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Mediated Collusion: Online Appendix

Proof of Lemma 1

To see that $\chi(p)$ is well defined, note that the LHS of (12) is strictly concave in χ over the range $(0, \hat{\pi} + x)$, and attains its maximum at $\chi = 2x$. Moreover, for $\chi = \hat{\pi}$, the LHS of (12) is no more than the RHS for all $p \in [\hat{\pi}, 1]$. Hence, as long as

$$\frac{2x \log(2x) + (\hat{\pi} - x) \log(\hat{\pi} - x)}{x + \hat{\pi}} \geq \log p + \frac{2x \log(\hat{\pi}) + (\hat{\pi} - x) \log(x)}{x + \hat{\pi}} - \log(\hat{\pi}) \quad \forall p \in [\hat{\pi}, 1], \tag{18}$$

equation (12) admits a unique solution $\chi \in [2x, \hat{\pi} + x]$. Since the LHS of (18) is independent of p and the RHS is increasing in p, it suffices that (18) holds for $p = 1$. In turn, this holds by the definition of $\hat{\pi}$. Finally, that $\chi(p)$ is decreasing and differentiable, with derivative satisfying (13) on $[\hat{\pi}, 1)$, follows from the implicit function theorem.

Proof of Lemma 2

For all $p \in [\hat{\pi}, 1]$, define $\alpha(p) = (\chi(p) - 2x)p$. Consider the differential equation

$$h'(p)\alpha(p) + \alpha'(p)h(p) = -\chi(p)h(p), \quad \text{with } h(\hat{\pi}) = 1. \tag{19}$$

The solution to (19) is $h(p) = \exp\left(-\int_{\hat{\pi}}^{p} \gamma(p')dp'\right)$, with $\gamma(p) = (\chi(p) + \alpha'(p))/\alpha(p)$.

We now construct density function $f_i(p)$. For $p \in [\hat{\pi}, 1)$, $f_i(p) = K \times h(p)$ for some constant $K > 0$ to be determined shortly. For $p \in [2x, \pi)$,

$$f_i(p) = \frac{1}{p^2} \int_{\chi^{-1}(p)}^{1} \chi(p')Kh(p')dp' = \frac{1}{p^2} \int_{\chi^{-1}(p)}^{1} \chi(p')f_i(p')dp'.$$

(Note that $h(p)$ is defined over $[\chi^{-1}(p), 1)$ for all $p \in [2x, \pi)$, since $\chi^{-1}(p) \in [\hat{\pi}, 1)$ for all p in this range). Finally, for $p < 2x$, $f_i(p) = 0$.

We now show that $f_i(p)$ satisfies (14). By construction, $f_i(p)$ satisfies (14) for all $p < \hat{\pi}$. Next, note that $f_i(p)$ solves (19) for $p \in [\hat{\pi}, 1)$, with $f_i(\hat{\pi}) = K$. Hence, for all $p \in [\hat{\pi}, 1)$,

$$\int_{p}^{1} (f_i'(p')\alpha(p') + \alpha'(p)f_i(p'))dp' = -\alpha(p)f_i(p) = -\int_{p}^{1} \chi(p')f_i(p')dp'$$
$$\iff f_i(p) = \frac{1}{(\chi(p) - 2x)p} \int_{p}^{1} \chi(p')f_i(p')dp',$$

where the first line uses $\alpha(1) = 0$. We now pin down constant $K > 0$. Since $F_i(\cdot)$ should satisfy $\int_{2x}^{1} f_i(p)dp = F_i(1) - F_i(2x) = 1$, we have that K is the solution to:

$$1 = K \left[\int_{2x}^{\hat{\pi}} \frac{1}{p^2} \int_{\chi^{-1}(p)}^{1} \chi(p')h(p')dp'dp + \int_{\hat{\pi}}^{1} \frac{1}{(\chi(p) - 2x)p} \int_{p}^{1} \chi(p')h(p')dp'dp \right].$$

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Proof of Lemma 3

Without loss, let $p < p'$. We consider two cases: $p \in [2x, \hat{\pi})$ and $p \in [\hat{\pi}, 1]$.

Suppose that $p \in [2x, \hat{\pi})$. We have

$$\begin{aligned}
F(p, p') &= \int_{\hat{p} \leq p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} = \int_{\chi(p')}^{p} \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi(\tilde{p})}{\hat{p}^2} f_i(\tilde{p}) d\tilde{p} d\hat{p} \\
&= \int_{\chi^{-1}(p)}^{p'} \int_{\chi(\tilde{p})}^{p} \frac{\chi(\tilde{p})}{\hat{p}^2} d\hat{p} f_i(\tilde{p}) d\tilde{p} = \int_{\chi^{-1}(p)}^{p'} \left(1 - \frac{\chi(\tilde{p})}{p}\right) f_i(\tilde{p}) d\tilde{p} \\
&= \int_{\tilde{p} \leq p'} F_j(p|\tilde{p}) f_i(\tilde{p}) d\tilde{p} = F(p', p),
\end{aligned}$$

where the second equality follows from (15) and the fact that $F(p'|\hat{p}) = 0$ for all $\hat{p} < \chi(p')$; the third equality follows from reversing the order of integration; and the fifth equality again follows from (15).

