}). \tag{A.21}$$

In addition, for each allocation $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we can find a corresponding supply from the intermediary such that $b(\mathbf{w}) = \sum_h \alpha^h x^h(\mathbf{w})$ and $\tilde{b}(\mathbf{w}) = \sum_h \alpha^h \tilde{x}^h(\mathbf{w})$. Since both $\mathbf{x}$ and $\tilde{\mathbf{x}}$ satisfy all feasibility conditions, $\mathbf{b}$ and $\tilde{\mathbf{b}}$ both satisfy the clearing constraints (54)-(56). As a result, (A.21) can be rewritten as $\sum_{\mathbf{w}} P(\mathbf{w}) b(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{b}(\mathbf{w})$. On the other hand, the intermediary's profit maximization implies that $\sum_{\mathbf{w}} P(\mathbf{w}) b(\mathbf{w}) \geq \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{b}(\mathbf{w})$. This is a contradiction! □

Proof of the Existence Theorem 3. For notational convenience, we put the endowment $\mathbf{e}^h$ onto the grid. Let $\mathbf{P} = [P(\mathbf{w})]_{\mathbf{w}}$ be the prices of all bundles. As in Prescott and Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{\mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P} \cdot \mathbf{P}} \leq 1\right\}, \tag{A.22}$$

where "$\cdot$" is the inner product operator. Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$. The local non-satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^* \cdot \mathbf{P}^*}.$$

Note that $\mathbf{P}' \cdot \mathbf{P}' = 1$. In order to preserve the convexity of the mapping with prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max\left\{0, \lambda^h + \frac{\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)}{A}\right\}, \tag{A.23}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h}, \tag{A.24}$$

where A is a positive number such that $\sum_h \left|\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)\right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda, \mathbf{x}, \mathbf{P}) \rightarrow (\lambda', \mathbf{x}', \mathbf{P}')$ is a mapping from $S^{H-1} \times \overline{X} \times S^{n-1} \longrightarrow S^{H-1} \times \overline{X} \times S^{n+1}$. Since each set is non-empty, compact, and convex, so is its cross-product. In addition, the overall mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda, \mathbf{x}, \mathbf{P})$.

Proved in the Second Welfare Theorem 2 in the Appendix in the main text of the main text, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the nonsatiation and the positive endowment assumptions ensure that there is a cheaper point as in the proof of Theorem 2. As a result, a compensated equilibrium is a competitive equilibrium with transfers.

We now need to show that there is no need for wealth transfers in equilibrium, i.e., the budget constraint without transfers $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ holds for every agent h. It is not difficult to show that $\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$. In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint (without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. $\square$

D Definition of Competitive Equilibrium (with Externality) Corresponding to the General Model in Section of 3 the Main Text

Definition A.1 (Competitive Equilibrium with Externality)**.** A competitive equilibrium is a specification of prices $(p_1, \mathbf{Q}, \mathbf{p})$, and an allocation $\left(\mathbf{c}_1^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ such that

- for any agent type h as a price taker, $\left(\mathbf{c}_1^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ solves

$$\max_{\mathbf{c}_1^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + \beta \sum_s \pi_s u^h\left(e_{w2s}^h + \sum_j D_{js}\theta_j^h + \tau_{w2s}^h, e_{z2s}^h + R_s k^h + \tau_{z2s}^h\right) \quad \text{(A.25)}$$

  subject to the budget constraints in the first period

$$c_{w1}^h + p_1\left(c_{z1}^h + k^h\right) + \sum_j Q_j \theta_j^h \quad \leq \quad e_{w1}^h + p_1 e_{z1}^h, \quad \text{(A.26)}$$

  the spot budget constraint in state s

$$\tau_{w2s}^h + p_{2s}\tau_{z2s}^h \quad = \quad 0, \text{ for } s = 1, \ldots, S, \quad \text{(A.27)}$$

  the collateral constraint in state s

$$p_{2s}R_s k^h + \sum_j D_{js}\theta_j^h \geq 0, \forall s = 1, \ldots, S, \quad \text{(A.28)}$$

  and the non-negativity constraint for saving

$$k^h \geq 0, \quad \text{(A.29)}$$

- markets clear for good w and good z at $t = 1$, for θ_j^h for all $j = 1, \ldots, J$, and for spot trade $\tau_{\ell s}^h$ in state s, respectively:

$$\sum_h \alpha^h c_{w1}^h \quad = \quad \sum_h \alpha^h e_{w1}^h, \quad \text{(A.30)}$$

$$\sum_h \alpha^h \left(c_{z1}^h + k^h\right) \quad = \quad \sum_h \alpha^h e_{z1}^h, \quad \text{(A.31)}$$

$$\sum_h \alpha^h \theta_j^h \quad = \quad 0, \forall j, \quad \text{(A.32)}$$

$$\sum_h \alpha^h \tau_{\ell 2s}^h \quad = \quad 0, \forall s; \ell = w, z. \quad \text{(A.33)}$$

E Public Finance Interpretation for the Saving Economy in the Main Text

The budget constraint with the prices of the rights to trade

$$\sum_{p_2} \delta^h(p_2)\left[c^h_{w1}(p_2) + p_1\left[c^h_{z1}(p_2) + k^h(p_2)\right] + P_\Delta(p_2)\,\Delta^h\left(k^h(p_2), p_2\right)\right] \le e^h_{w1} + p_1 e^h_{z1}, \quad \text{(A.34)}$$

which is the same as (28) in the main text, has a public finance interpretation, as if we were to try to implement the optimum solution by taxes and subsidies. With the constant relative risk aversion (CRRA) utility function, $u^h(c_w, c_z) = -\frac{1}{c_w} - \frac{1}{c_z}$ for $h = a, b$, the right to trade in exchange p_2 is

$$\Delta^h\left(k^h, p_2\right) = \left(\frac{\sqrt{p_2}}{1+\sqrt{p_2}}\right)\left[\sqrt{p_2}\left(e^h_{z2} + Rk^h\right) - e^h_{w2}\right]. \quad \text{(A.35)}$$

Substituting the rights to trade (A.35) into the budget constraint for an agent type h (A.34) gives

$$\sum_{p_2} \delta^h(p_2)\Bigg\{c^h_{w1}(p_2) + p_1\left[c^h_{z1}(p_2) + k^h(p_2)\right] \le e^h_{w1} + p_1 e^h_{z1} - \left[\left(\frac{p_2}{1+\sqrt{p_2}}\right)P_\Delta(p_2)\,R\right]k^h$$
$$- \left[\left(\frac{p_2}{1+\sqrt{p_2}}\right)P_\Delta(p_2)\right]e^h_{z2} + \left[\left(\frac{\sqrt{p_2}}{1+\sqrt{p_2}}\right)P_\Delta(p_2)\right]e^h_{w2}\Bigg\} \quad \text{(A.36)}$$

We can now see that we need to have three types of taxes/subsidies, (i) saving/collateral tax of $\left(\frac{p_2}{1+\sqrt{p_2}}\right)P_\Delta(p_2)\,R$ per unit of saving/collateral, k^h, (ii) collateral good endowment tax of $\left(\frac{p_2}{1+\sqrt{p_2}}\right)P_\Delta(p_2)$ per unit of collateral good z endowment at date $t = 2$, e^h_{z2}, and (iii) subsidy, negative tax $-\left(\frac{\sqrt{p_2}}{1+\sqrt{p_2}}\right)P_\Delta(p_2)$ per unit of consumption good endowment of good w at date $t = 2$, e^h_{w2}. The last two parts can be implemented in the form of lump-sum taxes and redistributions based on ownership of endowments. Endowments matter because they are part of excess demand. The tax/subsidy rate on endowments also depends on the exchange p_2 chosen. That is, the exchange p_2 itself is a choice as far as the household is concerned, so these are not lump sum endowment taxes.[1]

But again we do not need the taxes. We let the markets decide. Markets determine prices, and prices determine allocations.

[1]This is like looking up marginal rates in a big tax book and settling on which page (or pages) to use, indexed by p_2 that the agent chooses.

F The Presence of Rights has No Effect on the Standard Classical Economy

We can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when segregated markets are introduced. More specifically, start in the extended commodity space with markets for rights at various prices, writing down the programming problem. Then guess that a solution to the first order conditions of the Lagrangian problem (which are both necessary and sufficient) is the solution, quantities and Lagrange multipliers, of the standard classical economy, without externalities, in the standard commodity space. Rights to trade in the extended commodity space are simply again the excess demands of the classical economy, and there are no additional obstacle to trade constraints in either. The guess is verified to be correct. This implies that the (spot) prices in an active segregated exchange must be the same as the shadow prices from the planning problem without the segregated exchanges. That, in turn, ensures that the Lagrange multipliers for the rights constraints for those active spot market exchanges have to be zero. In the analogue decentralized equilibrium, this implies that the prices of the rights to trade in those are zero as well. This makes sense since there is no reason to restrict ("tax" or "subsidize") trade, as that trade is not imposing an externality. However, the Lagrange multipliers or decentralized prices of the rights in inactive exchanges are not necessarily zero. In fact, they should not be zero to help guide agents to choose the optimal exchanges in equilibrium. This is what is preventing the emergence of new equilibria that might feature some kind of price discrimination.

G Exogenous Incomplete Markets Economy: Geanakoplos and Polemarchakis (1986); Greenwald and Stiglitz (1986)

This section presents a slightly different model economy from the saving economy in Section 2 of the main text. There is more than one state in the second period here, whereas there is no uncertainty in the main text, though states of the world were included in the general environment of Section 3. In addition, for the most part, there is no saving here. An exception is the numerical example of G.4 where there is no insurance for idiosyncratic shocks and savings as a buffer stock against uncertainty plays a role. Yet, an inefficiency remains, regardless, coming from incomplete markets, and that is the focus of this section.

Consider an exogenously imposed incomplete markets economy. It is an economy with two periods $t = 1, 2$. There are S possible states of nature in the second period, $t = 2$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. There are 2 goods, labeled good w and good z, in each date and in each state. There are H types with fractions $\alpha^h > 0$, for $h = 1, 2, ..., H$ such that $\sum_h \alpha^h = 1$. Each agent type h is endowed with $\left(e^h_{w1}, e^h_{z1}\right)$ at date $t = 1$ and $\left(e^h_{w2s}, e^h_{z2s}\right)$ in state s at date $t = 2$. Preferences of each type h agent are represented by utility u^h.

There are J securities available for purchase or sale in the first period, $t = 1$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets in the second period $t = 2$ where D_{js} is the payoff of asset j in units of good w (the numeraire good) in state $s = 1, 2, \ldots, S$. Here we do not include securities paying in good z as there is trade in the two goods in spot markets, so these are not needed. Let θ^h_j denote the amount of the j^{th} security acquired by an agent of type h at $t = 1$ with $\boldsymbol{\theta}^h \equiv \left[\theta^h_j\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise. Let Q_j denote the price of security j with $\mathbf{Q} \equiv [Q_j]_j$. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is, $J < S$. Thus agents type h cannot achieve arbitrary targets for consumption, which enter utility as $u^h\left(e^h_{w2s} + \sum_j D_{js}\theta^h_j + \tau^h_{w2s}, e^h_{z2s} + \tau^h_{z2s}\right)$, where τ^h_{w2s} and τ^h_{z2s} are spot trades in good w and z in state s at date $t = 2$, respectively. The u^h are strictly concave with other regularity conditions.

For this exogenous incomplete markets economy, the key set of obstacle-to-trade constraints generating the pecuniary externality are the spot budget constraints

$$C^h_s\left(\boldsymbol{\tau}^h, \mathbf{p}\right) \equiv \tau^h_{w2s} + p_{2s}\tau^h_{z2s} = 0, \forall s, h, \tag{A.37}$$

where p_{2s} is the spot market price in state s of good z in terms of good w, and $\mathbf{p} \equiv [p_{2s}]_s$ is the vector of the spot prices. The constraints here are simple spot market budget constraints, but with incomplete markets this is how prices create externalities. To be consistent with the rights to trade $\Delta^h_s(\mathbf{p})$ defined below, we keep the vector of spot prices $\mathbf{p}$ in the general constraint $C^h_s\left(\boldsymbol{\tau}^h, \mathbf{p}\right)$. We simplify the notation by restricting ourselves here to two periods, two goods, S states, but it is easy to generalize. Likewise, we can easily incorporate intertemporal savings, for example the storage of good z, k^h, as in the numerical example below.

G.1 The Definition of Competitive Equilibrium with Exogenous Incomplete Markets

Definition A.2 (Competitive Equilibrium with Exogenous Incomplete Markets)**.** A competitive equilibrium is a specification of prices $(p_1, \mathbf{Q}, \mathbf{p})$, and an allocation $\left(\mathbf{c}^h_1, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ such that

- for any agent type h as a price taker, $\left(\mathbf{c}_1^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ solves

$$\max_{\mathbf{c}_1^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + \beta \sum_s \pi_s u^h\left(e_{w2s}^h + \sum_j D_{js}\theta_j^h + \tau_{w2s}^h, e_{z2s}^h + \tau_{z2s}^h\right) \tag{A.38}$$

subject to the budget constraints in the first period

$$c_{w1}^h + p_1 c_{z1}^h + \sum_{j=1}^J Q_j \theta_j^h \quad \leq \quad e_{w1}^h + p_1 e_{z1}^h, \tag{A.39}$$

and the spot budget constraint in state s

$$\tau_{w2s}^h + p_{2s}\tau_{z2s}^h \quad = \quad 0, \text{ for } s = 1, \ldots, S; \tag{A.40}$$

- markets clear for good $\ell = w, z$ at $t = 1$, for θ_j^h for all $j = 1, \ldots, J$, and for spot trade $\tau_{\ell 2s}^h$ in state s, respectively:

$$\sum_h \alpha^h c_{\ell 1}^h \quad = \quad \sum_h \alpha^h e_{\ell 1}^h, \forall \ell = w, z, \tag{A.41}$$

$$\sum_h \alpha^h \theta_j^h \quad = \quad 0, \forall j, \tag{A.42}$$

$$\sum_h \alpha^h \tau_{\ell 2s}^h \quad = \quad 0, \forall s; \ell = w, z. \tag{A.43}$$

The key constraints that generate the externality in this problem are the spot-budget constraints (A.40) for an agent of type h. Note that the spot price p_{2s} is determined by pretrade position of endowments and securities where endowments are exogenous but securities are endogenous, and we write this as $p_{2s} = p_{2s}(\boldsymbol{\theta}, \mathbf{e})$. As in Geanakoplos and Polemarchakis (1986), the dependency generates an indirect price effect from security reallocations. This indirect effect then produces an externality when the security markets are incomplete.

