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B Online Appendix B: Differentiated Products

In this appendix, we retain Assumptions 1–4 but relax the assumption that competition is homogeneous-product Bertrand. In particular, we exhibit a demand system with differentiated products where nonetheless market-segmentation maximizes industry profits and a version of Theorem 1 continues to hold.

There are two firms and two markets, and demand in market h is determined as follows: when the home firm's price is p_h, the foreign firm's price is p_f, and the market demand state is s_h, demand for the two firms' products is given by

$$\begin{aligned} D_h^h\left(p_h, p_f, s_h\right) &= s_h \frac{\left(p_h\right)^{-\sigma}}{\left(p_h\right)^{\gamma-\sigma}+\left(p_f\right)^{\gamma-\sigma}} \text{ and} \\ D_f^h\left(p_h, p_f, s_h\right) &= s_h \frac{\left(p_f\right)^{-\sigma}}{\left(p_h\right)^{\gamma-\sigma}+\left(p_f\right)^{\gamma-\sigma}}, \end{aligned}$$

where D_j^i is firm j's demand in market i and $\sigma > \gamma > 1$ are constants.[37] We assume that $(\sigma - 1)(\gamma - 1) > 1$. Note that, for each γ, these conditions are satisfied for sufficiently large σ, and the limit as $\sigma \to \infty$ corresponds to the case of homogeneous goods/perfect substitutes. Assume also that the distribution of demand states is symmetric across the two markets, and that costs are symmetric and constant with $c_h < c_f$.

With this market structure, we will show that industry profits are maximized under the home-market principle, and that—under the further assumption that deviations are punished by reversion to static Nash equilibrium pricing—the home-market principle is easier to sustain under a less informative information structure.

We first show that the home-market principle is optimal:

Proposition 6 *For every demand state s_h, industry profits $D_h^h\left(p_h, p_f, s_h\right) + D_f^h\left(p_h, p_f, s_h\right)$ are maximized at price vector $(p_h, p_f) = \left(\frac{\gamma}{\gamma-1}c_h, \infty\right)$. That is, the home-market principle whereby the home firm prices at $\frac{\gamma}{\gamma-1}c_h$ and the foreign firm prices at ∞ is optimal.*

We next characterize static Nash pricing. Since demand is linear in s_h, a pure strategy Nash equilibrium price profile $\left(p_h^{Nash}, p_f^{Nash}\right)$ satisfies

$$\begin{aligned} p_h^{Nash} &\in \arg\max_{p_h} \frac{\left(p_h\right)^{-\sigma}\left(p_h - c_h\right)}{\left(p_h\right)^{\gamma-\sigma}+\left(p_f^{Nash}\right)^{\gamma-\sigma}}, \text{ and} \\ p_f^{Nash} &\in \arg\max_{p_h} \frac{\left(p_f\right)^{-\sigma}\left(p_f - c_f\right)}{\left(p_h\right)^{\gamma-\sigma}+\left(p_h^{Nash}\right)^{\gamma-\sigma}}. \end{aligned}$$

Lemma 6 *A pure strategy Nash equilibrium exists, is unique, and satisfies $p_h^{Nash} \in \left(\frac{\sigma}{\sigma-1}c_h, \frac{\gamma}{\gamma-1}c_h\right)$ and $p_f^{Nash} \in \left(\frac{\sigma}{\sigma-1}c_f, \frac{\gamma}{\gamma-1}c_f\right)$.*

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[37] With $\gamma = 1$, this is the usual CES demand function, with elasticity of substitution σ. We instead assume $\gamma > 1$ to ensure that optimal prices are finite, as will be seen. We also impose the natural convention that $(\infty)^{\gamma-\sigma} = 0$, so that in particular $D_h^h\left(p_h, \infty, s_h\right) = s_h/\left(p_h\right)^{\gamma}$.

The main conclusion of this example is as follows:

Proposition 7 *Under Assumptions 1–4, for any information structure π there is a discount factor $\hat{\delta}(\pi)$ such that the home-market principle can be sustained with Nash reversion if and only $\delta \geq \hat{\delta}(\pi)$; furthermore, if π' is more informative than π then $\hat{\delta}(\pi') \geq \hat{\delta}(\pi)$.*

The proof is sketched in Online Appendix C. Given Lemma 6, the logic is exactly as in Theorem 1.

C Online Appendix C: Omitted Proofs

C.1 Proof of Lemma 1

The first claim is immediately implied by Assumption 3, and v_i^δ is increasing in δ because $D\left(p_i^m\left(c_{i,\tau}^i, s_{\tau-1}^i\right), s_\tau^i\right)\left(p_i^m\left(c_{i,\tau}^i, s_{\tau-1}^i\right) - c_{i,\tau}^i\right) > 0$ (by the assumptions that $D\left(c_i^i, s^i\right) > 0$ and $D\left(p^i, s^i\right)$ is continuous).