Now suppose that $p \in [\hat{\pi}, 1]$. We have

$$\begin{aligned}
F(p, p') &= \int_{\hat{p} \leq p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} \\
&= \int_{\hat{p} \leq p} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_{\hat{p} \leq p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p} \\
&= F(p, p) + \int_{\hat{p} \leq p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p}. \qquad (20)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
F(p', p) &= \int_{\hat{p} \leq p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\
&= \int_{\hat{p} \leq p} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_{p}^{p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\
&= F(p, p) + \int_{p}^{p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\
&= F(p, p) + \int_{p}^{p'} \left(1 - \frac{\chi(\hat{p})}{p}\right) f_i(\hat{p}) d\hat{p}, \qquad (21)
\end{aligned}$$

where the last equality uses (15). Since $F_j(p'|\hat{p}) = 0$ for all $\hat{p} < \chi(p')$ and $F_j(p|\hat{p}) = 0$ for all $\hat{p} < \chi(p)$, and again using (15), we have

$$\begin{aligned}
\int_{\hat{p}\leq p} \left(F_j\left(p'|\hat{p}\right) - F_j\left(p|\hat{p}\right)\right) f_i\left(\hat{p}\right) d\hat{p} &= \int_{\chi(p')}^{\chi(p)} F_j\left(p'|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} + \int_{\chi(p)}^{p} \left(F_j\left(p'|\hat{p}\right) - F_j\left(p|\hat{p}\right)\right) f_i\left(\hat{p}\right) d\hat{p} \\
&= \int_{\chi(p')}^{\chi(p)} \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} f_i\left(\tilde{p}\right) d\tilde{p} d\hat{p} + \int_{\chi(p)}^{p} \int_{p}^{p'} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} f_i\left(\tilde{p}\right) d\tilde{p} d\hat{p} \\
&= \int_{p}^{p'} \int_{\chi(\tilde{p})}^{\chi(p)} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} d\hat{p} f_i\left(\tilde{p}\right) d\tilde{p} + \int_{p}^{p'} \int_{\chi(p)}^{p} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} d\hat{p} f_i\left(\tilde{p}\right) d\tilde{p} \\
&= \int_{p}^{p'} \left(1 - \frac{\chi\left(\tilde{p}\right)}{p}\right) f_i\left(\tilde{p}\right) d\tilde{p}.
\end{aligned}$$

Together with (20) and (21), we have $F\left(p, p'\right) = F\left(p', p\right)$.

Proof of Lemma 4

The lemma follows from checking the various cases in (15). In particular, if $p \geq \hat{\pi}$ and $p' \in [\chi(p), p)$, then $F_j\left(p|p\right) = 1 - \frac{\chi(p)-2x}{p}$ and $F_j\left(p'|p\right) = 1 - \frac{\chi(p)}{p'}$, and hence

$$\begin{aligned}
IC(p, p'; F) &= p\left(1 - \frac{1}{2}\left(F_j^-\left(p|p_i = p\right) + F_j\left(p|p_i = p\right)\right)\right) - p'\left(1 - F_j\left(p'|p_i = p\right)\right) + x \\
&= p\left(\frac{\chi\left(p\right) - x}{p}\right) - p'\left(\frac{\chi\left(p\right)}{p'}\right) + x = 0.
\end{aligned}$$

If $p \geq \hat{\pi}$ and $p' < \chi(p)$, then $F_j\left(p'|p\right) = 0$, and hence

$$IC(p, p'; F) = p\left(\frac{\chi\left(p\right) - x}{p}\right) - p' + x = \chi\left(p\right) - p' > 0.$$

Finally, if $p < \hat{\pi}$ and $p' < p$, then $F_j\left(p|p\right) = F_j\left(p'|p\right) = 0$, and hence $IC(p, p'; F) = x > 0$.

Now suppose that $x \geq 1/3$. If $p \in [2x, \hat{\pi})$ and $p' > p$, then

$$IC(p, p'; F) = p - p'(1 - F_j(p'|p)) + x \geq 3x - 1 \geq 0,$$

where the first inequality uses $p \geq 2x$ and $p' \leq 1$, and the second uses $x \geq 1/3$. If $p \geq \hat{\pi}$ and $p' > p$, then

$$\begin{aligned}
IC(p, p'; F) &= p\frac{\chi(p) - x}{p} - p'\left(\frac{\chi(p) - 2x}{p} - \frac{1}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p}\right) + x \\
&= \chi(p)\frac{p - p'}{p} + 2x\frac{p'}{p} + \frac{p'}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} \\
&\geq \hat{\pi}\frac{\hat{\pi} - 1}{\hat{\pi}} + 2x + \frac{p'}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} > 0,
\end{aligned}$$

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where the weak inequality uses $p \geq \hat{\pi}$, $p' \in (p, 1]$, and $\chi(p) \leq \hat{\pi}$, and the strict inequality follows as $\hat{\pi} > 2x$ and hence $\hat{\pi} - 1 + 2x > 4x - 1 > 0$ (since $x \geq 1/3$).