G.2 Source of Inefficiency in the Incomplete Markets Example

Proposition B.1. *The competitive equilibrium with exogenous security markets is (constrained) efficient if and only if the equilibrium allocation* $\left(\mathbf{c}_1^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ *is first-best optimal or the spot price is independent of security positions, i.e.,* $\frac{\partial p_{2s}}{\partial \theta_j^h} = 0$ *for every state* s*, every security* j *and every agent of type* h*.*

Proof. We begin the proof by deriving the necessary and sufficient conditions for the first-best optimality. The social planner's problem for the first-best optimality is as follows:

Program A.1.

$$\max_{\left(\theta^h_{\ell 1},\theta^h_{\ell s}\right)_{\ell,s,h}} u^1\left(e^1_{w1}+\theta^1_{w1},e^1_{z1}+\theta^1_{z1}\right)+\beta\sum_s \pi_s u^1\left(e^1_{w2s}+\theta^1_{w2s},e^1_{z2s}+\theta^1_{z2s}\right) \tag{A.44}$$

subject to the participation constraints and the resource constraints, respectively,

$$\begin{aligned} u^h\left(e^h_{w1}+\theta^h_{w1},e^h_{z1}+\theta^h_{z1}\right)+\beta\sum_s \pi_s u^h\left(e^h_{w2s}+\theta^h_{w2s},e^h_{z2s}+\theta^h_{z2s}\right) &\geq \overline{\mathscr{U}}^h, \text{for } h=2,...,H;\\ \sum_h \alpha^h\theta^h_{\ell 1} &= 0, \text{for } \ell=w,z;\\ \sum_h \alpha^h\theta^h_{\ell 2s} &= 0, \text{for } \ell=w,z; s=1,...,S. \end{aligned}$$

Lemma A.1. *The necessary and sufficient conditions for the first-best optimality are as follows:*

$$\frac{\gamma^h_u u^h_{\ell 1}}{\alpha^h} = \frac{\gamma^{\tilde{h}}_u u^{\tilde{h}}_{\ell 1}}{\alpha^{\tilde{h}}}, \forall h,\tilde{h}=1,...,H;\ell=w,z \tag{A.45}$$

$$\frac{\gamma^h_u \beta\pi_s u^h_{\ell 2s}}{\alpha^h} = \frac{\gamma^{\tilde{h}}_u \beta\pi_s u^{\tilde{h}}_{\ell 2s}}{\alpha^{\tilde{h}}}, \forall h,\tilde{h}=1,...,H;\ell=w,z;s=1,...,S, \tag{A.46}$$

where γ^h_u is the Lagrange multipliers for the participation constraints for h (normalized by setting $\gamma^1_u=1$), and $u^h_{\ell 1}=\frac{\partial u^h}{\partial c^h_{\ell 1}}$ and $u^h_{\ell 2s}=\frac{\partial u^h}{\partial c^h_{\ell 2s}}$ are the marginal utility of an agent of type h with respect to $c_{\ell 1}$ and $c_{\ell 2s}$, respectively.

We now consider the following social planner's problem for the economy with exogenous security markets.

Program A.2.

$$\max_{\left(\theta^h_{w1},\theta^h_{z1},\theta^h_j,\tau^h_{w2s},\tau^h_{z2s}\right)_h} u^1\left(e^1_{w1}+\theta^1_{w1},e^1_{z1}+\theta^1_{z1}\right)+\beta\sum_s \pi_s u^1\left(e^1_{w2s}+\sum_j D_{js}\theta^h_j+\tau^1_{w2s},e^1_{z2s}+\tau^1_{z2s}\right) \tag{A.47}$$

subject to the participation constraints, the resource constraints, and the obstacle-to-trade constraints, respectively,

$$u^h\left(e^h_{w1}+\theta^h_{w1},e^h_{z1}+\theta^h_{z1}\right)+\beta\sum_s \pi_s u^h\left(e^h_{w2s}+\sum_j D_{js}\theta^h_j+\tau^h_{w2s},e^h_{z2s}+\tau^h_{z2s}\right)\geq \overline{\mathscr{U}}^h, \forall h, \tag{A.48}$$

$$\sum_h \alpha^h\theta^h_{\ell 1}=0, \forall \ell; \tag{A.49}$$

$$\sum_h \alpha^h\theta^h_j=0, \forall j, \tag{A.50}$$

$$\sum_h \alpha^h\tau^h_{w2s}=0, \forall s, \tag{A.51}$$

$$\tau^h_{w2s} + p_{2s}\left(\theta^1_s, ..., \theta^H_s\right)\tau^h_{z2s} = 0, \forall s, h. \tag{A.52}$$

Note that the resource (market-clearing) constraints for τ^h_{z2s} are omitted due to its linear dependence on (A.51) and (A.52). A solution to this social planner's problem is called a constrained optimal allocation.

The first order conditions for $\theta^h_{w1}, \theta^h_{z1}, \tau^h_{w2s}, \tau^h_{z2s}, \theta^h_j$ are as follows:

$$\begin{aligned}
\mu^h_u \beta \pi_s u^h_{w1} + \alpha^h \mu^\theta_{w1} &= 0, && \text{(A.53)}\\
\mu^h_u \beta \pi_s u^h_{z1} + \alpha^h \mu^\theta_{z1} &= 0, && \text{(A.54)}\\
\mu^h_u \beta \pi_s u^h_{w2s} + \alpha^h \mu^\tau_{w2s} + \mu^h_s &= 0, \forall s = 1, ..., S, && \text{(A.55)}\\
\mu^h_u \beta \pi_s u^h_{z2s} + p_{2s}\mu^h_s &= 0, \forall s = 1, ..., S, && \text{(A.56)}\\
\mu^h_u \beta \sum_s \pi_s u^h_{w2s} D_{js} + \alpha^h \mu^\theta_j + \sum_s \frac{\partial p_{2s}}{\partial \theta^h_j} \sum_{\tilde{h}} \mu^{\tilde{h}}_s \tau^{\tilde{h}}_{z2s} &= 0, \forall j = 1, ..., J, && \text{(A.57)}
\end{aligned}$$

where $\mu^h_u, \mu^\theta_{\ell 1}, \mu^\theta_j, \mu^\tau_{w2s}, \mu^h_s$ are the Lagrange multipliers for the participation constraints for h (normalize by setting $\gamma^1_u = 1$), for the resource constraints for $\theta^h_{\ell 1}$, for the resource constraints for θ^h_j, for the resource constraints for τ^h_{w2s} and for the obstacle to trade or spot-market constraints in state s for agent h.

The proof is divided into two parts as follows:

(i) ($\Leftarrow$) We now show that an allocation that satisfies the necessary and sufficient conditions for the first-best optimality (A.45)-(A.46) must satisfy the first order conditions (A.53)-(A.57). It is not difficult to see that this will be the case if the externality term, the last term of (A.57), is zero, i.e.,

$$\sum_s \frac{\partial p_{2s}}{\partial \theta^h_j} \sum_{\tilde{h}} \gamma^{\tilde{h}}_s \tau^{\tilde{h}}_{z2s} = 0 \tag{A.58}$$

It is obvious that if the spot price is independent of security positions, i.e., $\frac{\partial p_{2s}}{\partial \theta^h_j} = 0$ for every state s, every security j and every agent of type h, then condition (A.58) holds.

We now need to show that if the constrained optimal allocation is first-best optimal, then the no-externality condition (A.58) must hold. Since the allocation is first-best optimal, it must satisfy conditions (A.45) and (A.46), which imply that $\left(\frac{\mu^{\tilde{h}}_s}{\alpha^{\tilde{h}}}\right)$ must be constant across agents, i..e, for each s

$$\frac{\mu^h_s}{\alpha^h} = \frac{\mu^{\tilde{h}}_s}{\alpha^{\tilde{h}}} = \mathit{\Gamma}_s, \forall h, \tilde{h}. \tag{A.59}$$

Using these conditions, we can then show that

$$\begin{aligned}\sum_s \frac{\partial p_{2s}}{\partial \theta_j^h} \sum_{\tilde{h}} \mu_s^{\tilde{h}} \tau_{z2s}^{\tilde{h}} &= \sum_s \frac{\partial p_{2s}}{\partial \theta_j^h} \sum_{\tilde{h}} \left(\frac{\mu_s^{\tilde{h}}}{\alpha^{\tilde{h}}} \right) \alpha^{\tilde{h}} \tau_{z2s}^{\tilde{h}} \\ &= \sum_s \frac{\partial p_{2s}}{\partial \theta_j^h} \sum_{\tilde{h}} \mathit{\Gamma}_s \alpha^{\tilde{h}} \tau_{z2s}^{\tilde{h}} = \sum_s \frac{\partial p_{2s}}{\partial \theta_j^h} \mathit{\Gamma}_s \sum_{\tilde{h}} \alpha^{\tilde{h}} \tau_{z2s}^{\tilde{h}} = 0,\end{aligned}$$

where the last equation results from the resource constraints for τ_{z2s}^{h}. This proves that the no-externality condition (A.58) holds. To sum up, we prove that there is no externality *if* the constrained optimal allocation is first-best optimal or the spot price is independent of security positions, i.e., $\frac{\partial p_{2s}}{\partial \theta_j^h} = 0$ for every state s, every security j and every agent of type h.

(ii) ($\Rightarrow$) Unfortunately, we cannot generally prove the reversed statement but, as shown in Geanakoplos and Polemarchakis (1986), it is true generically (it is true except for some unlikely cases). The key idea is that the indirect price effects could be canceling each other out only if the equilibrium allocation is first-best optimal in most cases. But this does not happen generally.

$\square$

G.3 Remedy for the Externality in the Incomplete Markets Economy

With potentially incomplete security markets, a given security traded at $t = 0$ has implications in general for most if not all spot prices at $t = 1$. This is one source of externalities. To internalize these externalities, we thus need rights to trade indexed by the vector of spot prices $\mathbf{p} = [p_s]_s$ over all states s. That is, what we now term $\mathbf{p}$-exchanges must naturally deal with S spot markets as a bundle. As a result, all objects are indexed by vector $\mathbf{p}$. This is where there is a subtle difference from the saving economy.

As already noted, a key step is to define type h's rights to trade $\Delta_s^h(\mathbf{p})$ in the spot markets at prices p_{2s}. Type h chooses both the amount of these rights to trade, that is, the trades at p_{2s}, and the vector $\mathbf{p} = (p_{2s})_{s=1}^S$ itself. To repeat, there is in effect a market place exchange indexed by prices $\mathbf{p}$ where security trades will be entered into and priced at $t = 1$ and where goods will be exchanged in spot markets at state s at the same price p_{2s}. For these rights to have meaning these exchanges must be segregated and choices of the agents must be exclusive.

In more detail, the quantity of rights purchased over states $s = 1, \ldots, S$ is a vector $\mathbf{\Delta}^h(\mathbf{p}) = \left[\Delta_s^h(\mathbf{p})\right]_{s=1}^S$. In a particular state s, $\Delta_s^h(\mathbf{p})$ is defined as the standard excess demand for the numeraire, good w, of an agent type h in spot markets in state s. Namely, $\Delta_s^h(p_{21}, \ldots, p_{2s}, \ldots, p_{2S}) = \tau_{w2s}^{h*}(\mathbf{e}_{2s}^h, \boldsymbol{\theta}^h, p_{2s})$ is the solution to the state s utility maximization problem at price p_{2s}:

$$\left(\tau_{w2s}^{h*}\left(\mathbf{e}_{2s}^h, \boldsymbol{\theta}^h, p_{2s}\right), \tau_{z2s}^{h*}\left(\mathbf{e}_{2s}^h, \boldsymbol{\theta}^h, p_{2s}\right)\right) = \underset{\tau_{w2s}^h, \tau_{z2s}^h}{\operatorname{argmax}}\, u^h\left(e_{w2s}^h + \sum_{j=1}^{J} D_{js}\theta_j^h + \tau_{w2s}^h, e_{z2s}^h + \tau_{z2s}^h\right) \tag{A.60}$$

subject to the spot-budget constraints (A.37). This choice of rights could be costly to buy or alternatively it could generate revenue. In particular, let $P_\Delta(\mathbf{p}, s)$ denote the market price of rights to spot trade in exchange $\mathbf{p}$ in state s, with components as desired spot prices and the vector running over all states s. Then the net cost is this per unit price times the quantity of rights $\Delta_s^h(\mathbf{p})$ just defined. Namely, $\sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})$ if $\mathbf{p}$ is chosen. Let $\delta^h(\mathbf{p})$ be the indicator variable which is equal to 1 for the chosen $\mathbf{p}$ and is zero otherwise. Thus the budget term will be $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})$.

The key tie-in is that security trades $\boldsymbol{\theta}^h(\mathbf{p})$ are also tied to the choice of exchange $\mathbf{p}$. That is, let $Q_j(\mathbf{p})$ denote the price of security j traded in exchange $\mathbf{p}$. This then has a net cost in the budget $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_j Q_j(\mathbf{p})\theta_j^h(\mathbf{p})$. Both costs of rights to trade and the tie-in to securities are subtracted from the value of endowments at $t = 1$ leaving consumption as a residual. The entire budget is the following

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{w1}^h + p_1 c_{z1}^h + \sum_j Q_j(\mathbf{p})\theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})\right] \leq e_{w1}^h + p_1 e_{z1}^h. \tag{A.61}$$

Finally, the spot prices $\mathbf{p}$ and security prices $Q_j(\mathbf{p})$ will have to be such as to attain market clearing in rights to trade:

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}, \tag{A.62}$$

and market clearing in securities

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}. \tag{A.63}$$

Also the spot market in each state s in exchange $\mathbf{p}$ must be cleared, consistent with the agent types who have chose to trade there, validating their choice of $\mathbf{p}$.

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau_{\ell 2s}^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = w, z. \tag{A.64}$$

Note that due to the maximization of (A.60) subject to (A.37) that the chosen τ_{w2s}^h at $\mathbf{p}$ that appear in (A.64) will be the τ_{w2s}^{h*} in (A.60), in turn equal to the rights $\Delta_s^h(\mathbf{p})$ purchased. Finally, equations (A.62) can be satisfied trivially for inactive exchanges where $\delta^h(\mathbf{p}) = 0$ for all h.

In general, spot prices p_{2s} can be a complex mapping from pre-trade endowments and security holdings. There are few a priori restrictions on individual and aggregate excess demands. But, conceptually, for the individual this does not matter, as all she cares about are the chosen prices at which she will trade and the associated implication for rights, security, and spot trades. Finding an equilibrium is the economist's problem, not the agent's problem.

Let $\mathbf{x}^h(\mathbf{p}) = \left(\mathbf{c}_1^h, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right)$ denote a typical bundle or allocation for an agent type h, where again $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta_s^h(p_{2s})\right]_s$. If $\delta^h(\mathbf{p}) = 0$, then the rest of the $\mathbf{p}$ contingent choices need not be specified, as agent h is choosing not to trade at $\mathbf{p}$.

