Online Appendix for

"Signaling in a Global Game: Coordination and Policy Traps"

George-Marios Angeletos MIT and NBER

Christian Hellwig UCLA

Alessandro Pavan Northwestern University

In this Online Appendix we provide two auxiliary results for "Signaling in a Global Game: Coordination and Policy Traps." The first is Lemma 2; this lemma was used without proof in Proposition 4 (unbounded policy noise). The second is an example where the payoff that the policy maker enjoys from maintaining the status quo is negatively correlated with its strength; this case was briefly referred to in Section 5 of the paper.

A1. Proof of Lemma 2

Here we prove Lemma 2 from Proposition 4 in the paper. For convenience, we first restate the lemma below.

Lemma 2. *For any* $r^* \in (\underline{r}, \tilde{r})$ *and* $\varepsilon > 0$, *there exist* $\bar{\eta} > 0$ *and* $\bar{\rho} < \underline{r}/r^*$ *such that for any* $(\eta, \rho) < (\bar{\eta}, \bar{\rho})$, *conditions (3)-(2) below admit a solution* $(x', \hat{x}, \theta', \theta'')$ *that satisfies* $\theta' \leq \theta''$, $|x' - x^*| < \varepsilon$, $|\theta' - \theta^*| < \varepsilon$, $|\theta'' - \theta^{**}| < \varepsilon$, *and* $\hat{x} < -1/\varepsilon$.

$$1 - \Psi(\tfrac{x'-\theta'}{\sigma}) = \underline{r} - (\underline{r} - \rho r^*)[\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{x'-\theta''}{\sigma})] \tag{1}$$

$$1 - \Psi(\tfrac{\hat{x}-\theta'}{\sigma}) = \underline{r} + [r^*\rho + r^*(1-\rho)\exp(\tfrac{r^*-\underline{r}}{\eta}) - \underline{r}][\Psi(\tfrac{\hat{x}-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta''}{\sigma})] \tag{2}$$

$$\theta' = \rho[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})]\Psi(\tfrac{x'-\theta'}{\sigma}) + [1 - \rho + \rho\exp(-\tfrac{r^*-\underline{r}}{\eta})]\Psi(\tfrac{\hat{x}-\theta'}{\sigma}) + C(r^*) \tag{3}$$

$$C(r^*) = (1-\rho)[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta''}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta''}{\sigma})] \tag{4}$$

$$C(r^*) \leq (1-\rho)[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta'}{\sigma})] \tag{5}$$

Proof. It is useful to change variables as follows. Let

$$W \equiv \Psi(\tfrac{\hat{x}-\theta'}{\sigma}), \qquad Z \equiv \Psi(\tfrac{\hat{x}-\theta''}{\sigma}), \qquad Y \equiv \Psi(\tfrac{x'-\theta''}{\sigma}). \tag{6}$$

Conditions (1)-(4) can then be restated as follows:

$$\delta - \gamma Y = \Psi\left(\Psi^{-1}(Y) - \Psi^{-1}(Z) + \Psi^{-1}(W)\right) \tag{7}$$

$$W = \alpha + \beta Z \tag{8}$$

$$\theta' = \rho[1-\exp(-\tfrac{r^*-\underline{r}}{\eta})]\left[\Psi\left(\Psi^{-1}(Y) - \Psi^{-1}(Z) + \Psi^{-1}(W)\right) - W\right] + W + C(r^*) \tag{9}$$

$$Y = Z + \frac{C(r^*)}{(1-\rho)[1-\exp(-\frac{r^*-\underline{r}}{\eta})]} \tag{10}$$

where $\alpha, \beta, \delta \in (0,1)$ and $\gamma > 0$ are given by

$$\alpha \equiv \tfrac{1-\underline{r}}{1-\underline{r}+r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]}, \quad \beta \equiv \tfrac{r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]-\underline{r}}{1-\underline{r}+r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]}, \quad \gamma \equiv \tfrac{\underline{r}-\rho r^*}{1-\underline{r}+\rho r^*}, \quad \delta \equiv \tfrac{1-\underline{r}}{1-\underline{r}+\rho r^*}.$$