Proof of Lemma 5

For each $p, q \in [0,1]^2$ and $p' < p$, define

$$\phi(p, q, p') = p - p' + x - \frac{p}{2}\left(\mathbf{1}_{q \leq p} + \mathbf{1}_{q<p}\right) + p'\mathbf{1}_{q<p'}.$$

Intuitively, $\phi(p, q, p')$ is the payoff loss incurred by a bidder who deviates from p to p' when the opponent bids q. Note that

$$\begin{aligned}
IC(p, p'; F) &= \int_q \phi(p, q, p') dF_j(q|p) \qquad \forall p > p', \\
G(p, p; \Lambda, \mu) &= -\int_{p' \leq p} \phi(p, p, p') d\Lambda(p'|p) + \mu - p \qquad \forall p, \qquad \text{and} \\
G(p, q; \Lambda, \mu) &= -\int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) - \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) + 2(\mu - q) \qquad \forall p > q.
\end{aligned}$$

Now let F be feasible for **P**, and let (Λ, μ) be feasible for **D**. Note that

$$\begin{aligned}
0 &\leq \int_{p, q \leq p} G(p, q; \Lambda, \mu)\, dF(p, q) \\
&= -\int_{p, q \leq p} \left(\int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) + \mathbf{1}_{q<p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q)\right) dF(p, q) \\
&\quad + \int_{p, q \leq p} \left(2(\mu - q) - \mathbf{1}_{q=p}(\mu - p)\right) dF(p, q) \\
&= -\int_{p, q \leq p} \left(\int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) + \mathbf{1}_{q<p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q)\right) dF(p, q) + \mu - \Pi(F),
\end{aligned} \tag{22}$$

where the last equality uses $\int_{p,q \leq p} (2 - \mathbf{1}_{q=p}) dF(p, q) = 1$ (by symmetry of F) and $\int_{p,q \leq p} (2q - \mathbf{1}_{q=p} p) dF(p, q) = \Pi(F)$.

Since F is symmetric,

$$\begin{aligned}
\int_{p, q \leq p} \mathbf{1}_{q<p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q)\, dF(p, q) &= \int_{p, q<p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q)\, dF(p, q) \\
&= \int_{p, q>p} \int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p)\, dF(p, q).
\end{aligned}$$

Using this in (22), we get

$$
\begin{aligned}
0 &\leq -\int_{p,q}\int_{p'\leq p}\phi(p,q,p')d\Lambda\left(p'|p\right)dF(p,q)+\mu-\Pi(F)\\
&=-\int_{p}\int_{q}\int_{p'\leq p}\phi(p,q,p')d\Lambda\left(p'|p\right)dF(q|p)dF_i(p)+\mu-\Pi(F)\\
&=-\int_{p}\int_{p'\leq p}\int_{q}\phi(p,q,p')dF(q|p)d\Lambda\left(p'|p\right)dF_i(p)+\mu-\Pi(F)\\
&=-\int_{p}\int_{p'\leq p}IC(p,p';F)d\Lambda\left(p'|p\right)dF_i(p)+\mu-\Pi(F)\\
&\leq \mu-\Pi(F),
\end{aligned}
\tag{23}
$$

where the first two equalities follow from Fubini's theorem, which applies as ϕ, Λ, and F are bounded and measurable. This establishes part (i).

Suppose next that F and (Λ,μ) satisfy the conditions in part (ii). Then, the inequalities in (22) and (23) hold with equality, and so $\Pi(F)=\mu$. By part (i), it follows that F is optimal for **P**, and (Λ,μ) is optimal for **D**.

Proof of Lemma 6

To show that $\Lambda\in\mathcal{M}$, it suffices to show that Λ is bounded and measurable: given this, $\Lambda\left(\cdot|p\right)$ is a Borel measure on $\mathcal{B}\left([0,1]\right)$ by construction. Boundedness follows because, for each $p\in[0,1]$, we have

$$
\begin{aligned}
\Lambda([0,1]\left|p\right) &\leq \int_{\chi(p)}^{\hat{\pi}}\frac{2}{q}dq+\int_{\hat{\pi}}^{p}\frac{2\left(\chi\left(q\right)-x\right)\left(-\chi'\left(q\right)\right)}{x\chi(q)}dq\\
&\leq \int_{\chi(p)}^{\hat{\pi}}\frac{2}{q}dq+\int_{\hat{\pi}}^{p}\frac{2(-\chi'(q))}{x}dq\\
&\leq 2(\log(\hat{\pi})-\log(2x))+\frac{2}{x}(\hat{\pi}-2x),
\end{aligned}
$$

where the first inequality is by definition of λ, the second follows since $\chi\left(q\right)\geq x$, and the third follows since $\chi\left(p\right)\geq 2x$ and $\chi\left(\hat{\pi}\right)-\chi\left(p\right)\leq\hat{\pi}-2x$. Measurability follows because $\Lambda([p',p'']\left|p\right)$ is continuous on $[0,1]^3$ in the weak topology, which suffices for continuity on $[0,1]\times\mathcal{B}\left([0,1]\right)$.

For the proof that G satisfies the desired properties, we ease notation by writing G as a function of p,q only, suppressing the arguments $\Lambda,\hat{\pi}$. We use the following equations, which follow immediately from the definitions of G and Λ.

1. For $p\geq\hat{\pi}$,

$$
G\left(p,p\right)=\int_{\chi(p)}^{p}\left(-\frac{p}{2}-x+p'\right)\lambda\left(p'|p\right)dp'+\hat{\pi}-p. \tag{24}
$$

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2. For $p \geq q \geq \hat{\pi}$,

$$G(p,q) = \int_{\chi(p)}^{p} \left(-x + p'\mathbf{1}_{\{p' \leq q\}}\right) \lambda(p'|p)\, dp' + \int_{\chi(q)}^{q} (-q - x + q')\, \lambda(q'|q)\, dq' + 2(\hat{\pi} - q). \tag{25}$$

3. For $p \geq \hat{\pi} \geq q \geq \chi(p)$,

$$G(p,q) = \int_{\chi(p)}^{p} \left(-x + p'\mathbf{1}_{\{p' \leq q\}}\right) \lambda(p'|p)\, dp' + 2(\hat{\pi} - q). \tag{26}$$

4. For $p \geq \hat{\pi} \geq \chi(p) \geq q$,

$$G(p,q) = 2(\hat{\pi} - q). \tag{27}$$

5. For $p \geq \hat{\pi}$ and $q \in [\hat{\pi}, p)$,

$$\lambda(q|p) = \frac{(\chi(q) - x)(-\chi'(q))\, \lambda(\chi(q)|p)}{x}. \tag{28}$$

We prove the lemma through a series of claims.