Definition A.3. A competitive equilibrium with segregated exchanges indexed by $\mathbf{p}$ is a specification of allocation $\left[\mathbf{x}^h(\mathbf{p})\right]_{h,\mathbf{p}} \equiv \left[\mathbf{c}_1^h, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{h,\mathbf{p}}$ and prices $(p_1, \mathbf{Q}, \mathbf{p}, \mathbf{P}_\Delta)$ such that

(i) for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\left[x^h(\mathbf{p})\right]_{\mathbf{p}}} \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u\left(c_{w1}^h, c_{z1}^h\right) + \sum_s \pi_s u \left(e_{w2s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{w2s}^h(\mathbf{p}), e_{z2s}^h + \tau_{z2s}^h(\mathbf{p}) \right) \right]$$

subject to the budget constraints in the first period

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{w1}^h + p_1 c_{z1}^h + \sum_j Q_j(\mathbf{p})\,\theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\,\Delta_s^h(\mathbf{p}) \right] \leq e_{w1}^h + p_1 e_{z1}^h,$$

and the spot-budget constraint in state s

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[\tau_{w2s}^h(\mathbf{p}) + p_{2s}\tau_{z2s}^h(\mathbf{p}) \right] = 0, \forall s,$$

(ii) markets clear for good $\ell = w, z$ in $t = 1$, for securities j paying good w, for good $\ell = w, z$ in state s, and for rights to trade in exchange $\mathbf{p}$ for state s, respectively,

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h,$$

$$\sum_h \alpha^h c_{z1}^h = \sum_h \alpha^h e_{z1}^h,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p},$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau_{\ell 2s}^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = w, z,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}.$$

G.4 Illustrative Example for the Exogenous Incomplete Markets Economy

This numerical example is a deviation from the saving economy in Section 2 of the main text. This example has only one physical good in the first period where there are two goods in the main text. We are featuring idiosyncratic shocks in the second period, and no insurance over those, the source of incomplete markets.

Consider an economy with two periods, $t = 1, 2$. There are two states $s = 1, 2$ in the second period. Let the probability of state s is $\pi_s = \frac{1}{2}$. For simplicity, there is only one good in the first period $t = 1$, while there are two goods $\ell = w, z$ in the second period $t = 2$. Each unit of storage of the single good in the first period becomes one unit of good z in the second period regardless of the state (no shocks on the return to storage). Notationally, k units of storage in $t = 1$ gives k units of good z in every state in the second period.

There are two types of agents, $h = a, b$, both of which have an identical logarithmic utility function

$$u^h(c) = \ln c, \tag{A.65}$$

which is homothetic. As a result, the equilibrium spot price p_{2s} is determined by the ratio of the commodity aggregates, as in the collateral economy. Note that the externality exists in this economy due to the interaction between the incompleteness of the markets and storage, as first period decisions impact the second period price. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor is $\beta = 1$.

The endowment profiles of the agents are shown in Table A.1 below. Note that all risk is idiosyncratic, with type a relatively well endowed in state $s = 1$ and vice versa for type b. Note also that the symmetry of the endowments and the homogeneity of the preferences imply that an equilibrium allocation is symmetric. In addition, the endowment is structured in such a way that both types would like to save ($k^h > 0$).

Table A.1: Endowment profiles of the agents.

	e^h_{w1}	e^h_{w21}	e^h_{z21}	e^h_{w22}	e^h_{z22}
$h = a$	10	5	5	1	1
$h = b$	10	1	1	5	5

Definition A.4. A competitive equilibrium is a specification of prices (p_1, p_{21}, p_{22}), and an allocation $\left(c^h_{w1}, c^h_{\ell 2s}, k^h\right)_h$ such that

- for any agent type h as a price taker, $\left(c_{w1}^h, c_{\ell 2s}^h, k^h\right)$ solves

$$\max_{c_{w1}^h, c_{\ell 2s}^h, k^h} u\left(c_{w1}^h\right) + \beta \sum_s \pi_s \left[u\left(c_{w2s}^h\right) + u\left(c_{z2s}^h\right)\right] \tag{A.66}$$

subject to the budget constraints in period $t = 1$ and state $s = 1, 2$, respectively

$$c_{w1}^h + k^h = e_{w1}^h \tag{A.67}$$

$$c_{w2s}^h + p_{2s} c_{z2s}^h = e_{w2s}^h + p_{2s}\left[e_{z2s}^h + k^h\right], \forall s. \tag{A.68}$$

- markets clear for good w at $t = 1$:

$$\sum_h \alpha^h \left(c_{w1}^h + k^h\right) = \sum_h \alpha^h e_{w1}^h, \tag{A.69}$$

for good z in each state $s = 1, 2$:

$$\sum_h \alpha^h c_{w2s}^h = \sum_h \alpha^h e_{w2s}^h, \tag{A.70}$$

and for good z in each state $s = 1, 2$:

$$\sum_h \alpha^h c_{z2s}^h = \sum_h \alpha^h \left(c_{z2s}^h + k^h\right), \tag{A.71}$$

With a homogeneous homothetic utility function, the spot price in each state s is determined by the ratio of the aggregate resources:

$$p_{2s} = \frac{\sum_h \alpha^h e_{w2s}^h}{\sum_h \alpha^h \left(e_{z2s}^h + k^h\right)}, \tag{A.72}$$

with the equilibrium spot price function $p_{2s} = p_{2s}\left(\frac{\sum_h \alpha^h e_{w2s}^h}{\sum_h \alpha^h \left(e_{z2s}^h + k^h\right)}\right)$.

With symmetry, the first best allocation ("fb") with state contingent transfers has both agents save $k^{fb} = 3.5$, which implies that the equilibrium spot price of good z, $p_{2s}^{fb} = 0.4615$ in all states. On the other hand, the competitive equilibrium with externality for this environment is numerically solved and presented in columns 1 and 2 of Table A.2. With externality, each agent type saves more $k^{ex} = 4.3077$ to try to cover some of the idiosyncratic risk, which leads to a lower equilibrium spot price $p_{2s}^{ex} = 0.4105$ in all states.

We now solve for the competitive equilibrium with rights to trade in segregated exchanges using the following Pareto program with equal Pareto weights, $\lambda^h = \frac{1}{2}$ for all $h = a, b$. Let $\mathbf{x}^h \equiv \left[x^h\left(c_{w1}, k, \mathbf{p}, \boldsymbol{\Delta}\right)\right]$ be a typical lottery for an agent of type h. We again impose on the grid of the lottery that a positive mass can be put on a grid with the following property only: for each bundle $(c_{w1}, k, \mathbf{p}, \boldsymbol{\Delta})$ and each agent type h,

$$\Delta_s(\mathbf{p}) = p_{2s}\left(e_{z2s}^h + k\right) - e_{w2s}^h, \tag{A.73}$$

Table A.2: Competitive equilibrium with incomplete markets and the corresponding constrained optimal solutions.

	equilibrium with externality		constrained optimality		equilibrium with rights to trade	
	$h=a$	$h=b$	$h=a$	$h=b$	$h=a$	$h=b$
c_{w1}^h	5.6923	5.6923	5.9767	5.9767	5.9767	5.9767
k^h	4.3077	4.3077	4.0233	4.0233	4.0233	4.0233
c_{w21}^h	4.3077	10.7437	4.4272	10.3643	4.4272	10.3643
c_{w22}^h	1.5895	3.8718	1.5728	3.6822	1.5728	3.6822
c_{z21}^h	10.7437	4.3077	10.3643	4.4272	10.3643	4.4272
c_{z22}^h	3.8718	1.5895	3.6822	1.5728	3.6822	1.5728
p_{2s}	0.4105	0.4105	0.4272	0.4272	0.4272	0.4272
Expected Utility	4.5768	4.5768	4.5791	4.5791	4.5791	4.5791

The Pareto program with rights to trade is

$$\max_{\mathbf{x}^h} \sum_h \lambda^h \alpha^h \sum_{c_{w1},k,\mathbf{p},\boldsymbol{\Delta}} x^h\left(c_{w1},k,\mathbf{p},\boldsymbol{\Delta}\right)\left\{u\left(c_{w1}\right)+\beta\sum_{s=1}^{S}\pi_s V\left(e_{w2s,}^h, e_{z2s}^h+k, p_{2s}\right)\right\} \tag{A.74}$$

subject to

$$\sum_{c_{w1},k,\mathbf{p},\boldsymbol{\Delta}} x^h\left(c_{w1},k,\mathbf{p},\boldsymbol{\Delta}\right)=1, \forall h; \tag{A.75}$$

$$\sum_h \alpha^h \sum_{c_{w1},k,\mathbf{p},\boldsymbol{\Delta}} x^h\left(c_{w1},k,\mathbf{p},\boldsymbol{\Delta}\right)\left\{c_{w1}+k-e_{w1}^h\right\}=0; \tag{A.76}$$

$$\sum_h \alpha^h \sum_{c_{w1},k,\boldsymbol{\Delta}} x^h\left(c_{w1},k,\mathbf{p},\boldsymbol{\Delta}\right)\Delta_s=0, \forall s; \mathbf{p} \tag{A.77}$$

where the indirect utility function for an agent type h in state s is defined by

$$V\left(e_{w2s,}^h, e_{z2s}^h+k, p_{2s}\right)=\max_{\tau_{w2s},\tau_{z2s}} u\left(e_{w2s}^h+\tau_{w2s}\right)+u\left(e_{z2s}^h+k+\tau_{z2s}\right) \tag{A.78}$$

subject to the spot market budget constraint

$$\tau_{w2s}+p_{2s}\tau_{z2s}=0 \tag{A.79}$$

According to the second welfare theorem, the constrained optimal allocation can be decentralized as the competitive equilibrium with rights to trade in segregated exchanges. In fact, we numerically solve the linear programming problem above in Matlab and then recover the equilibrium prices using part of the proof of the theorem. The equilibrium outcome is presented in the columns 5 and 6 of Table A.2.

The equilibrium allocation with rights to trade in segregated exchanges has only *one active spot market* $\mathbf{p}^{op}$ with $p_{2s}^{op} = 0.4272$ for all $s = 1, 2$. The equilibrium savings here is lower than the one in competitive equilibrium with the externality, i.e., $k^{op} = 4.0233$ (but still higher than in the first best). Table A.3 presents equilibrium prices/fees of rights to trade in exchange vector $\mathbf{p} = (p_{21}, p_{22})$ in each state $s = 1, 2$, the $P_{\Delta}(\mathbf{p}, s)$ with argument ranging over $\mathbf{p}$ and s that is, including over inactive exchanges. Note that the prices of the rights to trade with the spot price p_{2s} in different exchanges are clearly different, i.e., $P_{\Delta}((p_{21}, p_{22}), 1) \neq P_{\Delta}((p_{21}, \tilde{p}_{22}), 1)$ when $p_{22} \neq \tilde{p}_{22}$. Note that here the vector is different in the second component, yet this makes the rights price for trading in the first state different.

Table A.3: The equilibrium price of the rights to trade in exchange $\mathbf{p} = (p_{21}, p_{22})$ in each state s, $P_{\Delta}(\mathbf{p}, s)$. The only active exchange is $\mathbf{p} = (0.42715, 0.42715)$, which is presented in bold.

$\mathbf{p}$		$P_{\Delta}(\mathbf{p}, 1)$	$P_{\Delta}(\mathbf{p}, 2)$
p_{21}	p_{22}		
0.41609	0.41609	0.07456	0.07456
0.41609	0.42715	0.07807	0.09456
0.42715	0.41609	0.09455	0.07806
0.42715	**0.42715**	**0.13342**	**0.13342**
0.42715	0.46154	0.17094	0.23213
0.46154	0.42715	0.23213	0.17094

In this equilibrium, each agent type buys/sells the rights to trade in an exchange $\mathbf{p}^{op} = (0.42715, 0.42715)$. Due to symmetry, an agent type $h = a$ sells the rights $\Delta^a(\mathbf{p}^{op}, 1) = -1.1457$ in state $s = 1$ and the numeraire good w, and then buys good z in $s = 1$. Agent type $h = a$ buys the same amount of good w in state $s = 2$, $\Delta^a(\mathbf{p}^{op}, 2) = 1.1457$, and hence agent type a sells good z at $s = 2$. This is crucial as savings of type a is motivated by the shortfall of type a's endowment in state $s = 2$. That is, this is where the exposure to idiosyncratic risk for agent type a is doing damage, bringing too much good z to the second period, creating the externality. The markets for rights to trade in good w can remove the externality through its marginal impact on the decision to save of each agent type. In effect, each pays a "tax" when selling good z and buying good w in the state which motivated the saving. The situation is reversed for agent type b in terms of the ordering over goods and states, but the same in terms of saving. In total, the net trade in the rights to trade will be zero in net value for both agent types at $t = 1$. What agent type a buys agent type b sells and vice versa, and each trade has the same value. That is, each agent does actively trade rights but with no implication for wealth at $t = 1$. The key is the marginal impact of each rights to trade on saving decisions. Note that the optimal lottery is degenerated; that is, all agents of each type choose the only one exchange as the probability measure at the optimal allocation $x^h(\mathbf{w}^{op}) = 1.000$ for all $h = a, b$, as shown in Table A.4. In fact, it is identical to the solution of the planning problem below.

Constrained Planning Problem

The constrained optimal allocation can be computed by maximizing the expected utility of a representative consumer, exploiting the symmetry, subject to spot market constraints with price p_{2s} replaced by the appropriate clearing ratio of commodities. The solution to the following planning problem below is confirmed to be the solution to the equilibrium with rights. For notational purposes, we drop the type-specific index h.

$$\max_{c_{w1}, c_{w2s}, c_{z2s}, k} u(c_{w1}) + \beta \sum_s \pi_s \left[u(c_{w2s}) + u(c_{z2s})\right] \tag{A.80}$$

subject to

$$c_{w1} + k = e_{w1}, \tag{A.81}$$

$$c_{w2s} + \left(\frac{E_{w2s}}{E_{z2s} + k}\right) c_{z2s} = e_{w2s} + \left(\frac{E_{w2s}}{E_{z2s} + k}\right)(e_{z2s} + k), \forall s = 1, 2, \tag{A.82}$$

where the aggregate of good $\ell = w, z$ in state s is $E_{\ell 2s} = \sum_h \alpha^h e^h_{\ell 2s}$. We here write the spot budget constraints in terms of the ratio of commodity aggregates. The optimal allocation is numerically solved and presented in columns 3 and 4 of Table A.2. In fact, the necessary condition for the optimality can be formulated as a cubic function, which leads to the only one feasible (as a real number) solution.

