Useful PF Math Tools

Envelope Theorem and Comparative Statics

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1. Is there a meaningful difference between constraints and optimization frictions for the envelope theorem?

2. For which side of the Baily-Chetty formula does the envelope theorem matter?

3. If I’m interested in a comparative static for one choice variable, can I ignore other choice variables?

4. Which type of derivative should I expect to see in a comparative static? Why?
Outline

General Envelope Theorem

Baily-Chetty Envelope Theorem Application

Comparative Statics
Suppose you have an optimized value function

Two effects of a (marginal & exogenous) parameter change:

1. Direct effect on objective function/constraints
2. Indirect effect on objective function/constraints through re-optimization

FOC previously held $\Rightarrow$ 2\textsuperscript{nd} effect $= 0$ to first-order
Math Setup

Utility \[ u(x; \theta) \]

Constraint \[ g(x; \theta) = 0 \]

Indirect Utility \[ V(\theta) = \max_x u(x; \theta) \text{ s.t. } g(x; \theta) = 0 \]

Decision rule \[ x^*(\theta) \]
Solving the Initial Optimization Problem

1. Lagrangian: $\mathcal{L}(x, \lambda; \theta) = u(x; \theta) + \lambda g(x; \theta)$
   (Recall setup has equality constraint to avoid complementary slackness)

2. FOC for $x$: $\frac{\partial \mathcal{L}}{\partial x} = 0$

3. FOC for $\lambda$: $\frac{\partial \mathcal{L}}{\partial \lambda} = g(x; \theta) = 0$

4. Solution: $V(\theta) = \mathcal{L}(x^*(\theta), \lambda(\theta); \theta) = u(x^*(\theta), \lambda(\theta); \theta)$
Total differentiation w.r.t. $\theta$

\[
\frac{dV(\theta)}{d\theta} = \frac{dL(x^*(\theta), \lambda(\theta); \theta)}{d\theta} = \frac{\partial L}{\partial \theta} + \frac{\partial L}{\partial x} \frac{\partial x^*(\theta)}{\partial \theta} + \frac{\partial L}{\partial \lambda} \frac{\partial \lambda(\theta)}{\partial \theta} = 0 \text{ by } x \text{ FOC}
\]

\[
\frac{\partial L}{\partial \theta} + \frac{\partial L}{\partial x} \frac{\partial x^*(\theta)}{\partial \theta} + \frac{\partial L}{\partial \lambda} \frac{\partial \lambda(\theta)}{\partial \theta} = 0 \text{ by constraint}
\]

\[
\frac{\partial u(x^*(\theta); \theta)}{\partial \theta} + \frac{\partial g(x^*(\theta); \theta)}{\partial \theta} \lambda(\theta) = 0 \text{ by constraint}
\]

Direct effect on objective + Direct effect on constraint = Value of changing constraint

(1) (2) (3) (4)
Graphical Intuition

FOC w.r.t. $x$ satisfied wherever value function lies...

...so behavioral response $\frac{dx^*(\theta)}{d\theta}$ has no first-order effect.
Limits of the Envelope Theorem

1. **Local** statement about **first-order** effects of **marginal** changes

2. What if the FOC isn’t initially satisfied?
   
   2.1 Externalities: private FOC isn’t social FOC
   
   2.2 Internalities: choices don’t reveal preferences
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Big Picture Idea

1. Setup:
   - Agent problem: \( V(\text{UI benefits}) = \max_{\text{choices}} U(\text{choices}; \text{UI benefits}) \) s.t. private constraints
   - Govt problem: \( W = \max_{\text{UI benefits}} V(\text{UI benefits}) \) s.t. UI program budget balance

2. Proof strategy: \( \frac{dW}{d\text{UI benefits}} = 0 \) at optimum

3. Envelope theorem applied:
   - What behavioral responses you can ignore: endogenous variables the agent was already privately optimizing over
   - What behavioral responses you can't ignore: impact on UI program budget constraint the agent doesn't internalize
Step-by-Step

1. **Envelope Theorem**: UI benefit and tax changes matter for welfare only by changing private constraints

2. **Govt budget balance**: $1 in UI benefits has a (probability-weighted) $1 mechanical tax cost and possible additional costs from behavioral responses

3. **Standard agent optimization**: By definition, the value of changing a within-state budget constraint is (probability-weighted) marginal utility

4. **Putting it all together**: Combining the above describes optimal benefits
Interpreting the Final Expression

\[
\frac{u'(c_u)}{u'(c_e)} = \underbrace{1}_{\text{mechanical cost}} + \underbrace{\epsilon_{b,d}}_{\text{behavioral cost}}
\]

MRS = mechanical cost + behavioral cost

tax $ while employed to finance $1 while unemployed

Just the government in an Econ 101 optimization problem!
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Big Picture Idea

- **Setup**: Optimize objective s.t. constraints
- **Solving for choice**: Take FOC(s)
  - Optimizing agent will always satisfy FOC(s), which can depend on exogenous parameters
  - FOC with *marginal* utility determines choice *level*
- **Solving for how choice responds to parameters**: Differentiate FOC(s)
  - Differentiate FOC with utility *curvature* determines choice *responsiveness* to parameter *changes*
FOC: $g(x; \theta) = 0$
Totally differentiate and rearrange (i.e. apply IFT): $\frac{dx}{d\theta} = -\frac{\partial h}{\partial x}$
Single FOC Case with Multiple Parameters

FOC: $g(x; \theta, \gamma) = 0$

Same as before for each parameter (holding the other exogenous parameter fixed)
Multiple FOC Case

FOC 1: $g(x, y; \theta) = 0$
FOC 2: $h(x, y; \theta) = 0$

Possible strategies (that do the same thing):

1. First substitute to combine into single FOC with a single endogenous choice
2. Totally differentiate both FOCs and solve the system of equations
Aside: Solution Method Multiple FOC Case

Solving the system of differentiated FOCs with multiple parameters can be cumbersome

- **Cramer’s Rule** is a useful solution method
- See pg 4 of **David Card’s lecture notes**

Cramer’s Rule:

- Matrix system of equations: \( \mathbf{A} \mathbf{x} = \mathbf{b} \)
  - E.g. \( \mathbf{x} \) is vector of “d choices” and \( \mathbf{b} \) is vector of expressions with “d parameters”
- Formula for entry \( x_{ij} \) in row \( i \) and column \( j \) of \( \mathbf{x} \):

\[
x_{ij} = \frac{\det(A_{ij})}{\det(A)}
\]

where \( A_{ij} \) replaces column \( i \) of \( \mathbf{A} \) with column \( j \) of \( \mathbf{b} \)