Claim 1. For any sequence $(p_n, q_n) \to (\hat{\pi}, \hat{\pi})$ such that $p_n \geq \hat{\pi} \geq q_n$ for all n, we have $\lim_{n\to\infty} G(p_n, q_n) = 0$.

Proof. This follows from two observations. First, $G(\hat{\pi}, \hat{\pi}) = 0$, by (24) and $\chi(\hat{\pi}) = \hat{\pi}$. Second, for any $\pi \in [0,1]$ and any sequence $(p_n, q_n) \to (\pi, \pi)$ such that $p_n \geq \pi \geq q_n$ and $p_n \neq q_n$ for all n, we have $\lim_{n\to\infty} G(p_n, q_n) = 2G(\pi, \pi)$.

Claim 2. $G(p,q) = 0$ for all p, q such that $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$.

Proof. By Claim 1, it suffices to show that $\partial G(p,q)/\partial p = \partial G(p,q)/\partial q = 0$ whenever $p > q$ and $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$. By (26),

$$\begin{aligned} \frac{\partial G(p,q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\, \lambda(\chi(p)|p), \quad \text{and} \\ \frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - 2. \end{aligned}$$

By (17) and (28), both of these derivatives equal 0.

Claim 3. $G(p,q) = 0$ for all p, q such that $1 > p \geq q \geq \hat{\pi}$.

Proof. Suppose first that $p > q$. By Claim 1, it suffices to show that $\partial G(p,q)/\partial p = \partial G(p,q)/\partial q = 0$ whenever $1 > p > q \geq \hat{\pi}$. By (25),

$$\begin{aligned} \frac{\partial G(p,q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\, \lambda(\chi(p)|p), \quad \text{and} \\ \frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^{q} \lambda(q'|q)\, dq' - x\lambda(q|q) + (\chi(q) - x + q)(-\chi'(q))\, \lambda(\chi(q)|q) - 2. \end{aligned}$$

By (17) and (28), $\partial G(p,q)/\partial p = 0$. To show that $\partial G(p,q)/\partial q = 0$, note that $G(q, \chi(q)) = 0$

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by Claim 2, so (26) gives

$$\begin{aligned} G(q,\chi(q)) &= \int_{\chi(q)}^{q} \left(-x + q'\mathbf{1}_{\{q'\le\chi(q)\}}\right)\lambda(q'|q)\,dq' + 2(\hat{\pi}-q) = 0 \quad \iff \\ \int_{\chi(q)}^{q}\lambda(q'|q)\,dq' &= 2\frac{\hat{\pi}-\chi(q)}{x}. \end{aligned} \tag{29}$$

Note also that

$$-x\lambda(q|q) + (\chi(q)-x)(-\chi'(q))\lambda(\chi(q)|q) = 0, \tag{30}$$

since $\partial G(q,q')/\partial q = 0$ for all $q > q' \ge \chi(q)$ by Claim 1. In total, we have

$$\begin{aligned} \frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^{q}\lambda(q'|q)\,dq' - x\lambda(q|q) + (\chi(q)-q-x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\ &= q\lambda(q|p) - 2\frac{\hat{\pi}-\chi(q)}{x} - x\lambda(q|q) + (\chi(q)-q-x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\ &= -2\frac{\hat{\pi}-\chi(q)}{x} + q\left(\lambda(q|p) + \chi'(q)\lambda(\chi(q)|q)\right) - 2 \\ &= -2\frac{\hat{\pi}-\chi(q)}{x} + q\left(\frac{2(\chi(q)-x)(-\chi'(q))}{x\chi(q)} + \chi'(q)\frac{2}{\chi(q)}\right) - 2 \\ &= 2\left(-\frac{\hat{\pi}-\chi(q)}{x} - \frac{\chi(q)-2x}{x}q\frac{\chi'(q)}{\chi(q)} - 1\right) \\ &= 2\left(-\frac{\hat{\pi}-\chi(q)}{x} - \frac{\chi(q)-2x}{x}\frac{\chi(q)-(\hat{\pi}+x)}{\chi(q)-2x} - 1\right) = 0, \end{aligned}$$

where the second line is by (29), the third line is by (30), the fourth line uses $\lambda(q|p) = \frac{2(\chi(q)-x)(-\chi'(q))}{x\chi(q)}$ and $\lambda(\chi(q)|q) = \frac{2}{\chi(q)}$ (by (17)), the fifth line collects terms, and the sixth line is by (13). Finally, the claim also holds when $p = q$, as $\lim_{q\uparrow p} G(p,q) = 2G(p,p)$ for all p.

Claim 4. $G(p,q) > 0$ if either $p \in (\hat{\pi},1), q < \chi(p)$ or $p < \hat{\pi}, q \le p$.

Proof. If $p \in (\hat{\pi},1)$ and $q < \chi(p)$, then $\partial G(p,q)/\partial q = -2 < 0$ by (27). Since $G(p,\chi(p)) = 0$ by Claim 2, this implies that $G(p,q) > 0$.