G.5 Markets for Rights to Trade Do Not Complete the Securities Markets

Of course, one might wonder if our method solves the externality problem by simply completing the markets? By allowing agents to choose markets with pre-specified spot prices in each state s, we effectively create state-contingent transfers of wealth at least to some degree. But is it enough to achieve the first best allocation? The answer is generally, no. Exogenous incomplete markets

Table A.4: The equilibrium allocation.

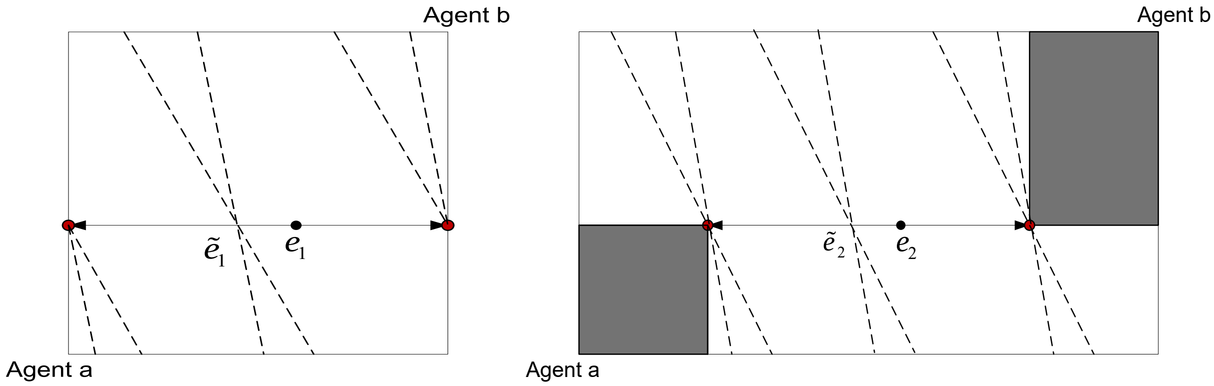
	$h = a$	$h = b$
c_{w1}^{op}	5.9767	5.9767
k^{op}	4.0233	4.0233
p_{21}^{op}	0.42715	0.42715
p_{22}^{op}	0.42715	0.42715
Δ_1^{op}	-1.1457	1.1457
Δ_2^{op}	1.1457	-1.1457
x^h	1.000	1.000
$P_{\Delta}\left(\mathbf{p}^{op}, 1\right)$	0.13342	0.13342
$P_{\Delta}\left(\mathbf{p}^{op}, 2\right)$	0.13342	0.13342

and the positivity of spot prices still restrict how much wealth transfers we can make in each state. Technically, the feasible set with incomplete markets and rights to trade is generically a strict subset of the feasible set with the complete markets.

See Figure 1 for an illustrative example. This example assumes the stereotypical debt contract, a bond that pays the same amount of good w in each two states. However, in state $s = 2$, there is more of good w and good z overall. Then, no matter what the price ratio p_{2s} in state $s = 2$, certain regions cannot be reached. The main point is that the scarcity in state $s = 1$ can affect the feasibility in state $s = 2$ because the markets are incomplete.

H A General Model with Price Externalities and Its Prototypical Economies

This section formulates a general model that captures key features regarding price externalities of 6 prototypical economies including a collateral economy (Kilenthong and Townsend, 2014b), an exogenous incomplete markets economy (Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986), a moral hazard with retrading economy (Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011), a liquidity constrained economy (Hart and Zingales, 2013), a fire sales economy (Lorenzoni, 2008), and a hidden information with retrading economy (Diamond and Dybvig, 1983; Jacklin, 1987). Each subsection presents a key ingredient of the model along with the relevant part of each prototypical economy. In order to map those models into the unified general



(a) A feasible set in state $s = 1$.

(b) A feasible set in state $s = 2$. The shaded areas are not feasible.

Figure 1: The set of feasible allocations when markets are incomplete markets with rights to trade is generically a strict subset of the feasible set with the complete markets. The figures (a) and (b) assume the stereotypical debt contract, a bond that pays one unit of good w in each two states, $s = 1$ and $s = 2$. However, in state $s = 2$, due to variation across states in underlying endowments, there is more of good w overall, hence the larger Edgeworth box. Endowments plus pre-trade positions in the bond, $\tilde{\mathbf{e}}_s^h \equiv \left(e_{w2s}^h + \theta^h, e_{z2s}^h\right)$, determine the starting point before spot trade, a point on the horizontal line, which must be feasible. Then, no matter what the price ratio p_{2s} in state $s = 2$, from very steep to very flat budget line, certain regions of the box $s = 2$ cannot be reached. The scarcity in state $s = 1$ can affect the feasibility in state $s = 2$ because the markets are incomplete, despite markets for rights.

framework and keep the notation for each model as close as possible to the original one, as just cited, the notation in this section is slightly different from the main text and earlier sections in that all indices over commodities and agent types here will be all numbered, and in some cases the first period is $t = 0$.

H.1 Basic Ingredients: Commodity Space, Preferences, Endowments, and Technology

There are L commodities. These can be basic underlying commodities and also date and/or state contingent where the date and/or state are public. In order to incorporate private information problems into this framework, we also allow a subset of commodities to be contingent on recommended but unobserved actions or on reported but unobserved states. For actions, let a be the recommended action and (with the incentive compatibility constraints in place) the actually taken action, and a' be potentially deviating action. For privately observed states, let a be the reported state and (with incentive compatibility constraints in place) the actual state, and let a' be some potentially counterfactual report. Let $A \in \mathbb{R}_+$ be the set of possible actions/states, i.e., $a, a' \in A$.

There is a continuum of agents of measure one. The agents are divided into H (ex-ante) types, each of which is indexed by $h = 1, 2, \cdots, H$. Each type h consists of $\alpha^h \in [0, 1]$ fraction of the population such that $\sum_h \alpha^h = 1$. In addition, this model allows for ex-post diversity denoted by ex-post (either observable or unobservable) type $\omega \in \Omega$. More formally, let $\zeta^h(\omega)$ be the fraction of agents of type h whose ex-post type is ω. An ex-post type ω may depend on an observed output, an unobserved action, and/or unobserved state of nature as well.

Each agent type h is endowed with an endowment $\mathbf{e}^h \in \mathbb{R}_+^L$. Note that $\mathbf{c}^h$ and $\mathbf{e}^h$ lie in the L-dimensional commodity space. The preferences of an agent of type h are represented by the utility function $U^h(\mathbf{c}^h)$, where $\mathbf{c}^h \in \mathbb{R}^L$ is the consumption allocation for an agent of type h.

Each agent of type h has an access to a production technology defined implicitly by

$$F^h\left(\mathbf{y}^h\right) \geq 0, \tag{A.83}$$

where $\mathbf{y}^h \in \mathbb{R}^L$ is the vector of its inputs and outputs in commodity space L. This production technology is generally a multidimensional vector of constraints with dimension O, i.e., $F^h(\mathbf{y}^h) \equiv \left[F_o^h(\mathbf{y}^h)\right]_{o=1}^O$.

H.1.1 Basic Ingredients for the Collateral Economy

This is a two-period economy, $t = 0, 1$. There are a finite S states of nature in the second period $t = 1$, i.e., $s = 1, 2, ..., S$. Let $0 < \pi_s \leq 1$ be the objective and commonly assessed probability of state s occurring, where $\sum_s \pi_s = 1$. There are two goods, called good 1 and good 2 in each period. These two goods can be traded in each date and in each state, and we refer to those markets as spot markets with good 1 as the numeraire good in every date and state. Thus, there are $L = 2(1 + S)$ commodities. There is no unobserved action or privately observed state.

Each agent of type $h = 1, 2, \ldots, H$ is endowed with good 1 and good 2, $\mathbf{e}_0^h = \left(e_{10}^h, e_{20}^h\right)$ in the first period and $\mathbf{e}_s^h = \left(e_{1s}^h, e_{2s}^h\right)$, in each state $s = 1, \cdots, S$. Let $\mathbf{e}^h = \left(\mathbf{e}_0^h, \cdots, \mathbf{e}_S^h\right)$ be the endowment profile of an agent of type h over the first period and all states s in the second period, respectively. There is no ex-post diversity in this economy, and therefore we simply omit all related notation.

The preferences of an agent of type h are represented by the utility function $u^h\left(c_1^h, c_2^h\right)$, and the discounted expected utility of h is defined by:

$$U^h\left(\mathbf{c}^h\right) \equiv u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u^h\left(c_{1s}^h, c_{2s}^h\right), \tag{A.84}$$

where β is the discount factor.

Good 1 is consumable but cannot be stored from $t = 0$ to $t = 1$ (is completely perishable), while good 2 is consumable and storable. The good 2 that is stored can be collateralizable, i.e., can serve as collateral to back promises. Henceforth, good 2 and the collateral good will be used interchangeably. Each unit of good 2 stored (as input) will become R_s units of good 2 in state s. As a result, the production function in our general framework can be written as follows:

$$F_s^h(\mathbf{y}^h) = -y_{2s}^h - R_s y_{20}^h \geq 0, \text{for } s = 1, \ldots, S, \tag{A.85}$$

where $y_{20}^h \in \mathbb{R}_-$ and $y_{2s}^h \in \mathbb{R}_+, s = 1, 2, \ldots, S$ are inputs and outputs, respectively. We use the standard convention under which an input must be non-positive and an output must be non-negative. This economy has $O = S$ production functions.

H.1.2 Basic Ingredients for the Exogenous Incomplete Markets Economy

Consider an economy with two periods, $t = 0, 1$. There are S possible states of nature in the second period $t = 1$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s such that $\sum_s \pi_s = 1$. There are 2 goods, labeled good 1 and good 2, in each date and in each state. Thus, there are $L = 2(1 + S)$ commodities. Because the endowment profiles are the same as specified in the collateral economy discussed above, we omit the details in this section for brevity.

The preferences of an agent of type h are represented by the utility function $u^h\left(c_1^h, c_2^h\right)$, and the discounted expected utility of h is defined by:

$$U^h\left(\mathbf{c}^h\right) \equiv u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u^h\left(c_{1s}^h, c_{2s}^h\right), \tag{A.86}$$

where β is the discount factor. There is no ex-post diversity in this economy, and endowments and preferences are known ex-ante, and therefore we simply omit all related notation.

For simplicity, we assume that there is no production. Thus, F_o^h can be suppressed. As a result, there would be no externalities if preferences were identically homothetic, as spot prices are determined by ratio of aggregate endowment only, which no one can influence. So we assume otherwise; that is, preferences are not identically homothetic.

H.1.3 Basic Ingredients for the Moral Hazard with Retrading Economy

There are two physical commodities, labeled as good 1 and good 2, in each states. These commodities can be produced using the sole input, called action, a. Let A be the number of possible actions. As in the literature, the random production technology is given by $f(\mathbf{q}|a)$, which is the probability density function of the output vector of good 1 and good 2, $\mathbf{q} = (q_1, q_2)$, conditional on an action a taken by an agent. In other words, the probability that the realized output will be $\mathbf{q}$ is $f(\mathbf{q}|a)$ when an agent takes an action a. The action that an agent takes is *private information*. Hence, there is a *moral hazard* problem. There is a continuum of ex ante identical agents of mass 1, i.e., no diversity in types so trivially $\alpha^1 = 1$. For simplicity, we assume that each agent is endowed with zero units of both goods.

We will now map this moral hazard economy into our general model with securities trading. Different combinations of outputs $\mathbf{q}$ define (idiosyncratic) states or indexes for contracting purposes. There is no loss of generality to assume that there are a finite Q states, $\mathbf{q} \in Q$. Following the mechanism design literature, an optimal consumption of the two goods under moral hazard depends on realized output $\mathbf{q}$ and recommended action a; that is, $c_1(\mathbf{q}, a)$ and $c_2(\mathbf{q}, a)$. Accordingly, we define commodity using both output/state $\mathbf{q}$ and recommended action a. In particular, for each recommended action a, there are Q states. There are two commodities in each state. In addition, actual action a itself is another commodity. Therefore, there are $L = 2QA + 1$ commodities in this model.

Each agent is endowed with the instantaneous common utility function for the two goods and action, $u(c_1, c_2, a)$. Again, let a be recommended action, and a' be taken (possibly out-of equilibrium) action. The discounted expected utility of an agent who is reported action a but took action a' is defined by:

$$U(\mathbf{c}) = \sum_{\mathbf{q}} \pi(\mathbf{q}|a') u\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a'\right) \tag{A.87}$$

where $\pi(\mathbf{q}|a)$ denote the probability of realizing outputs $\mathbf{q}$ given action a (actually taken), which satisfies the following probability constraint:

$$\sum_{\mathbf{q}} \pi(\mathbf{q}|a) = 1, \forall a. \tag{A.88}$$

Ex-post diversity in this model is determined by actual (ex ante) action and realized (ex post) outputs, i.e., $\omega = (\mathbf{q}, a')$. For generality, let $\delta(a')$ be the fraction of agents who took action a'. Recall that the fraction of agents who realized outputs $\mathbf{q}$ conditional on taking action a' is $f(\mathbf{q}|a')$. As a result, the fractions of agents of ex-post type $(\mathbf{q}, a')$ is $\zeta^1(\mathbf{q}, a') = f(\mathbf{q}|a')\,\delta(a')$.

As in the literature, the probability distribution across outputs/states depend on agent's choice of action a. This dependency is modeled as a general production function F whose input is actual action a and outputs are $\mathbf{q}$:

$$F(\mathbf{q}, a) = f(\mathbf{q}|a) - \pi(\mathbf{q}|a) = 0, \forall \mathbf{q}, a \tag{A.89}$$

In words, different actions will lead to different probability distributions. There are, as in (A.89), $O = QA$ production functions. Combining these production technologies with the probability conditions (A.88) leads to standard probability constraints of production function $f(\mathbf{q}|a)$:

$$\sum_{\mathbf{q}} \pi(\mathbf{q}|a) = 1 \Rightarrow \sum_{\mathbf{q}} f(\mathbf{q}|a) = 1, \forall a. \tag{A.90}$$

H.1.4 Basic Ingredients for the Liquidity Constrained Economy

Consider an economy with four periods, $t = 0, 1, 2, 3$. There are two types of agents, called "doctors" and "builders", each of which consists of $\alpha^h > 0$ for all $h = b, d$ fraction of the population with $\sum_{h=b,d} \alpha^h = 1$. Each agent $h = b, d$ is endowed with $e^h = e$ units of wheat at period $t = 0$. This is a simplified and deterministic version of Hart and Zingales (2013) in which we assume that the doctors will buy building services in period $t = 1$ first, and the builders will buy doctor services later in period $t = 2$. As in the collateral model in section H.1.1, there is no unobserved action or privately observed state, and therefore we simply omit all related notations.