For any given r^*, (7)-(10) is a system of four equations in four unknowns, (Y, Z, W, θ'). We first seek a solution to this system, proceeding as follows. Substituting (8) into (7) gives

$$\Psi^{-1}(\delta - \gamma Y) - \Psi^{-1}(Y) = \Psi^{-1}(\alpha + \beta Z) - \Psi^{-1}(Z). \tag{11}$$

Together with (10), this defines a (sub)system of two equations in two unknowns, (Y, Z). In Step 1, we establish the existence of a solution (Y^*, Z^*) to this (sub)system. Condition (8) then gives W^*, while (9) gives θ'. From (6) we can then back out $(x', \hat{x}, \theta'')$. In Step 2, we check that the thresholds $(x', \hat{x}, \theta', \theta'')$ satisfy the inequality in (5). Finally, Step 3 shows convergence to the corresponding thresholds of the benchmark game.

Step 1. We first prove that (10) and (11) admit a solution for (Y, Z). Let $LHS(Y)$ and $RHS(Z)$ denote, respectively, the left-hand and the right-hand side of (11). Note that $LHS(Y)$ and $RHS(Z)$ are defined for $Y \in (0, \min\{1, \delta/\gamma\})$ and $Z \in (0,1)$ and are continuous in Y and Z. Moreover, LHS is decreasing in Y, with $\lim_{Y\to 0} LHS(Y) = \infty$, $\lim_{Y\to\min\{1,\delta/\gamma\}} LHS(Y) = -\infty$ and $LHS(Y) \gtreqless 0$ if and only if $Y \lesseqgtr 1-\underline{r}$, whereas

$\lim_{Z\to 0} RHS(Z) = \infty$, $\lim_{Z\to 1} RHS(Z) = -\infty$, and $RHS(Z) \gtreqless 0$ if and only if $Z \lesseqgtr 1-\underline{r}$. It follows that (11) implicitly defines a continuous function $Y = g(Z;\eta,\rho)$, with $g:(0,1)\times\mathbb{R}^2 \to (0,\min\{1,\delta/\gamma\})$; note that $\lim_{Z\to 0} g(Z) = 0$, $\lim_{Z\to 1} g(Z) = \min\{1,\delta/\gamma\}$, and $g(Z) \lesseqgtr 1-\underline{r}$ if and only if $Z \lesseqgtr 1-\underline{r}$. Condition (10), on the other hand, defines explicitly a function $Y = f(Z;\eta,\rho)$. We thus seek a solution (Y^*, Z^*) to $Y = f(Z) = g(Z)$.

Note that $f(Z;\eta,\rho)$ is continuous and increasing in (Z,η,ρ) with $f(0;\eta,\rho) \to C(r^*) \in (0, 1-\underline{r})$ as $(\eta, r) \to (0,0)$. Then, take any (Z_0, η_0, ρ_0) such that $f(Z_0;\eta_0,\rho_0) < 1-\underline{r}$, and note that $g(Z;\eta,\rho)$ is also continuous in (Z,η,ρ) with $g(Z_0;\eta,\rho) \to 1-\underline{r}$ as $\eta\to 0$ and $g(Z;\eta,\rho)\to 0$ for any (η,ρ) as $Z\to 0$. It follows that there exist $\tilde{\eta}\in(0,\eta_0)$, $\tilde{\rho} < \min\{\rho_0, \underline{r}/r^*\}$ and $Z_1 < Z_0$ such that for any $(\eta,\rho) < (\tilde{\eta},\tilde{\rho})$, $g(Z_0;\eta,\rho) > f(Z_0;\eta,\rho)$ and $g(Z_1;\eta,\rho) < f(Z_1;\eta,\rho)$. The graphs of g and f thus intersect at least twice for (η,ρ) sufficiently small, implying that the system $Y = f(Z) = g(Z)$ admits at least two solutions, as illustrated in Figure A1.

---insert figure A1 about here---

Consider the lowest solution (Z^*, Y^*), let $W^* = \alpha + \beta Z^*$ and note that (Z^*, Y^*, W^*) are continuous in (η,ρ) and satisfy $Z^* \in (0, 1-\underline{r})$, $Y^* \in (Z^*, 1-\underline{r})$ and $W^* \in (Z^*, 1-\underline{r})$. The thresholds $(x', \hat{x}, \theta', \theta'')$ are then the unique solutions to (6) and (9). That $W^* > Z^*$ and $Y^* > Z^*$ imply that $0 < \theta' < \theta''$.