If $p < \hat{\pi}$ and $q \le p$, then $d\Lambda(p'|p) = d\Lambda(q'|q) = 0$ for all $p' < p, q' < q$. Hence, $G(p,q) = 2(\hat{\pi}-q) > 0$.

Claims 1–4 establish the desired conclusion when $p < 1$. The $p = 1$ case follows by continuity, since $\chi(p)$ and $G(p,q)$ are continuous in p for all $q < 1$.

Proof of Proposition 7

For any positive integer $N > 1/x$, let $\mathcal{F}^N$ denote the set of all symmetric cdfs on $\mathbf{N} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\right\}^2$. Let $\mathbf{P}^N$ denote the primal program with $\mathcal{F}^N$ in place of $\mathcal{F}$. Note that in $\mathbf{P}^N$ firms can still contemplate deviations to arbitrary prices, so $\mathbf{P}^N$ is a strictly more constrained program than $\mathbf{P}$. Let π^N and π denote the values of $\mathbf{P}^N$ and $\mathbf{P}$, respectively. Let

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$\mathcal{F}_{IC} \subset \mathcal{F}$ and $\mathcal{F}_{IC}^{N} \subset \mathcal{F}^{N}$ denote the set of symmetric cdfs that satisfy incentive compatibility in $\mathbf{P}$ and $\mathbf{P}^N$. Note that $\mathcal{F}_{IC}^{N} = \mathcal{F}^{N} \cap \mathcal{F}_{IC}$. We show that there exists a solution to $\mathbf{P}^N$, and that in every such solution the highest winning bid equals 1 with positive probability.

Let f be an optimal pmf in $\mathbf{P}^N$. Without loss, assume that f is symmetric. Such an f exists as $\mathcal{F}_{IC}^{N}$ is compact and profits are continuous in f. Let $\bar{p}$ denote the highest bid in the support of the marginal of f. Suppose toward a contradiction that $\bar{p} < 1$. Let $\beta = f(\bar{p}, \bar{p})$ and $\gamma_p = f(\bar{p}, p)$ for $p \in [0, \bar{p})$. Let $\gamma = \sum_{p<\bar{p}} \gamma_p$ and $\tilde{\gamma}_p = \sum_{p \le p' < \bar{p}} \gamma_{p'}$ (and let $\tilde{\gamma}_{\bar{p}} = 0$).

Lemma 7 *$\beta > 0$.*

Proof. Suppose that $\beta = 0$. Then $\bar{p}$ is never a winning bid. Let $p^* < \bar{p}$ denote the highest bid that ever wins. Consider the bid distribution that everywhere replaces all bids above p^* with bids of p^*. Clearly, this variation leaves profit unchanged. We argue that it remains an equilibrium.

First, conditional on each recommended bid, the distribution of the opponent's bid has shifted down. This makes any upward deviation weakly less profitable. So upward IC still holds.

Next, if one's recommended bid is below p^*, the payoff from following the recommendation as well as from any downward deviation are unchanged. So downward IC holds for recommendations below p^*.

It remains to establish downward IC for a recommended bid of p^*. Let Pr denote probability under the original distribution, and $\widehat{\Pr}$ denote probability under the new distribution. We must show that, for all $p < p^*$,

$$\frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*)\, p^* \ge \widehat{\Pr}(p_i = p^* \wedge p_j \ge p)\, p - \widehat{\Pr}(p_i = p^*)\, x. \tag{31}$$

First, note that

$$\begin{aligned}
& \frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*) \\
= \;& \frac{1}{2}\left(\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*) + \Pr(p_i > p^* \wedge p_j = p^*)\right) \\
= \;& \frac{1}{2}\left(\Pr(p_i = p_j = p^*) + 2\Pr(p_i = p^* \wedge p_j > p^*)\right) \\
= \;& \frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*), \qquad (32)
\end{aligned}$$

where the second equality uses symmetry. Second, note that

$$\widehat{\Pr}(p_i = p^* \wedge p_j \ge p) = \Pr(p_i = p^* \wedge p_j \ge p) + \Pr(p_i > p^* \wedge p_j \ge p) \quad \text{and} \tag{33}$$

$$\widehat{\Pr}(p_i = p^*) = \Pr(p_i = p^*) + \Pr(p_i > p^*). \tag{34}$$

Third, by $IC_{p^*,p}$, we have

$$\left(\frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*)\right) p^* \ge \Pr(p_i = p^* \wedge p_j \ge p)\, p - \Pr(p_i = p^*)\, x,$$

and by $IC_{p',p}$ for all $p' > p^*$, we have

$$0 \geq \Pr(p_i > p^* \wedge p_j \geq p)\, p - \Pr(p_i > p^*)\, x.$$

Adding these inequalities, we have

$$\begin{aligned}&\left(\frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*)\right) p^* \\ \geq\ & (\Pr(p_i = p^* \wedge p_j \geq p) + \Pr(p_i > p^* \wedge p_j \geq p))\, p - (\Pr(p_i = p^*) + \Pr(p_i > p^*))\, x. \quad (35)\end{aligned}$$