There are two commodities in period $t = 0$, wheat w_0^h, and storage f_0^h, where the latter is formally defined below. There are three commodities in period $t = 1$, storage f_1^h, building services b^d and labor supply of the doctors l^d. Similarly, there are three commodities in period $t = 2$, storage f_2^h, doctor services d^b, and labor supply of the builders l^b. There is one commodity, wheat w_3^h, in the last period $t = 3$. Therefore, there are $L = 9$ commodities in this model.

The preferences of doctors and builders are represented by

$$U^d(\mathbf{c}) = u^d\left(w^d, d^d, b^d, l^d\right) = w_3^d + b^d - \frac{\left(l^d\right)^2}{2}, \tag{A.91}$$

$$U^b(\mathbf{c}) = u^b\left(w^b, d^b, b^b, l^b\right) = w_3^b + d^b - \frac{\left(l^b\right)^2}{2}, \tag{A.92}$$

respectively. Note that doctors do not consume doctor services, and vice versa for builders. We can write the utility function in a more general from as follows:

$$U^h(\mathbf{c}) = u^h\left(w^h, d^h, b^h, l^h\right) = w_3^h + \delta_b^h b^h + \left(1 - \delta_d^h\right) d^h - \frac{\left(l^h\right)^2}{2}, \tag{A.93}$$

where $\delta_b^h = 1$ if $h \neq b$, and zero otherwise.

There are two technologies or assets available in period $t = 0$. First, the collateralizable asset is a storage technology, whose return from $t = 0$ to $t = 3$ is 1 unit of wheat, i.e., saving one unit of wheat in the first period $t = 0$ will return 1 unit of wheat in the last period $t = 3$. In addition, the claim on the output of this technology is transferable, and therefore can be used as private money (or collateral) during periods $t = 1$ and $t = 2$. The second asset is an investment project, whose return from $t = 0$ to $t = 3$ is $\bar{R} > 1$ units of wheat. However, this asset cannot be used as collateral. For simplicity, we consider only a deterministic return case here. Let f_0^h be the amount of wheat stored by an agent type $h = b, d$, and accordingly, the agent type h invests $e - f_0^h$ units of wheat in the investment project.

The production technologies are irreversible; that is, their outputs will be realized in the last period $t = 3$ only. The production function of the storage technology (denoted by subscript "s") for an agent type h is defined as follows:

$$F_s^h\left(f_2^h, y_{31}^h\right) \quad = \quad y_{s3}^h - f_2^h = 0, \forall h = b, d, \tag{A.94}$$

where f_2^h is the number of claims on the storage technology held by the agent type h at the end of period $t = 2$, and y_{s3}^h is the output in unit of wheat in period $t = 3$ received by the agent type h from the storage technology. Similarly, the production function for investment project (denoted by subscript "i") is defined by

$$F_i^h\left(e - f_0^h, y_{i3}^h\right) \quad = \quad y_{i3}^h - \bar{R}\left(e - f_0^h\right) = 0, \forall h, \tag{A.95}$$

where f_0^h is the amount of wheat stored by an agent type h in period $t = 0$, and y_{i3}^h is the output in unit of wheat in period $t = 3$ received by the agent type h from the investment technology .

In addition, the builders and the doctors produce building and doctor services (denoted by subscript "o"), respectively, using the following simple linear technologies:

$$F_o^h\left(y_h^h, l^h\right) \quad = \quad y_h^h - l^h = 0, \forall h = b, d, \tag{A.96}$$

which use labor as the only input. For notational convenience, we also set

$$F_0^h\left(y_{-h}^h\right) = y_{-h}^h = 0, \forall h = b, d, \tag{A.97}$$

where $y_{-h}^h = y_b^d, y_d^b$ denote building services produced by doctors and vice versa. To sum up, there are $O = 4$ production functions.

H.1.5 Basic Ingredients for the Fire Sales Economy

Consider an economy with three periods, $t = 0, 1, 2$. There are two states, $s = 1, 2$, realized in period $t = 1$, with probability π_1 and π_2, respectively. We use histories of these states to define states in period $t = 2$; that is, if the state $s = 1$ is realized in period $t = 1$, then the state in period $t = 2$ will automatically be $s = 1$. Therefore, there are two states $s = 1, 2$ in the last period $t = 2$. There are two types of agents, called "consumers" and "entrepreneurs", each of which consists of equal mass. A consumer receives an endowment of e units of consumption goods in each period while an entrepreneur is endowed with n units of consumption goods in the first period $t = 0$ only.

There are $L = 8$ commodities in this model, i.e., two physical goods, namely consumption and capital goods, in period $t = 0$, two physical goods, namely consumption and capital goods, at each state $s = 1, 2$ in period $t = 1$, and one physical good, namely consumption good, at each state $s = 1, 2$ in period $t = 2$.

The preferences of a consumer is represented by

$$U^c(\mathbf{c}) = E\left[c_0^c + c_1^c + c_2^c\right] = c_0^c + \sum_{s=1,2} \pi_s \left(c_{1s}^c + c_{2s}^c\right), \tag{A.98}$$

where superscript "c" stands for consumer, c_0^c is the consumer's consumption in period 0, and c_{ts}^c is the consumer's consumption at state $s = 1, 2$ in period $t = 1, 2$. The preferences of an entrepreneur is represented by

$$U^e(\mathbf{c}) = E\left[c_2^e\right] = \sum_{s=1,2} \pi_s c_{2s}^e, \tag{A.99}$$

where superscript "e" stands for entrepreneur, c_{2s}^e is the entrepreneur's consumption at state $s = 1, 2$ in period $t = 2$.

The following production functions, and inputs/outputs are generally written as $F_{\ell ts}^h$, and $y_{\ell ts}^h$, where $h = c, e$ denotes an agent type, $\ell = c, i, k, n, o, p, r$ denotes an input/output type of commodities ("c" stands for consumption goods, "i" stands for input for new capital production, "k" stands for capital input, "n" stands for new capital, "o" stands for old capital, "r" stands for repairing input for capital maintenance) or a constraint type ("p" stands for weakly positive constraints), $t = 0, 1, 2$ denotes a period, and $s = 0, 1, 2$ denotes a state with state $s = 0$ for period $t = 0$.

Each entrepreneur can turn a unit of consumption good into a unit of (new) capital good at any period and any state of nature. This constitutes the first set of production functions:

$$F_{n00}^e\left(y_{n00}^e, y_{i00}^e\right) = y_{n00}^e + y_{i00}^e = 0, \tag{A.100}$$

$$F_{n1s}^e\left(y_{n1s}^e, y_{i1s}^e\right) = y_{n1s}^e + y_{i1s}^e = 0, \forall s = 1, 2, \tag{A.101}$$

where $y_{n00}^e \in \mathbb{R}_+$ ($y_{i00}^e \in \mathbb{R}_-$) and $y_{n1s}^e \in \mathbb{R}_+$ ($y_{i1s}^e \in \mathbb{R}_-$) are the outputs in unit of capital goods (inputs in unit of consumption goods) in period $t = 0$, and at state s in period $t = 1$, respectively. These specifications with profit maximization limit the price of capital not to be larger than 1 at any point in time. On the other hand, the capital investment is irreversible; that is, it is not feasible to directly turn a capital good into consumption goods. This irreversibility leads to fire sales, which can cause the price of capital to be significantly below one.

In addition, each entrepreneur has access to an entrepreneurial production technology, which transforms y^e_{n00} units of the capital goods in period $t = 0$ into $a_s y^e_{n00}$ units of the consumption goods in period $t = 1$, where $s = 1, 2$ is the aggregate state. This technology can be represented by the following production function:

$$F^e_{c1s}\left(y^e_{c1s}, y^e_{n00}\right) \quad = \quad y^e_{c1s} - a_s y^e_{n00} = 0, \forall s = 1, 2, \tag{A.102}$$

where y^e_{c1s} is the output in unit of consumption goods at state $s = 1, 2$ in period $t = 1$.

The capital must be repaired at the cost $\gamma > 0$ units of consumption goods at state $s = 1, 2$ in period $t = 1$ per unit of capital chosen to be repaired. Non-repaired part will be fully depreciated. This maintenance technology can be represented by the following production function:

$$F^e_{r1s}\left(y^e_{o1s}, y^e_{r1s}\right) \quad = \quad \gamma y^e_{o1s} + y^e_{r1s} = 0, \forall s = 1, 2, \tag{A.103}$$

where $y^e_{o1s} \in \mathbb{R}_+$ is the output in unit of (old) capital goods at state s in period $t = 1$ from this maintenance process, and $y^e_{r1s} \in \mathbb{R}_-$ is the input in unit of consumption goods for the maintenance process. The production technology also requires that old repaired capital cannot be larger (in absolute value) than the original capital from period $t = 0$, i.e.,

$$F^e_{p1s}\left(y^e_{o1s}, y^e_{r1s}\right) \quad = \quad y^e_{o1s} + y^e_{r1s} \geq 0, \forall s = 1, 2. \tag{A.104}$$

Further, an entrepreneur can use all capital available in period $t = 1$, y^e_{k1s}, to produce Ay^e_{k1s} units of consumption goods in period $t = 2$, with $A > 1$. This technology can be represented by the following production function:

$$F^e_{c2s}\left(y^e_{c2s}, y^e_{k1s}\right) \quad = \quad y^e_{c2s} - Ay^e_{k1s} = 0, \forall s = 1, 2, \tag{A.105}$$

where y^e_{c2s} is the output in unit of consumption goods at state $s = 1, 2$ in period $t = 1$.

Each consumer owns a traditional production technology, which produces consumption goods in period $t = 2$ using capital goods in period $t = 1$, y^c_{k1s}, as the input. The traditional technology is represented by the production function $f\left(y^c_{k1s}\right)$, which is assumed to be increasing, strictly concave, twice differentiable, and satisfies the following properties $f(0) = 0, f'(0) = 1, f'\left(y^c_{1s}\right) \geq \bar{q}$. Strict concavity and $f'(0) = 1$ assumptions imply that consumers would not produced (new) capital using technology (A.100)-(A.101) even if they were be able to do so. They will own capital only when there is fire sales, under which the price of capital wold be below one. The capital good is fully depreciated at the end of the last period $t = 2$. This traditional technology can be represented by the following production function:

$$F^c_{c2s}\left(y^c_{c2s}, y^c_{k1s}\right) = y^c_{c2s} - f\left(y^c_{k1s}\right) = 0, \forall s = 1, 2, \tag{A.106}$$

where y^c_{c2s} is the output in unit of consumption goods in period $t = 2$.

To sum up, there are $O = 13$ production functions in this model.

H.1.6 Basic Ingredients for the Hidden Information with Retrading Economy

This is an economy with unobserved states or preference/liquidity shocks and retrading possibilities (e.g., Allen and Gale, 2004; Diamond and Dybvig, 1983; Farhi et al., 2009; Jacklin, 1987). Similar to the moral hazard problem, if there were no retrading possibility, then the Prescott-Townsend equilibria would have been equivalent to Pareto optima. However, this liquidity problem features externalities when agents can trade in spot/private markets ex-post creating the interaction of binding incentive constraints and the spot prices. As in Prescott and Townsend (1984b) and Farhi et al. (2009), we focus only on incentive compatible allocations (rather than sequential service constraints and no bank runs).

There is a continuum of ex-ante identical agents with total mass 1, i.e., no diversity in types so trivially $\alpha^1 = 1$. There are three periods, $t = 0, 1, 2$. There is one physical commodity in each period $t = 1, 2$. Each agent is endowed with e units of the good in the contracting period, $t = 0$, and this will be an input into production functions.

Let η be a ex-post preference/liquidity shock which defines a (idiosyncratic) state in this model. There is no loss of generality to assume that there are a finite Q states, $\eta \in Q$. The shock/state is drawn at $t = 1$ with $\pi(\eta)$ as the probability that an agent will receive η shock such that

$$\sum_{\eta=1}^{Q} \pi(\eta) = 1. \tag{A.107}$$

In this sense there is ex-post diversity. Henceforth, we represent an ex-post type of an agent by his shock η. The fraction of agents of ex-post type $\omega = \eta$ is $\zeta^1(\omega) = \pi(\eta)$. With a continuum of agents, we also interpret $\pi(\eta)$ as the fraction of agents receiving η shock.

To sum up, each state η has two dated commodities; that is, the physical good in period $t = 1$ or good 1, and the physical good in period $t = 2$ or good 2. In addition, an investment decision at $t = 0$, ρ, is also a commodity. Therefore, there are $L = 2Q + 1$ commodities in this model.

The utility function conditional on a shock η is given by $u(c_1, c_2, \eta)$, where (c_1, c_2) is the vector of consumption allocations in period $t = 1$ and $t = 2$, respectively. For example, in the Diamond-Dybvig model, the shock will dictate if an agent would like to consume now or later. The utility function is assumed to be differentiable, concave, increasing in c_1 and c_2, and satisfies the usual Inada conditions with respect to c_1 and c_2. The discounted expected utility of an agent is define by:

$$U(\mathbf{c}) = \sum_{\eta} \pi(\eta)\, u(c_1(\eta), c_2(\eta), \eta)\,. \tag{A.108}$$

Following the literature, there are two technologies or assets. First, the short-term asset is a storage technology, whose return from t to $t+1$ is R_1, i.e., saving one unit of the good today at t will return R_1 units of the good in the next period at $t+1$, $t = 0, 1$. The second asset is the long-term asset. The long-term investment must be taken at $t = 0$, and its return R_2 will be realized at $t = 2$. We assume that the long-term asset is more productive than the short-term asset, i..e, $R_2 > R_1$. For simplicity, returns here are deterministic, i.e., no aggregate shocks.

There is no loss of generality to assume that an agent must decide how much to invest in the short-term and the long-term assets at the beginning $t = 0$. Let ρ be the fraction of initial endowment invested in the short-term asset; that is, ρe is the total amount invested in the short-term asset, and $(1-\rho)\, e$ is the total amount invested in the long-term asset. In addition, we assume that there is no option to liquidate at $t = 1$, and there is no short term investment between $t = 1$ and $t = 2$ without loss of generality[2]. The production functions of short-term asset, F_1 (between $t = 0$ and $t = 1$), and long-term asset F_2 (between $t = 0$ and $t = 2$) are as follows:

$$F_1(\rho e, y_1) = y_1 - R_1 \rho e = 0, \tag{A.109}$$

$$F_2((1-\rho)\, e, y_2) = y_2 - R_2 (1-\rho)\, e = 0, \tag{A.110}$$

where y_t is an output in unit of the physical good in period $t = 1, 2$ regardless of the state of nature. Note however that outputs do not really vary with the preference/liquidity shocks since the shocks η are liquidity, not productivity shocks and the distribution of shocks in the population is constant. To sum up, there are $O = 2$ production functions.