Step 2. We now show that (5) holds, if (η,ρ) are sufficiently small. Using $\Psi(\frac{x'-\theta'}{\sigma}) = \Psi(\Psi^{-1}(Y^*) - \Psi^{-1}(Z^*) + \Psi^{-1}(W^*))$ and (7), we have that $\Psi(\frac{x'-\theta'}{\sigma}) = \delta - \gamma Y^*$ and hence $\Psi(\frac{x'-\theta'}{\sigma}) - \Psi(\frac{\hat{x}-\theta'}{\sigma}) = \delta - \gamma Y^* - W^*$. As $(\eta,\rho)\to(0,0)$, $Z^*\to 0$, $Y^*\to C(r^*)$, $W^*\to 0$, $\delta\to 1$, $\gamma \to \frac{\underline{r}}{1-\underline{r}}$ and $\exp(-\frac{r^*-\underline{r}}{\eta}) \to 0$, implying that

$$(1-\rho)[1-\exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta'}{\sigma})] \to 1 - \tfrac{\underline{r}}{1-\underline{r}} C(r^*).$$

Since $C(r^*) < 1-\underline{r}$, necessarily $1 - \frac{\underline{r}}{1-\underline{r}}C(r^*) > 1-\underline{r} > C(r^*)$, which implies that there exist $\eta' \in (0,\tilde{\eta})$ and $\rho' \in (0,\tilde{\rho})$ such that $(x', \hat{x}, \theta', \theta'')$ satisfies (5) for all $(\eta,\rho) < (\eta',\rho')$.

Step 3. We conclude by showing convergence. As $(\eta,\rho)\to(0,0)$, $Y^*\to C(r^*)$, $W^*\to 0$, and $Z^*\to 0$. Using (6), (9) and (1), we then have that $\theta' \to C(r^*)$, $\hat{x} = \theta' + \sigma\Psi^{-1}(W) \to$

$-\infty$, $x' \to x^*$, and $\theta'' \to \theta^{**}$. Hence, for any $\varepsilon > 0$, there exist $\hat{\eta} \in (0, \eta')$ and $\hat{\rho} \in (0, \rho')$ such that $(\eta, \rho) < (\hat{\eta}, \hat{\rho})$ suffices for $|x' - x^*| < \varepsilon$, $|\theta' - \theta^*| < \varepsilon$, $|\theta'' - \theta^{**}| < \varepsilon$ and $\hat{x} < -1/\varepsilon$, where $(x^*, \theta^*, \theta^{**})$ are as in Proposition 2. *QED*

A2. Alternative payoff structures (continued)

Proposition 5 in the paper assumes that the payoff that the policy maker enjoys from maintaining the status quo is positively correlated with (or independent of) its strength. The following example shows that such a positive correlation is not essential.

Proposition 6. *Suppose* $R(\theta, A, r) = \theta - A$ *and* $U(\theta, A, r) = v(\theta) - A - C(r)$, *where* v *is not necessarily monotonic, but satisfies* $v(\theta) > 1 - \underline{r}$ *for all* $\theta \in [0, 1)$. *There exists* $\hat{r} > \underline{r}$ *such that, for any* $r^* \in [\underline{r}, \hat{r})$, *there is an equilibrium in which the status quo is abandoned if and only if* $\theta < 0$ *and the policy maker sets* r^* *for* $\theta \in [0, \theta^{**}]$ *and* $\underline{r}$ *otherwise.*

