Useful PF Math Tools

Envelope Theorem and Comparative Statics

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Questions you should be able to answer at the end

1. Is there a meaningful difference between constraints and optimization frictions for the envelope theorem?
2. For which side of the Baily-Chetty formula does the envelope theorem matter?
3. If I'm interested in a comparative static for one choice variable, can I ignore other choice variables?
4. Which type of derivative should I expect to see in a comparative static? Why?

Outline

Big Picture Idea

- Suppose you have an optimized value function
- Two effects of a (marginal & exogenous) parameter change:
    1. Direct effect on objective function/constraints
    2. Indirect effect on objective function/constraints through re-optimization
- FOC previously held $\Rightarrow$ 2nd effect $= 0$ to first-order

Math Setup

Utility	$u(x;\theta)$
Constraint	$g(x;\theta) = 0$
Indirect Utility	$V(\theta) = \max_x u(x;\theta)$ s.t. $g(x;\theta) = 0$
Decision rule	$x^*(\theta)$

Solving the Initial Optimization Problem

1. Lagrangian: $\mathcal{L}(x, \lambda; \theta) = u(x; \theta) + \lambda g(x; \theta)$

   (Recall setup has equality constraint to avoid complementary slackness)

2. FOC for x: $\frac{\partial \mathcal{L}}{\partial x} = 0$

3. FOC for λ: $\frac{\partial \mathcal{L}}{\partial \lambda} = g(x; \theta) = 0$

4. Solution: $V(\theta) = \mathcal{L}(x^*(\theta), \lambda(\theta); \theta) = u(x^*(\theta), \lambda(\theta); \theta)$

Total differentiation w.r.t. θ

$$\frac{\mathrm{d}V(\theta)}{\mathrm{d}\theta} = \frac{\mathrm{d}\mathcal{L}(x^*(\theta), \lambda(\theta); \theta)}{\mathrm{d}\theta} \tag{1}$$

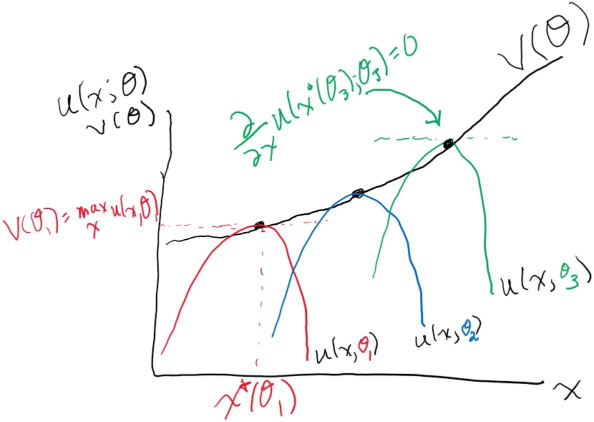
$$= \frac{\partial \mathcal{L}}{\partial \theta} + \underbrace{\frac{\partial \mathcal{L}}{\partial x}}_{=0 \text{ by } x \text{ FOC}} \frac{\partial x^*(\theta)}{\partial \theta} + \underbrace{\frac{\partial \mathcal{L}}{\partial \lambda}}_{=0 \text{ by constraint}} \frac{\partial \lambda(\theta)}{\partial \theta} \tag{2}$$

$$= \underbrace{\frac{\partial u(x^*(\theta); \theta)}{\partial \theta}}_{\text{direct effect on objective}} + \underbrace{\frac{\partial g(x^*(\theta); \theta)}{\partial \theta}}_{\text{direct effect on constraint}} \underbrace{\lambda(\theta)}_{\text{value of changing constraint}} \tag{3}$$

$$\tag{4}$$

Graphical Intuition

FOC w.r.t. x satisfied wherever value function lies...



...so behavioral response $\frac{\mathrm{d}x^*(\theta)}{\mathrm{d}\theta}$ has no first-order effect

Limits of the Envelope Theorem

1. **Local** statement about **first-order** effects of **marginal** changes
2. What if the FOC isn't initially satisfied?
   2.1 Externalities: private FOC isn't social FOC
   2.2 Internalities: choices don't reveal preferences

Outline

Big Picture Idea

1. Setup:
   - Agent problem: V(UI benefits)=$\max_{\text{choices}} U$(choices; UI benefits) s.t. private constraints
   - Govt problem: $W = \max_{\text{UI benefits}} V$(UI benefits) s.t. UI program budget balance
2. Proof strategy: $\frac{\mathrm{d}W}{\mathrm{d}\text{ UI benefits}} = 0$ at optimum
3. Envelope theorem applied:
   - What behavioral responses you **can** ignore: endogenous variables the agent was already privately optimizing over
   - What behavioral responses you **can't** ignore: impact on UI program budget constraint the agent doesn't internalize

Step-by-Step

1. **Envelope Theorem:** UI benefit and tax changes matter for welfare only by changing private constraints
2. **Govt budget balance:** $1 in UI benefits has a (probability-weighted) $1 mechanical tax cost *and* possible additional costs from behavioral responses
3. **Standard agent optimization:** By definition, the value of changing a within-state budget constraint is (probability-weighted) marginal utility
4. **Putting it all together:** Combining the above describes optimal benefits

Interpreting the Final Expression

$$\underbrace{\frac{u'(c_u)}{u'(c_e)}}_{MRS} = \underbrace{\overbrace{1}^{\text{mechanical cost}} + \overbrace{\epsilon_{b,d}}^{\text{behavioral cost}}}_{\text{tax \$ while employed to finance \$1 while unemployed}}$$

Just the government in an Econ 101 optimization problem!

Outline

General Envelope Theorem

Baily-Chetty Envelope Theorem Application

Comparative Statics

Big Picture Idea

- **Setup:** Optimize objective s.t. constraints
- **Solving for choice:** Take FOC(s)
  - Optimizing agent will always satisfy FOC(s), which can depend on exogenous parameters
  - FOC with *marginal* utility determines choice *level*
- **Solving for how choice responds to parameters:** Differentiate FOC(s)
  - Differentiate FOC with utility *curvature* determines choice *responsiveness* to parameter *changes*

Single FOC Case w/ Single Parameter

FOC: $g(x;\theta) = 0$

Totally differentiate and rearrange (i.e. apply IFT): $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{\frac{\partial h}{\partial \theta}}{\frac{\partial h}{\partial x}}$

Single FOC Case with Multiple Parameters

FOC: $g(x; \theta, \gamma) = 0$

Same as before for each parameter (holding the other exogenous parameter fixed)

Multiple FOC Case

FOC 1: $g(x, y; \theta) = 0$

FOC 2: $h(x, y; \theta) = 0$

Possible strategies (that do the same thing):

1. First substitute to combine into single FOC with a single endogenous choice
2. Totally differentiate *both* FOCs and solve the system of equations

Aside: Solution Method Multiple FOC Case

Solving the system of differentiated FOCs with multiple parameters can be cumbersome

- Cramer's Rule is a useful solution method
- See pg 4 of David Card's lecture notes

Cramer's Rule:

- Matrix system of equations: $\mathbf{Ax} = \mathbf{b}$
  - E.g. $\mathbf{x}$ is vector of "$\mathrm{d}$choices" and $\mathbf{b}$ is vector of expressions with "$\mathrm{d}$parameters"
- Formula for entry x_{ij} in row i and column j of $\mathbf{x}$:

$$x_{ij} = \frac{det(\mathbf{A}_{ij})}{det(\mathbf{A})}$$

where $\mathbf{A}_{ij}$ replaces column i of $\mathbf{A}$ with column j of $\mathbf{b}$