H.2 Market Structure: Security and Spot Markets

There are J securities. Let $\theta_j^h \in \mathbb{R}$ denote the amount of security j acquired (negative if sold) by an agent of type h, and $\mathbf{D}_j = [D_{j\ell}]_{\ell=1}^{L} \in \mathbb{R}_+^L$ denote its payoff vector. Thus securities have payoffs of goods in the L-dimensional space of underlying commodities. Notationally, let $\mathbf{D} = [\mathbf{D}_j]_{j=1}^{J}$ be

[2] This economy is equivalent to the one in Diamond and Dybvig (1983) where banks invest in the long-term asset only, and then liquidate a fraction of the projects at $t = 1$. In Allen and Gale (2004) with stochastic returns, some short term investment may be necessary at $t = 1$.

the payoff matrix of all securities. Let $\mathbf{Q} \in \mathbb{R}_+^J$ be the price vector of all securities, that is, $Q_j \geq 0$ for $j = 1, \ldots, J$.

In addition, agents can trade in each of M spot markets of subsets of commodities. Let $L^m \subset L$ be the subset of commodities that can be traded in spot markets m. Let $\boldsymbol{\tau}^h$ denote the set of trades in these markets with $\tau_{\ell m}^h$ denoting the amount of good ℓ in market L^m acquired (negative if surrendered) by an agent of type h. Note again that these spot trades $\tau_{\ell m}^h$ are restricted to be traded with commodities in L^m only. Let $\mathbf{p}_m \equiv [p_{\ell m}]_{\ell \in L^m} \in \mathbb{R}_+^{L^m}$ be the price vector of commodities in L^m.

The relationship between consumption, endowments, securities, spot trades, and outputs for an agent of type h is defined implicitly by

$$g^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) = 0. \tag{A.111}$$

These will be obvious identities or accounting formulas in the examples which follow. This condition is generally multidimensional vector with dimension N, i.e.,

$$g^h \equiv \left(g_n^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right)\right)_{n=1}^N.$$

H.2.1 Market Structure for the Collateral Economy

Let $\theta_{\ell s}^h$ denote securities paying in good $\ell = 1, 2$ in state s or net transfers of good $\ell = 1, 2$ in state s acquired by an agent type h. If this is negative, it is a promise to pay. Also $\theta_{\ell 0}^h$ is spot purchase of good ℓ at $t = 0$ but for convenience we refer to this as a security trade. Thus there are $J = 2(1 + S)$ securities. Let $Q_{\ell 0}$ and $Q_{\ell s}$ denote the security (spot) price of good ℓ at period $t = 0$ and the price of a security paying in good ℓ in state s, respectively. We take good $\ell = 1$ as the numeraire.

Let $\tau_{\ell s}^h$ denote spot trade amount of good $\ell = 1, 2$ in spot markets in state s, L^s ($m = s$ here), acquired by an agent of ex-ante type h. With abuse of notation, let $\tau_{\ell 0}^h$ denote spot trade amount of good ℓ in spot markets L^0 in period $t = 0$ acquired by an agent of ex-ante type h. Each spot market has two commodities, namely good 1 and good 2, i.e., $L^m = 2$ for all $m = 0, 1, \ldots, S$. There are $M = S + 1$ spot markets here. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_s denote the spot-market-clearing price of good 2 in each spot market L^s.

The consumption-relationship constraints in this case are defined as follows:

$$g_{\ell s}^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) = e_{\ell s}^h + y_{\ell s}^h + \theta_{\ell s}^h + \tau_{\ell s}^h - c_{\ell s}^h = 0, \text{ for } \ell = 1, 2; s = 0, 1, \ldots, S, \tag{A.112}$$

where we set $y_{10}^h = 0$ and $y_{1s}^h = 0$ to represent the fact that good 1 cannot be stored. There are $N = 2(1+S)$ consumption-relationship constraints. As proved in Kilenthong and Townsend (2014b), with complete collateralized contracts, there is no need for restricted/spot trades $\boldsymbol{\tau}$ in this case. All trades can be accomplished in ex-ante security markets. As a result, the consumption-relationship constraints can be rewritten as follows:

$$g_{\ell s}^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \mathbf{y}^h\right) = e_{\ell s}^h + y_{\ell s}^h + \theta_{\ell s}^h - c_{\ell s}^h = 0, \text{ for } \ell = 1, 2; s = 0, 1, \ldots, S. \tag{A.113}$$

Nevertheless, we can define what the ex-post spot price p_s would be that would clear these markets (without active trade).

H.2.2 Market Structure for the Exogenous Incomplete Markets Economy

There are $J < S$ securities available for purchase or sell in the first period $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets where D_{js} be the payoff of asset j in unit of good 1 (the numeraire good) in state s in the second period $t = 1$, $s = 1, 2, \ldots, S$. Let θ_j^h denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$, and Q_j denote the price of security j. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is again, $J < S$. This is crucial.

Let $\tau_{\ell s}^h$ denote spot trade amount of good $\ell = 1, 2$ in spot markets in state s, L^s ($m = s$ here), acquired by an agent of ex-ante type h. With abuse of notation, let $\tau_{\ell 0}^h$ denote spot trade amount of good $\ell = 1, 2$ in spot markets L^0 in period $t = 0$ acquired by an agent of ex-ante type h. Each spot market has two commodities, namely good 1 and good 2, i.e., $L^m = 2$ for all $m = 0, 1, \ldots, S$. There are $M = S + 1$ spot markets here. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_0 and p_s denote the spot-market-clearing price of good 2 in spot market L^0 in period $t = 0$, and the spot-market-clearing price of good 2 in spot market L^s at state s in period $t = 1$, respectively.

The consumption-relationship functions in the first period $t = 0$ is defined as follows:

$$g_\ell^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h\right) = e_{\ell 0}^h + \tau_{\ell 0}^h - c_{\ell 0}^h = 0, \text{ for } \ell = 1, 2. \tag{A.114}$$

The consumption-relationship function for good 1 and good 2, respectively, in the state s in the second period is defined as follows:

$$g_{1+2s}^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right) = e_{1s}^h + \sum_j D_{js}\theta_j^h + \tau_{1s}^h - c_{1s}^h = 0, \text{ for } s = 1, \ldots, S, \tag{A.115}$$

$$g_{2+2s}^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right) = e_{2s}^h + \tau_{2s}^h - c_{2s}^h = 0, \text{ for } s = 1, \ldots, S. \tag{A.116}$$

Note here there will be active spot market trades $\boldsymbol{\tau}$ to support the equilibrium allocation. To sum up, there are $N = 2(1+S)$ consumption-relationship constraints.

H.2.3 Market Structure the Moral Hazard with Retrading Economy

To be consistent with the general model, one can imagine that there are state-contingent securities paying in good $\ell = 1, 2$ when state/output is $\mathbf{q}$ and the recommended action is a, namely $\theta_\ell(\mathbf{q}, a)$. That is, security j is indexed by $\mathbf{q}$, ℓ, and a. Even though there are $J = 2QA$ securities available to trade, each agent can trade only $2Q$ securities depending on his recommended action a only. In particular, an agent recommended action a will be able to trade only securities $\theta_a \equiv [\theta_\ell(\mathbf{q}, a)]_{\ell,\mathbf{q}}$. Let $Q_\ell(\mathbf{q}, a)$ denote the price of a security paying in good ℓ conditional on output $\mathbf{q}$ and recommended action a. Recall that actual action $a' = a$ is an input of the production technology, and there is no loss of generality to assume that it is non-tradable. Therefore, there is no price for that commodity action. Note also that as in the literature, these equilibrium securities prices are fair prices.

There is the possibility of retrade in ex post spot markets. One can think of two subperiods: the first with the application of inputs, securities, and production; the second for output and possible retrading with final consumption. Without aggregate uncertainty, there is only one set of spot markets ($M = 1$) for good 1 and good 2 ($L^1 = 2$). Let $\tau_\ell(\mathbf{q}, a)$ be spot trade of an agent of ex-post type $(\mathbf{q}, a)$ when a is both the recommended and taken action. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p denote the spot-market-clearing price of good 2, which depends on agents' action a (recommended and taken) and securities $\theta_a = [\theta_\ell(\mathbf{q}, a)]_{\ell,\mathbf{q}}$ as a function of recommended action a (as if the markets can be partitioned by action a); that is, $p = p(\theta_a, a)$.

As in Kilenthong and Townsend (2011) and the collateral example in Section H.2.1, the spot markets are redundant (with complete contracts), however. Anything that can be done with spot markets can be done without them with altered security holdings. Therefore, we can omit spot trades, henceforth, though there is still an implicit shadow spot price. The consumption-relationship in this case is defined as follows:

$$g_{\ell\mathbf{q}a} = q_\ell + \theta_\ell(\mathbf{q}, a) - c_\ell(\mathbf{q}, a) = 0, \forall \mathbf{q}, a; \ell = 1, 2. \quad \text{(A.117)}$$

There are $N = 2QA$ consumption-relationship constraints.

H.2.4 Market Structure the Liquidity Constrained Economy

To be consistent with the general model, there is no security in this model; that is, $J = 0$. All trades occur in the spot markets. There are 2 sets of spot markets in period $t = 1$ and $t = 2$; that is, $M = 2$. Agents can trade storage claim τ^h_{f1} and building services τ^h_b in the spot markets in period $t = 1$ at price p_b; that is, there are two commodities in the spot markets in period $t = 1$ ($L^1 = 2$). Similarly, agents can trade storage claim τ^h_{f2} and doctor services τ^h_d in the spot markets in period $t = 2$ at price p_d; that is, there are two commodities in the spot markets in period $t = 2$ ($L^2 = 2$).

The consumption-relationship constraints are as follows:

$$
\begin{aligned}
g_b^h \left(b^h, y_b^h, \tau_b^h\right) &= b^h - \left(y_b^h + \tau_b^h\right) = 0, \forall h, && \text{(A.118)}\\
g_d^h \left(d^h, y_d^h, \tau_d^h\right) &= d^h - \left(y_d^h + \tau_d^h\right) = 0, \forall h, && \text{(A.119)}\\
g_w^h \left(w^h, y_{i3}^h, y_{s3}^h\right) &= w_3^h - y_{i3}^h - y_{s3}^h = 0, \forall h, && \text{(A.120)}\\
g_{ft}^h \left(f_t^h, f_{t-1}^h, \tau_{ft}^h\right) &= f_t^h - f_{t-1}^h - \tau_{ft}^h = 0, \forall h; t = 1, 2. && \text{(A.121)}
\end{aligned}
$$

To sum up, there are $N = 5$ consumption-relationship constraints.

H.2.5 Market Structure the Fire Sales Economy

Let θ_0^h and θ_{ts}^h denote securities paying in unit of consumption goods in period $t = 0$, and securities paying in consumption goods at state $s = 1, 2$ in period $t = 1, 2$ acquired by an agent type $h = c, e$, respectively. There are $J = 5$ securities. Let Q_{ts} denote the contract/security price of a security paying in consumption goods at state $s = 1, 2$ in period $t = 1, 2$, and $Q_0 = 1$ the price of the contract/security paying in unit of consumption goods in period $t = 0$, which is the numeraire good.

There are also spot markets at each state $s = 1, 2$ in period $t = 1$. There are $L^m = 2$ commodities in each spot markets $m = 1, 2$. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_s denote the spot-market-clearing price of good 2. As in Lorenzoni (2008), the market clearing conditions for the spot markets in each state s (for τ_{ks}^h and τ_{cs}^h) imply that the spot price p_s is determined by capital input for the traditional technology y_{k1s}^c, i.e., $p_s = f' \left(y_{k1s}^c\right)$.

The consumption-relationship functions for consumers and entrepreneurs are given by

$$g_{c0}^c \left(e, \theta_0^c, c_0^c\right) = e + \theta_0^c - c_0^c = 0, \tag{A.122}$$

$$g_{c1s}^c \left(e, \theta_{1s}^c, \tau_{cs}^c, c_{1s}^c\right) = e + \theta_{1s}^c + \tau_{cs}^c - c_{1s}^c = 0, \forall s = 1, 2, \tag{A.123}$$

$$g_{c2s}^c \left(e, \theta_{2s}^c, y_{c2s}^c, c_{2s}^c\right) = e + \theta_{2s}^c + y_{c2s}^c - c_{2s}^c = 0, \forall s = 1, 2, \tag{A.124}$$

$$g_{ks}^c \left(\tau_{ks}^c, y_{k1s}^c\right) = \tau_{ks}^c - y_{k1s}^c = 0, \forall s = 1, 2, \tag{A.125}$$

$$g_{c0}^e \left(n, \theta_0^e, y_{i00}^e\right) = n + \theta_0^e - y_{i00}^e = 0, \tag{A.126}$$

$$g_{c1s}^e \left(\theta_{1s}^e, \tau_{cs}^e, y_{c1s}^e, y_{r1s}^e, y_{i1s}^e\right) = \theta_{1s}^e + \tau_{cs}^e + y_{c1s}^e + y_{r1s}^e + y_{i1s}^e = 0, \forall s = 1, 2, \tag{A.127}$$

$$g_{ks}^e \left(y_{n1s}^e, y_{o1s}^e, \tau_{ks}^e, y_{k1s}^e\right) = y_{n1s}^e + y_{o1s}^e + \tau_{ks}^e - y_{k1s}^e = 0, \forall s = 1, 2, \tag{A.128}$$

$$g_{c2s}^e \left(y_{c2s}^e, \theta_{2s}^e, c_{2s}^e\right) = y_{c2s}^e + \theta_{2s}^e - c_{2s}^e = 0, \forall s = 1, 2, \tag{A.129}$$

where a function with superscript "c" ("e") is a consumption-relationship function for consumers (for entrepreneurs). There are $N = 14$ consumption-relationship conditions.

H.2.6 Market Structure the Hidden Information with Retrading Economy

To be consistent with the general model, one can imagine that there are state-contingent securities $\theta_t(\eta')$ paying the single good at $t = 1, 2$ conditional on reported shock/state $\eta' \in Q$. That is, security j is indexed by t and η'. There are $J = 2Q$ (the number of states times the number of dates) securities available.

There is the possibility of retrade in ex post spot markets as in the moral hazard with retrading economy above. Without aggregate uncertainty, there is only one set of spot markets ($M = 1$) for good 1 and good 2 ($L^m = 2$), in which everyone participates. Let $\tau_t(\eta)$ be spot trade of an agent of ex-post type η when η is both truthfully reported and realized shock. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p denote the spot-market-clearing price of good 2, which depends on securities $\boldsymbol{\theta} = [\theta_t(\eta)]_{t,\eta}$ and investment decision ρ; that is, $p = p(\theta, \rho)$.