Finally, (32), (33), (34), and (35) imply (31). ■

We now construct another distribution $\hat{f}$, which we will show is an equilibrium bid distribution that yields strictly higher profit than f. This will establish the desired contradiction, completing the proof. First, note that $2(\beta+\gamma)x \geq \beta\bar{p}$, by $IC_{\bar{p},\bar{p}}$. Let $\eta = 1/N < x$, and fix $\rho \in \left(0, \min\left\{\frac{\bar{p}}{\bar{p}+\eta}, \frac{\bar{p}+\eta}{\bar{p}+2\eta}\right\}\right)$. Define

$$\hat{f}(p,q) = \begin{cases} f(p,q) & \text{if } \max\{p,q\} < \bar{p} \text{ or } \max\{p,q\} > \bar{p}+\eta \\ (1-\rho) f(p,q) & \text{if } \min\{p,q\} < \bar{p} = \max\{p,q\} \\ \rho f(\min\{p,q\}, \bar{p}) & \text{if } \min\{p,q\} < \bar{p}, \max\{p,q\} = \bar{p}+\eta \\ \left(1 - \frac{\bar{p}+2\eta}{\bar{p}+\eta}\rho\right)\beta & \text{if } p = q = \bar{p} \\ \frac{\eta\rho}{\bar{p}+\eta}\beta & \text{if } \min\{p,q\} = \bar{p}, \max\{p,q\} = \bar{p}+\eta \\ \frac{\bar{p}\rho}{\bar{p}+\eta}\beta & \text{if } p = q = \bar{p}+\eta \end{cases}.$$

Since $\beta > 0$, industry profit under $\hat{f}$ exceeds that under f. Thus, it remains to verify incentive compatibility. Define

$$\begin{aligned}\widehat{IC}_{p,p'} &= \left(\sum_{\tilde{p}>p}\hat{f}(\tilde{p},p) + \frac{1}{2}\hat{f}(p,p)\right)p - \left(\sum_{\tilde{p}\geq p'}\hat{f}(\tilde{p},p)\right)p' + \sum_{\tilde{p}}\hat{f}(\tilde{p},p)\,x \quad \text{and} \\ IC_{p,p'} &= \left(\sum_{\tilde{p}>p}f(\tilde{p},p) + \frac{1}{2}f(p,p)\right)p - \left(\sum_{\tilde{p}\geq p'}f(\tilde{p},p)\right)p' + \sum_{\tilde{p}}f(\tilde{p},p)\,x \geq 0.\end{aligned}$$

Lemma 8 *Downward IC from $\bar{p}+\eta$ is satisfied: $\widehat{IC}_{\bar{p}+\eta,p} \geq 0$ for all $p \leq \bar{p}+\eta$.*

Proof. First,

$$\begin{aligned}\widehat{IC}_{\bar{p}+\eta,\bar{p}+\eta} &= \frac{1}{2}\frac{\bar{p}\rho}{\bar{p}+\eta}\beta(\bar{p}+\eta) - \frac{\bar{p}\rho}{\bar{p}+\eta}\beta(\bar{p}+\eta) + \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\gamma\right)x \\ &= \rho\left(-\frac{1}{2}\beta\bar{p} + (\beta+\gamma)x\right) \geq 0,\end{aligned}$$

where the inequality holds because $2(\beta+\gamma)x \geq \beta\bar{p}$.

Second, for $p \in (\bar{p}, \bar{p}+\eta)$, it is clear that $\widehat{IC}_{\bar{p}+\eta,p} \geq \widehat{IC}_{\bar{p}+\eta,\bar{p}+\eta}$, and hence $\widehat{IC}_{\bar{p}+\eta,p} \geq 0$.

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Third, for $p \leq \bar{p}$, we have

$$\begin{aligned}\widehat{IC}_{\bar{p}+\eta,p} &= \frac{1}{2}\frac{\bar{p}\rho}{\bar{p}+\eta}\beta\left(\bar{p}+\eta\right) - \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\tilde{\gamma}_p\right)p \\ &\quad + \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\gamma\right)x \\ &= \rho\left(\frac{1}{2}\beta\bar{p} - \left(\beta+\tilde{\gamma}_p\right)p + \left(\gamma+\beta\right)x\right) \geq 0,\end{aligned}$$

where the inequality holds by $IC_{\bar{p},p}$. ■

Lemma 9 *Downward IC from $\bar{p}$ is satisfied: $\widehat{IC}_{\bar{p},p} \geq 0$ for $p \leq \bar{p}$.*

Proof. By definition,

$$\begin{aligned}\widehat{IC}_{\bar{p},p} &= \left(\frac{1}{2}\beta - \frac{1}{2}\frac{\rho\bar{p}\beta}{\bar{p}+\eta} - \frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta\right)\bar{p} \\ &\quad - \left(\beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2\frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta + \left(1-\rho\right)\tilde{\gamma}_p\right)p \\ &\quad + \left(\left(1-\rho\right)\gamma + \beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2\frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta\right)x \\ &= \left(1-\rho\right)\left(\frac{1}{2}\beta\bar{p} - \left(\beta+\tilde{\gamma}_p\right)p + \left(\gamma+\beta\right)x\right) \\ &\quad + \rho\left(\left(\frac{1}{2}\beta - \frac{1}{2}\frac{1}{\bar{p}+\eta}\beta\bar{p}\right)\bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)x\right).\end{aligned}$$

Since the first line is non-negative by $IC_{\bar{p},p}$, the proof is completed by noting that

$$\begin{aligned}&\left(\frac{1}{2}\beta - \frac{1}{2}\frac{1}{\bar{p}+\eta}\beta\bar{p}\right)\bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)x \\ \geq\ & \frac{1}{2}\beta\frac{\bar{p}\eta}{\bar{p}+\eta} > 0.\end{aligned}$$

■

Lemma 10 *Upward IC from $p \leq \bar{p}$ to $p' \geq \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for $p < \bar{p}$ and $p' \geq \bar{p}$.*

Proof. For $p = \bar{p}$, this holds because $\eta \leq x$. So suppose that $p < \bar{p}$. Since $\left(1-\rho\right)\gamma_p + \rho\gamma_p = \gamma_p$, $\hat{f}\left(p\right) = f\left(p\right)$ for all $p < \bar{p}$. Moreover, given recommendation p, the conditional probability that the opponent's recommendation is no less than $\bar{p}$ is

$$\frac{\left(1-\rho\right)\gamma_p + \rho\gamma_p}{\hat{f}\left(p\right)} = \frac{\gamma_p}{f\left(p\right)},$$

and hence $\widehat{IC}_{p,\bar{p}}$ follows from $IC_{p,\bar{p}}$.