Proof. Let $\hat{r} \in (\underline{r}, \tilde{r})$ be the unique solution to

$$C(\hat{r}) = \sigma \left[\Psi^{-1} \left(1 - \tfrac{\underline{r}}{1-\underline{r}} C(\hat{r}) \right) - \Psi^{-1}\left(C(\hat{r})\right) \right]$$

and note that $C(\hat{r}) < 1 - \underline{r}$. For any $r^* \in (\underline{r}, \hat{r})$, let $x^* = \sigma \Psi^{-1}(1 - \frac{\underline{r}}{1-\underline{r}} C(r^*))$ and

$$\theta^{**} = x^* - \sigma \Psi^{-1}\left(C(r^*)\right) = \sigma \left[\Psi^{-1} \left(1 - \tfrac{\underline{r}}{1-\underline{r}} C(r^*) \right) - \Psi^{-1}\left(C(r^*)\right) \right],$$

and note that $\theta^{**} \geq 0$ for any $r^* < \hat{r}$ and solves $\Psi(\frac{x^* - \theta^{**}}{\sigma}) = C(r^*)$. Finally, let $\hat{\theta} \in (0, 1)$ be the unique solution to $\Psi(\frac{x^* - \hat{\theta}}{\sigma}) = \hat{\theta}$ and observe that $\theta^{**} > \hat{\theta}$ since $\Psi(\frac{x^* - \theta^{**}}{\sigma}) < \theta^{**}$ when $r^* < \hat{r}$.

We next prove that the following is part of an equilibrium: the policy maker sets $r(\theta) = r^*$ if $\theta \in [0, \theta^{**}]$ and $r(\theta) = \underline{r}$ otherwise; agents attack if and only if $(x, r) < (x^*, r^*)$, or $x < \underline{x}$; and the status quo is abandoned if and only if $\theta < 0$.

Consider the agents. For $r = \underline{r}$, beliefs are pinned down by Bayes' rule (this is immediate when noise is unbounded, whereas with bounded noise, it follows from the fact that $\theta^{**} < 2\sigma$)

and satisfy $\mu\left(0|x,\underline{r}\right) > \underline{r}$ if and only if $x < x^*$, where x^* solves

$$\frac{1-\Psi(\frac{x^*}{\sigma})}{1-\Psi(\frac{x^*}{\sigma})+\Psi(\frac{x^*-\theta^{**}}{\sigma})} = \underline{r}.$$

For any (x,r) such that $r = r^*$ and $\Theta\left(x\right) \cap [0,\theta^{**}] \neq \varnothing$, μ is also determined by Bayes' rule and satisfies $\mu\left(0|x,\underline{r}\right) = 0$. For any (x,r) such that either $r = r^*$ and $\Theta\left(x\right) \cap [0,\theta^{**}] = \varnothing$, or $r > r^*$, $\Theta\left(x\right) \subseteq \Theta(r)$, take any beliefs such that $\mu\left(0|x,r^*\right) = 1$ if $x < \underline{x}$ and $\mu\left(0|x,r^*\right) = 0$ otherwise. Finally, for any $r \in (\underline{r}, r^*)$, note that $[0,\theta^{**}] \cap \Theta(r) = \varnothing$. Then take any beliefs such that $\mu(\hat{\theta}|x,r) > \underline{r}$ if and only if $x < x^*$ and $\mu(\{\theta \in \Theta(x) \cap \Theta(r)\}|x,r) = 0$ if $\Theta(x) \nsubseteq \Theta(r)$. Given these beliefs, the strategy of the agents is sequentially rational for any (x,r).

Consider the policy maker. Given the agents' strategy, it is optimal to set either $\underline{r}$ or r^*. The payoff from setting $\underline{r}$ is zero for $\theta \leq \hat{\theta}$ and $v\left(\theta\right) - \Psi(\frac{x^*-\theta}{\sigma})$ for $\theta > \hat{\theta}$, whereas the payoff from setting r^* is negative for $\theta < 0$ and $v\left(\theta\right) - C\left(r^*\right)$ for $\theta \geq 0$. Since $\Psi(\frac{x^*-\theta^{**}}{\sigma}) = C\left(r^*\right) \leq C(\hat{r}) < 1 - \underline{r} < v\left(\theta\right)$, it follows that r^* is optimal if and only if $\theta \in [0,\theta^{**}]$. *QED*

The above result assumes that v is sufficiently high. Multiplicity, however, survives even if v is negative for all θ : there exists a continuum of equilibria in which an intermediate set of θ who would maintain the status quo even by setting $\underline{r}$, prefer to raise the policy at r^*, because the cost of the policy is lower than that of the attack at $\underline{r}$ (i.e., $C\left(r^*\right) \leq A\left(\theta,\underline{r}\right)$). These equilibria differ with respect to both the level of the policy and the regime outcome.