As in the moral hazard with retrading economy, the spot markets are redundant (with complete contracts), however. Anything that can be done with spot markets can be done without them with altered security holdings. Therefore, we can omit spot trades, henceforth, though there is still an implicit shadow spot price. The consumption-relationship in this case is defined as follows:

$$g_{t\eta'} = y_t + \theta_t(\eta') - c_t(\eta') = 0, \forall \eta'; t = 1, 2. \tag{A.130}$$

There are $N = 2Q$ consumption-relationship constraints.

H.3 Trade Frictions: Obstacle-to-Trade Constraints

There are I sets of obstacle-to-trade constraints, each of which contains potentially multiple conditions, indexed by $a = 1, \ldots, A$ and $a' = 1, \ldots, A$. Each set of obstacle-to-trade constraints $i = 1, 2, ...I$ depends on the spot prices of a particular subset of commodities $\mathbf{p}^i$ or the prices of a particular subset of securities denoted $\mathbf{Q}^i$, or both. Each can depend on the same set of prices $(\mathbf{p}^i, \mathbf{Q}^i)$. In their general form, each obstacle to trade constraint (a, a') in set i can be written as:

$$C^h_{i,a,a'}\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right) \geq 0, \text{ for } i = 1, \ldots, I; a \in A; a' \in A. \tag{A.131}$$

These obstacle-to-trade constraints could be in the form of collateral constraints, retrading in exogenous incomplete-market constraints, incentive compatibility constraints under moral hazard with retrading, incentive compatibility constraints under hidden information with retrading, liquidity constraints, and no-default constraints. The total number of obstacle-to-trade constraints is $V \leq IA^2$, where the inequality results from the fact that the maximum number of constraints for each agent type is IA^2 but it is possible for some types to have less. Note that an action as in a moral hazard model, or privately observed state indexes the commodities, and therefore is included in $\mathbf{c}^h$. In addition to actions or preference shocks, the index (a, a') also denote each individual constraint in each set i of the obstacle-to-trade constraints sharing the same set of prices $\left(\mathbf{p}^i, \mathbf{Q}^i\right)$.

The dependency on market-clearing prices of these obstacle-to-trade constraints is the source of price externalities in this paper. Most of the literature focuses only on the dependency on the restricted/spot prices. This paper explicitly puts security prices into the constraints in order to emphasize that price externalities could arise even when we shut down the spot markets. In other words, the spot markets/prices are not fundamental to the externality problem. It is an obstacle to trade itself, which can not be removed, that is key to the problem. As shown in the collateral economy below, one can get rid of the spot markets there since they are redundant. The collateral constraints (the need to back promises by collateral) then depend on security prices only, but the price externality still occurs.

H.3.1 Trade Frictions for the Collateral Economy

As in Kilenthong and Townsend (2014b), the collateral constraints or obstacle-to-trade constraints state that the value of collateral y^h_{2s} must weakly exceed value of promises to pay $\left(\theta^h_{1s}, \theta^h_{2s}\right)$:

$$p_s y^h_{2s} \geq p_s \left(-\theta^h_{2s}\right) + \left(-\theta^h_{1s}\right), \text{for } s = 1, \ldots, S, \tag{A.132}$$

which can be rewritten as follow:

$$p_s \left(y^h_{2s} + \theta^h_{2s}\right) + \theta^h_{1s} \quad \geq \quad 0, \text{ for } s = 1, \ldots, S, \tag{A.133}$$

where again p_s is the spot price of good 2 in units of good 1 in state s.

These collateral constraints can be rewritten in terms of security prices as following:

$$C^h_s \left(\theta^h, \mathbf{y}^h, \mathbf{Q}_s\right) = Q_{2s} \left(y^h_{2s} + \theta^h_{2s}\right) + Q_{1s}\theta^h_{1s} \quad \geq \quad 0, \text{ for } s = 1, \ldots, S, \tag{A.134}$$

which results from the fact that, with complete state contingent contracts at $t = 0$ and the possibility of retrading, the spot price ratio p_s equals to the ratio of security prices $\frac{Q_{2s}}{Q_{1s}}$. This formulation emphasizes that we can shut down the spot markets, but the collateral constraints still depend on security prices, which still generate externalities. In other words, the spot markets/prices are not fundamental to the externality problem. It is an obstacle to trade itself, which can not be removed, that is key to the problem.

Each agent of type h faces $I = S$ sets of obstacle-to-trade constraints, each of which contains only one constraint, i.e., technically $A = 1$. Therefore, there are $V = S$ obstacle-to-trade constraints in total.

H.3.2 Trade Frictions for the Exogenous Incomplete Markets Economy

The obstacle-to-trade or spot-budget constraint for an agent of type h in each state s is simply the budget constraint in that state:

$$C_s^h\left(\boldsymbol{\tau}_s^h, \mathbf{p}\right) = \tau_{1s}^h + p_s \tau_{2s}^h = 0, \text{ for } s = 1, \ldots, S, \tag{A.135}$$

Note that the spot price p_s is determined by pre-trade position of endowments and securities where endowments are exogenous but securities are endogenous.

Each agent of type h faces $I = 1$ set of obstacle-to-trade constraints, which contains $A = S$ constraints. Therefore, there are $V = S$ obstacle-to-trade constraints in total.

H.3.3 Trade Frictions for the Moral Hazard with Retrading Economy

The possibility of retrade in ex post spot markets creates obstacle to trade in this model. With the possibility of retrade, the ex-post utility maximization problem of an agent who was recommended action a receiving compensation $(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a))$, but took action a' when the spot market price is p is as follows:

$$v\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a', p\right) = \max_{\tau_1, \tau_2} u\left(c_1(\mathbf{q}, a) + \tau_1, c_2(\mathbf{q}, a) + \tau_2, a'\right) \tag{A.136}$$

subject to the budget constraint:

$$\tau_1 + p\tau_2 = 0, \tag{A.137}$$

taking spot-market-clearing price p as given.

As in Kilenthong and Townsend (2011), the possibility of retrade in ex post spot markets and the moral hazard problem imply that the incentive compatibility constraints (IC) are as following: $\forall a, a'$,

$$C_{1,a,a'}\left(\mathbf{c}, p\right) = \sum_{\mathbf{q}} u\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a\right) f(\mathbf{q}|a) - \sum_{\mathbf{q}} v\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a', p\right) f(\mathbf{q}|a') \geq 0, \tag{A.138}$$

Here the agent takes the recommended action a and so $a = a'$. There is only one set of obstacle-to-trade constraints, $I = 1$, and there are A^2 constraints for this one i. Therefore, there are $V = A^2$ incentive compatibility constraints in total.

H.3.4 Trade Frictions for the Liquidity Constrained Economy

The obstacle-to-trade or spot market constraints for an agent type $h = b, d$ in period t are as follows:

$$C_1^h\left(\tau_{f1}^h, \tau_b^h, \mathbf{p}\right) = \tau_{f1}^h + p_b\tau_b^h = 0, \forall h = b, d, \quad (A.139)$$

$$C_2^h\left(\tau_{f2}^h, \tau_d^h, \mathbf{p}\right) = \tau_{f2}^h + p_d\tau_d^h = 0, \forall h = b, d, \quad (A.140)$$

where p_b and p_d are the spot-market-clearing prices of building and doctor services in period $t = 1$ and $t = 2$, respectively; that is, p_b is such that $\sum_{h=b,d}\alpha^h\tau_b^h = 0$, and vice versa. Note that the spot price p_b and p_d are determined by storage positions of all agents which are endogenous. Each agent of type h faces $I = 1$ set of obstacle-to-trade constraints, which contains $A = 2$ constraints. Therefore, there are $V = 2$ obstacle-to-trade constraints in total.

H.3.5 Trade Frictions for the Fire Sales Economy

Each consumer faces the following sets of obstacle-to-trade constraints. First, the participation constraint for a consumer is given by

$$C_{pc}^c\left(\theta_0^c, \theta_{1s}^c, \theta_{2s}^c\right) = \theta_0^c + \sum_s \pi_s\left(\theta_{1s}^c + \theta_{2s}^c\right) \geq 0. \quad (A.141)$$

This constraint states that a consumer would not enter the contract at period $t = 0$ and would be at autarky unless the contract offers a non-negative expected return from $t = 0$ to $t = 2$. Second, the no-default conditions for a consumer are as follows:

$$C_{cd1}^c\left(\theta_{1s}^c, \theta_{2s}^c\right) = \theta_{1s}^c + \theta_{2s}^c \geq 0, \forall s = 1, 2, \quad (A.142)$$

$$C_{cd2}^c\left(\theta_{2s}^c\right) = \theta_{2s}^c \geq 0, \forall s = 1, 2. \quad (A.143)$$

These constraints imply that a consumer would default (not pay when $\theta_{ts}^c < 0$) at state $s = 1, 2$ in period $t = 1, 2$ unless the return from that period on is non-negative.

Each entrepreneur faces the following obstacle-to-trade constraints or no-default conditions:

$$C_{ed1}^e\left(y_{n00}^e, \theta_{1s}^e, \theta_{2s}^e, p_s\right) = \left(\eta a_s + \max\left\{p_s - \gamma, 0\right\}\right)y_{n00}^e + \theta_{1s}^e + \theta_{2s}^e \geq 0, \forall s = 1, 2, \quad (A.144)$$

$$C_{ed2}^e\left(y_{k1s}^e, \theta_{2s}^e\right) = \eta A y_{k1s}^e + \theta_{2s}^e \geq 0, \forall s = 1, 2, \quad (A.145)$$

where $1-\eta \in (0,1)$ is the fraction of the firm's current profit that the entrepreneur could keep if he decided to default. Constraints (A.144) imply that the entrepreneur is better off not defaulting at state $s = 1, 2$ in period $t = 1$. In particular, he would get $(1-\eta)\, a_s y^e_{n00}$ if he defaulted. On the other hand, his net income would be $(a_s + \max\{p_s - \gamma, 0\})\, y^e_{n00} + \theta^e_{1s} + \theta^e_{2s}$ in case of no default. Similarly, constraints (A.145) imply that net income of the entrepreneur at state $s = 1, 2$ in period $t = 2$ in case of no default, $Ay^e_{k1s} + \theta^e_{2s}$, is larger than his net income in case of default, $(1-\eta)\, Ay^e_{k1s}$.

In addition, both agent types also face the following spot market budget constraints:

$$C^h_{spot}\left(\tau^h_{cs}, \tau^h_{ks}, p_s\right) \;=\; \tau^h_{cs} + p_s \tau^h_{ks} = 0, \forall h = c, e; s = 1, 2, \tag{A.146}$$

To be consistent with the general model, there are $I = 7$ sets of obstacle-to-trade constraints for the consumer c and 6 sets of obstacle-to-trade constraints for the entrepreneur e. Each set contains only one constraint, i.e., $A = 1$. Therefore, there are 7 obstacle-to-trade constraints in total for the consumer c and 6 obstacle-to-trade constraints in total for the entrepreneur e.

It is worthy of emphasis that the spot market budget constraints (A.146) are not the sources of the externality here because this model has a complete contingent contracting structure. See a similar result in Proposition B.1. On the other hand, the key obstacle-to-trade constraints that cause an inefficiency in this model is the first set of no-default conditions for an entrepreneur (A.144), which again depends on equilibrium prices which in turn are determined by collective ex-ante choices of the agents.

H.3.6 Trade Frictions for the Hidden Information with Retrading Economy

With abuse of notation, we refer to $(c_1(\eta'), c_2(\eta'))$ as pre-trade compensation condition on reported state/shock. Thus, the ex-post utility maximization problem at $t = 1$ of an agent who reported state η', realized state η, and received compensation $(c_1(\eta'), c_2(\eta'))$ is as follows:

$$v\left(c_1\left(\eta'\right), c_2\left(\eta'\right), \eta, p\right) = \max_{\tau_1, \tau_2} u\left(c_1\left(\eta'\right) + \tau_1, c_2\left(\eta'\right) + \tau_2, \eta\right) \tag{A.147}$$

subject to the budget constraint:

$$\tau_1 + p\tau_2 = 0, \tag{A.148}$$

taking spot price (interest rate) p as given.

In addition, the possibility of retrade in ex post spot markets and the hidden information problem imply that an incentive compatibility (IC) or obstacle-to-trade constraint:

$$C_{1,\eta,\eta'}(\mathbf{c}, p) = u\left(c_1(\eta), c_2(\eta), \eta\right) - v\left(c_1\left(\eta'\right), c_2\left(\eta'\right), \eta, p\right) \;\geq\; 0, \forall \eta, \eta'. \tag{A.149}$$

There are $I = Q^2$ constraints for each i, and therefore with only one i, there are $V = Q^2$ obstacle-to-trade constraints in total. This will be imposed so actual and reported states will be the same.

H.4 Competitive Equilibrium with Externalities of Prototypical Economies

This section presents the standard definitions of competitive equilibria that have externalities for a liquidity constrained economy (Hart and Zingales, 2013) and a fire sales economy (Lorenzoni, 2008). The definitions for an incomplete markets economy and a collateral economy are already displayed in online Appendix G.1 and Kilenthong and Townsend (2014b), respectively. The definitions of a moral hazard with retrading economy and a hidden information with retrading economy can be found in Kilenthong and Townsend (2011).