For $p' \in (\bar{p}, \bar{p}+\eta)$, it is clear that $\widehat{IC}_{p,p'} \geq \widehat{IC}_{p,\bar{p}+\eta}$. Thus, it remains to show that $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$. Given recommendation p, the conditional probability that the opponent's recommendation is no less than $\bar{p}+\eta$ is

$$\frac{\rho\gamma_p}{\hat{f}(p)} = \rho\frac{\gamma_p}{f(p)}.$$

Thus, $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$ is implied by $IC_{p,\bar{p}} \geq 0$ if

$$(\bar{p}+\eta)\,\rho\frac{\gamma_p}{f(p)} \leq \bar{p}\frac{\gamma_p}{f(p)},$$

which follows from $\rho \leq \frac{\bar{p}}{\bar{p}+\eta}$. ■

Lemma 11 *IC from $p < \bar{p}$ to $p' < \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for all $p < \bar{p}, p' < \bar{p}$.*

Proof. This follows from $\hat{f}(p) = f(p)$ and $\sum_{\tilde{p}\geq p'} \hat{f}(p,\tilde{p}) = \sum_{\tilde{p}\geq p'} f(p,\tilde{p})$ for all $p' < \bar{p}$. ■

Together, Lemmas 8–11 imply that all incentive constraints are satisfied. This completes the proof.

Proof of Proposition 8

First, suppose that $n \geq 3$. Note that π as a function of x is larger under (optimal) almost-uniform bids than under bid rotation, because the static bid distribution under bid rotation is almost-uniform (as at most one firm bids below 1). It thus suffices to show that x as a function of π is also larger under almost-uniform bids: that is, that

$$\begin{aligned} \frac{\delta}{(1-\delta)\,n} &\geq \frac{\delta^{n-1}}{1-\delta^n}, \qquad \text{or equivalently} \\ 1+\delta+\ldots+\delta^{n-3}+\delta^{n-1} &\geq (n-1)\,\delta^{n-2}. \end{aligned}$$

This holds whenever $n \geq 3$, because

$$1+\delta+\ldots+\delta^{n-3}+\delta^{n-1} \geq (n-3)\,\delta^{n-2}+\delta^{n-3}+\delta^{n-1} \geq (n-1)\,\delta^{n-2},$$

where the last line holds because $\delta^2+1 \geq 2\delta$ for all $\delta \in [0,1]$.

Now suppose that $n = 2$. Substituting for π in the system for (π, x) under bid rotation, we see that profit under bid rotation equals

$$\frac{1-\delta^2}{\delta}\exp\left(-\frac{1-\delta-\delta^2}{\delta}\right) = \frac{1-\delta}{\delta}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\delta\log\left(1+\delta\right)\right). \tag{36}$$

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Substituting for x in the system for (π, x) under almost-uniform bids, we have

$$\begin{aligned} \frac{2(1-\delta)}{\delta} &= \frac{1-2\log(2x)}{1-x}, \qquad \text{or equivalently} \\ x &= \frac{1}{2}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\frac{1}{2}\right)\exp\left(\frac{1-\delta}{\delta}x\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \pi &= \frac{2(1-\delta)}{\delta}x \\ &= \frac{1-\delta}{\delta}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\frac{1}{2}\right)\exp\left(\frac{1-\delta}{\delta}x\right). \end{aligned} \tag{37}$$

Since $x > 0$, we see that (37) is greater than (36) whenever

$$\frac{1}{2} \geq \delta \log(1+\delta).$$

This inequality holds for all $\delta < \frac{1}{2}$, which completes the proof.

Asymmetric Equilibrium without Mediation

Let $n = 3$ and $x < \frac{1}{4}$. Firms 1 and 2 mix independently, with

$$p_i = \begin{cases} x & \text{with prob } \frac{2}{7} \\ \frac{3}{2}x & \text{with prob } \frac{2}{7} \\ 4x & \text{with prob } \frac{3}{7} \end{cases}$$

Firm 3 stays out (or, equivalently, bids above $4x$). There are no on-path punishments. Any off-path bid is punished by x.

The winning bid distribution is given by

$$\min\{p_1, p_2\} = \begin{cases} x & \text{with prob } \frac{24}{49} \\ \frac{3}{2}x & \text{with prob } \frac{16}{49} \\ 4x & \text{with prob } \frac{9}{49} \end{cases}$$

Hence, industry profit equals $\frac{24}{49}(x) + \frac{16}{49}\left(\frac{3}{2}x\right) + \frac{9}{49}(4x) = \frac{12}{7}x$. This is greater than $\frac{3}{2}x$, which by Proposition 1, is the maximum cartel profit when all firms win with positive probability.

We check that this is an equilibrium. For firm 3, bidding just below x gives $x - x = 0$, bidding just below $\frac{3}{2}x$ gives $\frac{25}{49}\left(\frac{3}{2}x\right) - x < 0$, and bidding just below $4x$ gives $\frac{9}{49}(4x) - x < 0$, so it is optimal for firm 3 to stay out. For firm 1 or 2, bidding at x, $\frac{3}{2}x$, or $4x$ all give an

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expected payoff of $\frac{6}{7}x$, because

$$\underbrace{\frac{6}{7}}_{\text{prob win if } p_i = x} (x) = \underbrace{\frac{4}{7}}_{\text{prob win if } p_i = \frac{3}{2}x} \left(\frac{3}{2}x\right) = \underbrace{\frac{3}{14}}_{\text{prob win if } p_i = 4x} (4x) .$$

Finally, for firm 1 or 2, any off-path bid gives a strictly lower payoff, because bidding just below x gives $x - x = 0$, bidding just below $\frac{3}{2}x$ gives $\frac{5}{7}\left(\frac{3}{2}x\right) - x = \frac{1}{14}x$, and bidding just below $4x$ gives $\frac{3}{7}(4x) - x = \frac{5}{7}x$.

Optimal Bid Distributions with Only Downward Incentive Constraints

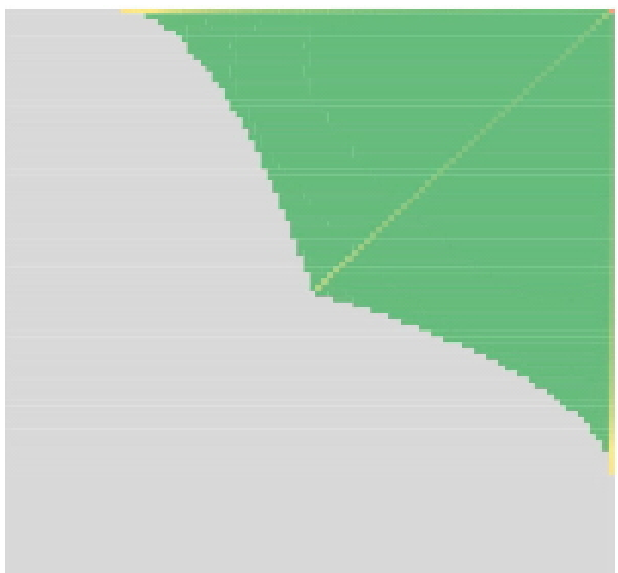

Figure 7: The optimal bid distribution when $x = .1$, with only downward incentive constraints. The color scheme in Figures 7–9 is the same as in Figure 1. Bid pair $p_1 = p_2 = 1$ is recommended with probability .06.

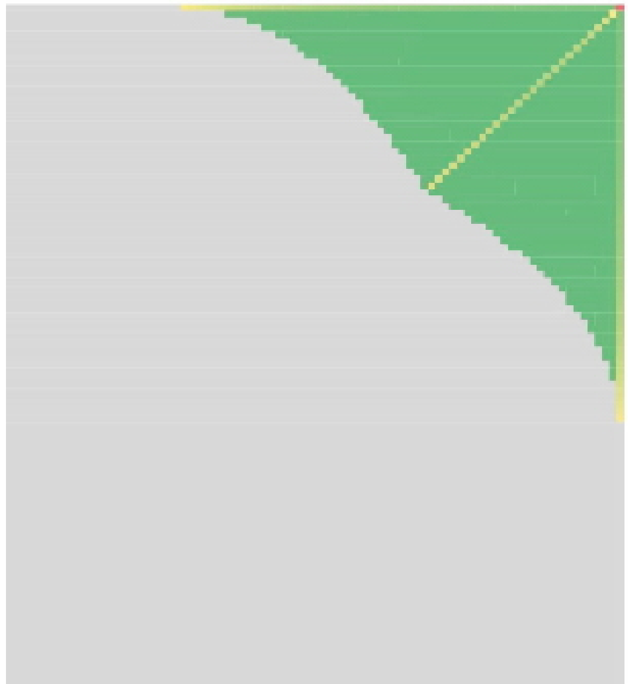

Figure 8: The optimal bid distribution when $x = .2$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .16.

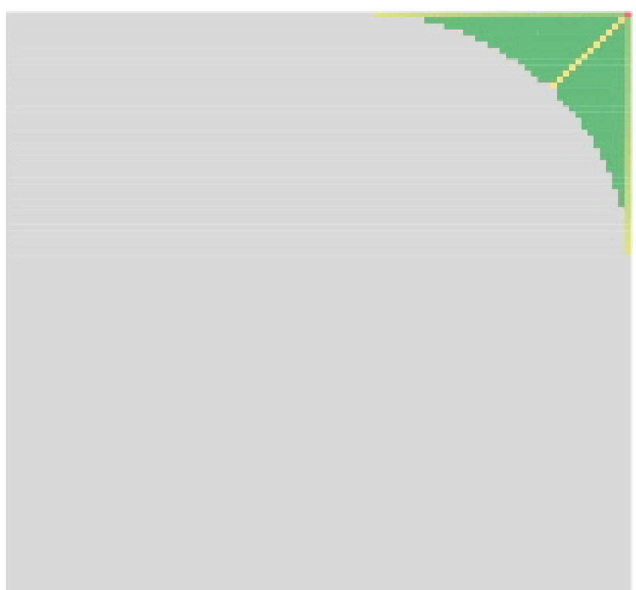

Figure 9: The optimal bid distribution when $x = .3$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .31.