H.4.1 Competitive Equilibrium for the Liquidity-Constrained Economy

Definition A.5. A competitive equilibrium with liquidity constraints is a specification of allocation $\left(f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, b^h, d^h, w_3^h, \mathbf{y}^h\right)_{h=b,d}$ and prices (p_b, p_d) such that

(i) for each h, $\left(f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, \mathbf{y}^h\right)$ solves the utility maximization problem

$$\max_{f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, \mathbf{y}^h} w_3^h + \delta_b^h b^h + \left(1 - \delta_d^h\right) d^h - \frac{\left(l^h\right)^2}{2} \tag{A.150}$$

subject to the production constraints (A.94)-(A.97), the consumption-relationship constraints (A.118)-(A.121), and the obstacle-to-trade constraints (A.139)-(A.140), taking prices (p_b, p_d) as given;

(ii) markets clear for storage claims in period $t = 1$

$$\sum_h \alpha^h \tau_{f1}^h = 0, \tag{A.151}$$

markets clear for building services in period $t = 1$

$$\sum_h \alpha^h \tau_b^h = 0, \tag{A.152}$$

markets clear for storage claims in period $t = 2$

$$\sum_h \alpha^h \tau_{f2}^h = 0, \tag{A.153}$$

markets clear for doctor services in period $t = 2$

$$\sum_h \alpha^h \tau_d^h = 0, \tag{A.154}$$

H.4.2 Competitive Equilibrium for the Fire Sales Economy

Definition A.6. A competitive equilibrium with fire sales is a specification of allocation $\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h\right)_{h=c,e}$ and prices $(p_s, Q_{ts})_s$ such that

(i) for each consumer c, $(\mathbf{c}^c, \theta^c, \tau^c, \mathbf{y}^c)$ solves the utility maximization problem

$$\max_{\mathbf{c}^c, \theta^c, \tau^c, \mathbf{y}^c} c_0^c + \sum_{s=1,2} \pi_s \left(c_{1s}^c + c_{2s}^c\right) \tag{A.155}$$

subject to the budget constraints

$$\theta_0^c + \sum_{s,t} Q_{ts} \theta_{ts}^c \quad \leq \quad 0, \tag{A.156}$$

the production constraints (A.106), the consumption-relationship constraints (A.122)-(A.125), and the obstacle-to-trade constraints (A.141)-(A.143), (A.146), taking prices (p_s, Q_{ts}) as given;

(ii) for each entrepreneur e, $(\mathbf{c}^e, \theta^e, \tau^e, \mathbf{y}^e)$ solves the utility maximization problem

$$\max_{\mathbf{c}^e, \theta^e, \tau^e, \mathbf{y}^e} \sum_{s=1,2} \pi_s c_{2s}^e \tag{A.157}$$

subject to the budget constraints

$$\theta_0^e + \sum_{s,t} Q_{ts} \theta_{ts}^e \quad \leq \quad 0, \tag{A.158}$$

the production constraints (A.100)-(A.105), the consumption-relationship constraints (A.126)-(A.129), and the obstacle-to-trade constraints (A.144)-(A.145), (A.146), taking prices (p_s, Q_{ts}) as given;

(iii) markets clear in contract paying in period $t = 0$

$$\theta_0^c + \theta_0^e = 0, \tag{A.159}$$

markets clear in security paying at state s in period t

$$\theta_{ts}^c + \theta_{ts}^e = 0, \forall t = 1, 2; s = 1, 2, \tag{A.160}$$

markets clear in consumption good at state s in period $t = 1$

$$\tau_{cs}^c + \tau_{cs}^e = 0, \forall s = 1, 2, \tag{A.161}$$

markets clear in capital good at state s in period $t = 1$

$$\tau_{ks}^c + \tau_{ks}^e = 0, \forall s = 1, 2. \tag{A.162}$$

H.5 Competitive Equilibrium and Pareto Problem of the Generalized Model: The Lottery Representation

Let $x^h(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta})$ be the probability measure on $(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta})$ for an agent of type h. In other words, $x^h(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta})$ is the probability of receiving allocation $(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y})$, and being in exchanges $(\mathbf{p}, \mathbf{Q})$ with the rights to trade $\boldsymbol{\Delta}$. It is worthy of emphasis that price externalities cannot simply be overcome using lotteries to convexify the problem, as shown in Kilenthong and Townsend (2011) where price externalities exist even in a setting with lotteries. For notational purposes, let $\mathbf{w} \equiv (\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta})$.

As a probability measure, a lottery of an agent of type h, x^h, satisfies the following probability constraint:

$$\sum_{\mathbf{w}} x^h(\mathbf{w}) = 1. \tag{A.163}$$

With a continuum of agents, $x^h(\mathbf{w})$ can be interpreted as the fraction of agents of type h assigned to a bundle $(\mathbf{w})$. More formally, with all choice object gridded up as an approximation, the commodity space is assumed to be a finite dimensional linear space[3]. Let $\mathbf{x}^h \equiv \left[x^h(\mathbf{w})\right]_{\mathbf{w}}$ be a typical lottery for an agent of type h.

We now consider a bundle $\mathbf{w}$ as a typical commodity. Each bundle will be feasible for an agent of type h with endowment $\mathbf{e}$ only if it satisfies the spot-market budget constraints, the consumption-relationship constraints, the technology constraints, the obstacle-to-trade constraints, and the right-to-trade requirements[4] as following:

$$\begin{aligned}
\sum_{\ell=1}^{L^m} p_{\ell m} \tau_{\ell m}^h(\omega) &\leq 0, \forall s, \omega, && \text{(A.164)} \\
g_n^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) &= 0, \forall n, && \text{(A.165)} \\
F_o^h\left(\mathbf{y}^h\right) &= 0, \forall o, && \text{(A.166)} \\
C_{i,a,a'}^h\left(\mathbf{c}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right) &\geq 0, \forall i, a, a' && \text{(A.167)} \\
\boldsymbol{\Delta}_i - \mathbf{d}_i^h(\mathbf{w}) &= 0, \forall i, && \text{(A.168)}
\end{aligned}$$

where $\mathbf{d}_i^h(\mathbf{w})$ is the excess demand functions corresponding to the i^{th} set of obstacle-to-trade constraints for an agent type h who holds a bundle $\mathbf{w}$.

[3] The limiting arguments under weak-topology used in Prescott and Townsend (1984a) can be applied to establish the results if the commodity space is not finite.

[4] In some cases, as in a moral hazard with retrading and a hidden information with retrading, the right-to-trade requirements may be embedded implicitly in the obstacle-to-trade constraints. As a result, they may not be explicitly written out as separate constraints as in Kilenthong and Townsend (2011).

Accordingly, we impose the following condition on the probability measure as follows:

$$\begin{aligned} x^h(\mathbf{w}) &\geq 0 \text{ if conditions (A.164) to (A.168) hold,} \\ &= 0 \text{ if otherwise.} \end{aligned} \tag{A.169}$$

In words, a positive measure can be defined only on feasible bundles, which satisfy the spot-market budget constraints, the consumption-relationship constraints, the technology constraints, the obstacle-to-trade constraints, have the required amount of the rights to trade.

In addition, in order to incorporate the moral hazard economy into the general framework with lotteries, we need to impose additional conditions, called mother-nature constraints, to ensure that the lottery is consistent with the production technology[5], as in Prescott and Townsend (1984b). There are W mother-nature constraints. The mother-nature constraints can be written as linear constraints as follows:

$$\sum_{\mathbf{w}} \Gamma_w(\mathbf{w}) = 0, \tag{A.170}$$

where $\Gamma_w(\mathbf{w}) \in \mathbb{R}$ is the coefficient for w^{th} constraint corresponding for $(\mathbf{w})$.

More formally, the consumption possibility set of an agent of type h is defined as follows:

$$X^h = \left\{ \mathbf{x}^h : x^h(\mathbf{w}) \text{ satisfies (A.163), (A.169), and (A.170)} \right\} \tag{A.171}$$

Let $\mathbf{x}^h$ be a typical element of X^h. Note that X^h is compact and convex.

H.5.1 Competitive Equilibrium with Segregated Security Exchanges for the Generalized Model

Let $P(\mathbf{w})$ be the price of a bundle or contract $(\mathbf{w})$. Each agent is infinitesimally small relative to the entire economy and will take all prices as given. The intermediaries also act competitively. Note as well that the rights to trade $\mathbf{\Delta}$ is also priced.

Consumers: Each agent of type h, taking prices $P(\mathbf{w})$ as given, chooses $\mathbf{x}^h$ to maximize its expected utility:

$$\max_{\mathbf{x}^h \in X^h} \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{w}) \tag{A.172}$$

subject to the budget constraint

$$\sum_{\mathbf{w}} P(\mathbf{w}) x^h(\mathbf{w}) \leq \sum_{\mathbf{w}} P(\mathbf{w}) e^h(\mathbf{w}) \tag{A.173}$$

[5]That is, the technology constraints will be replaced by the mother-nature constraints for the moral hazard with retrading economy.

Financial Intermediaries: There is no loss of generality to assume that there is a representative financial intermediary who issues (sells) $b(\mathbf{w}) \in \mathbb{R}_+$ units of each bundle at the unit price $P(\mathbf{w})$. Note that the intermediary can issue any non-negative number of a bundle; that is, the number of bundles issued does not have to be between zero and one and is not a lottery. Let $\mathbf{b}$ be the vector of the number of bundles issued as one move across bundles, arguments in $b(\mathbf{w})$. With constant returns to scale, the profit of a market-maker must be zero and the number of market-makers becomes irrelevant. Therefore, without loss of generality, we assume there is one representative market-maker, which takes prices as given.

In order to deal with ex post diversity in private information problems, we define some participation mechanism. Let ξ_j^h, called eligibility weight, denote the mass of agents of type h who are eligible to trade security j, and hence $\xi_j = \sum_h \alpha^h \xi_j^h$ be the total mass of agents of all types who are eligible to trade security j, adding up over ex-ante types. Note that ξ_j^h can depend on observable actions or unobserved states in A.

The intermediary's profit maximization problem is as follows:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} P(\mathbf{w})\, b(\mathbf{w}) \tag{A.174}$$

subject to

$$\sum_{j \in J^i} \sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} b(\mathbf{w})\, \Psi_{zj}\xi_j\theta_j \quad \leq \quad 0, \forall z = 1, \ldots, Z; i = 1, \ldots, I; \mathbf{p}^i; \mathbf{Q}^i, \tag{A.175}$$

where $\Psi = [\Psi_{zj}]_{z,j}$ is the matrix of security weights[6] for the j^{th} security in the z^{th} feasibility constraint, and J^i is the set of all securities corresponding to the i^{th} set of obstacle-to-trade constraints, and therefore will be exclusively available to agents in a particular exchange only. These constraints state that the financial intermediary must put together deals that execute all securities properly.

Market Clearing: The market-clearing conditions for contracts/lotteries are as follows:

$$\sum_h \alpha^h x^h(\mathbf{w}) = b(\mathbf{w}), \ \forall \mathbf{w}. \tag{A.176}$$

In order to define the market-clearing conditions for spot trades consistently, we need to define indicator functions ρ_{mi} whose value will be one if spot markets m are relevant to obstacle-to-trade constraints i, and zero otherwise. More formally, $\rho_{mi} = 1$ if $\mathbf{p}_m$ is part of obstacle-to-trade

[6]This extra notation is needed to capture trading structure in private information problems. See the examples of this matrix in the appendices of Kilenthong and Townsend (2014a).

constraint i, $C^h_{i,a,a'}$, and $\rho_{mi} = 0$ otherwise. The market-clearing conditions for spot trades are as follows:

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau}_{-\ell-m},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w})\, \tau_{\ell m}\rho_{mi} = 0, \forall m; \ell; i; \mathbf{p}^i, \mathbf{Q}^i, \tag{A.177}$$

where $\boldsymbol{\tau} = (\tau_{\ell m}, \boldsymbol{\tau}_{-\ell-m})$. Note that the ex-post diversity is already captured implicitly in the lottery x^h, and therefore there is no need for $\zeta^h(\omega)$ in the lottery representation. For example, as in the moral hazard case, $\zeta^h(\mathbf{q}, a) = \delta(a) f(\mathbf{q}|a)$ represents a fraction of population who took action a and received outputs $\mathbf{q}$. Under the lottery representation, the mother nature constraints ensure that these fractions are consistent with the production technology. That is, the ex-post diversity is already part of the lottery.

In addition, we again require that the markets for the rights to trade clear; that is, for each exchange $(\mathbf{p}^i, \mathbf{Q}^i)$,

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w})\, \boldsymbol{\Delta}^h_i = 0, \forall i; \mathbf{p}^i, \mathbf{Q}^i. \tag{A.178}$$

Definition A.7. A competitive equilibrium with segregated exchanges is a specification of allocation $(\mathbf{x}, \mathbf{b})$, and prices $P(\mathbf{w})$ such that

(i) for each agent of type h, $\mathbf{x}^h \in X^h$ solves (A.172) subject to (A.173), taking prices $P(\mathbf{w})$ as given;

(ii) for the financial intermediary, $\mathbf{b}$ solves (A.174) subject to (A.175), taking prices $P(\mathbf{w})$ as given;

(iii) markets for contracts/lotteries, spot trades, and rights to trade clear; that is, (A.176), (A.177) and (A.178) hold.

H.5.2 Constrained Optimal Allocations for the Generalized Model

An allocation $\mathbf{x} \equiv (\mathbf{x}^h)_h$ is attainable if $\mathbf{x}^h \in X^h$ for all h, and it satisfies the following resource constraints for securities corresponding to the i^{th} obstacle-to-trade constraint in exchange $(\mathbf{p}^i, \mathbf{Q}^i)$, for spot trades $\tau_{\ell m}$ in exchange $(\mathbf{p}^i, \mathbf{Q}^i)$, and the consistency constraints, respectively:

$$\sum_{j\in J^i} \sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w})\, \Psi_{zj}\xi_j\theta_j \leq 0, \forall z; i; \mathbf{p}^i; \mathbf{Q}^i, \tag{A.179}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau}_{-\ell-m},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w})\, \tau_{\ell m}\rho_{mi} = 0, \forall m; \ell; i; \mathbf{p}^i, \mathbf{Q}^i, \tag{A.180}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w})\, \boldsymbol{\Delta}^h_i = 0, \forall i; \mathbf{p}^i, \mathbf{Q}^i. \tag{A.181}$$

The key idea of our market-based solution is to formulate the consistency constraints as linear feasibility constraints. This formulation in fact makes the price externalities depletable, and therefore makes it possible to reach a constrained optimal allocation using a non-personalized price system as in a standard Walrasian equilibrium in contrast to a Lindahl equilibrium where prices are personalized. Technically, the linear consistency constraints correspond to market clearing conditions for the rights to trade in the competitive equilibrium.

Definition A.8. An allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_{h=1}^{H} \in X^1 \times \ldots \times X^H$ is said to be *attainable* if $\mathbf{x}^h \in X^h$ for every agent of type h, and it satisfies (A.179)-(A.181).

Let X denote the set of all attainable allocations. With finite linear weak-inequality constraints, the attainable set X is compact and convex. In addition, the assumption that the endowment is on the grids also ensures that X is nonempty.

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent type strictly better off without making any other agent type worse off. We characterize constrained optimality using the following Pareto program. Let $\lambda^h \geq 0$ be the Pareto weight of agent type h. There is no loss of generality to normalize the weights such that $\sum_h \lambda^h = 1$. A constrained Pareto optimal allocation $\mathbf{x}$ solves the following Pareto program.

Program A.3. The Pareto Program with Lotteries:

$$\max_{\mathbf{x} \in X} \sum_h \lambda^h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{w}) \tag{A.182}$$

subject to (A.179)-(A.181).

It is clear that the objective function now is linear in $\mathbf{x}$. Thus, it is continuous and weakly concave. As discussed earlier, the feasible set X is non-empty, compact, and convex. Therefore, a solution to the Pareto program for given positive Pareto weights exists and is a global maximum. The proof of the equivalence between Pareto optimal allocations and the solutions to the program is omitted for brevity (see Prescott and Townsend, 1984b, for a similar proof).

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