C.2 Proof of Lemma 2

To see that (2) is sufficient, consider the strategy profile where, on-path, each firm i prices at $p_i^m\left(c_{i,t}^i, s_{t-1}^i\right)$ in its home market and prices at ∞ in all foreign markets; and where, after either entering any foreign market or detecting entry into its home market, firm i sets price $p_{i,t}^j = c_{j,t}^j$ in every market j in every subsequent period t. (Note that this is possible by Assumption 4.) In addition, specify that, if a firm either enters a foreign market or detects entry into its home market, it believes that every other firm has also detected entry into its home market.[38]

Under this strategy profile, at on-path histories each firm i's continuation payoff and best deviation payoff are independent of $\left(x_{j,i,\tau}^k\right)_{\tau=0}^t$ with $j \neq k$, and the continuation payoff after entering a foreign market is 0. Hence, firm i does not have a profitable deviation at any on-path history if and only if (2) holds for all $I_i^t \in \mathcal{I}_i(\pi)$.

Finally, firms also do not have profitable deviations at off-path histories, as once a firm enters a foreign market or detects entry into its own market it receives its minmax payoff of 0 from any strategy that never prices strictly below the home firm's cost in any market. (Pricing exactly at the home firm's cost yields no sales due to the tie-breaking rule.)

C.3 Proof of Lemma 3

Recall that $\mathcal{B}_{i,\pi}^{c,s^i}\left(b|_{\hat{c},\hat{s}^i}, z_i\right) = \left(b|_{c,s^i}\left(s^{-i}\right)\right)_{s^{-i}}$ with

$$b|_{c,s^i}\left(s^{-i}\right) = \frac{\sum_{\hat{s}^{-i}} b_{\hat{c},\hat{s}^i}\left(\hat{s}^{-i}\right) M\left(c, s|\hat{c}, \hat{s}^i, \hat{s}^{-i}\right) \pi^m\left(z_i|\hat{s}, \hat{c}, s, c\right)}{\sum_{\tilde{s}_t^{-i}} \sum_{\hat{s}^{-i}} b_{\hat{c},\hat{s}^i}\left(\hat{s}^{-i}\right) M\left(c, s^i, \tilde{s}^{-i}|\hat{c}, \hat{s}^i, \hat{s}^{-i}\right) \pi^m\left(z_i|\hat{s}, \hat{c}, s^i, \tilde{s}^{-i}, c\right)}.$$

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[38] We implicitly assume here that each firm finds all combinations of signals possible, even if π does not have full support.

For notational convenience, given $(\hat{c}, \hat{s}^i, c, s^i)$, let

$$\Pr^{\pi}\left(z_i, s^{-i}\right) = \sum_{\hat{s}^{-i}} b_{\hat{c},\hat{s}^i}\left(\hat{s}^{-i}\right) M\left(c, s|\hat{c}, \hat{s}^i, \hat{s}^{-i}\right) \pi^m\left(z_i|\hat{s}, \hat{c}, s, c\right)$$

be the probability of (z_i, s^{-i}). Thus, $b|_{c,s^i}\left(s^{-i}\right) = \Pr^{\pi}\left(z_i, s^{-i}\right) / \left(\sum_{\tilde{s}^{-i}} \Pr^{\pi}\left(z_i, \tilde{s}^{-i}\right)\right)$.

If $\pi' \geq \pi$ then there exists $f_i : Z_i \times Z_i \to [0, 1]$ such that $\pi\left(z_i|\hat{s}, \hat{c}, s, c\right) = \sum_{z'_i} f_i\left(z_i, z'_i\right) \pi'\left(z'_i|\hat{s}, \hat{c}, s, c\right)$. One can then check that

$$b|_{c,s^i}\left(s^{-i}\right) = \sum_{z'_i} \alpha_{z'_i} \frac{\Pr^{\pi'}\left(z'_i, s^{-i}\right)}{\sum_{\tilde{s}^{-i}} \Pr^{\pi'}\left(z'_i, \tilde{s}^{-i}\right)},$$

where

$$\alpha_{z'_i} := f_i\left(z_i, z'_i\right) \frac{\sum_{\tilde{s}^{-i}} \Pr^{\pi'}\left(z'_i, \tilde{s}^{-i}\right)}{\sum_{\tilde{z}'_i} \sum_{\tilde{s}^{-i}} f_i\left(z_i, \tilde{z}'_i\right) \Pr^{\pi'}\left(\tilde{z}'_i, \tilde{s}^{-i}\right)} \geq 0$$

and $\sum_{z'_i} \alpha_{z'_i} = 1$. This proves the first claim.

Finally, if $\pi' > \pi$, $\pi'\left(z|p, c, s\right) > 0$ for all z, p, c, s, and $M\left(c, s|\hat{c}, \hat{s}\right) > 0$ for all $c, s, \hat{c}, \hat{s}$, then $\alpha_{z'_i} > 0$ for all z'_i. This proves the second claim.

C.4 Proof of Lemma 5

By Assumption 6 and Fact 5 of PS, for each $b_i \in \text{ext}\left(\mathcal{M}\left(T^U_{i,\pi}\right)\right)$ and (c, s^i), there exist $\hat{b}_i \in \mathcal{M}\left(T^U_{i,\pi}\right)$, $(\hat{c}, \hat{s}^i)$, and z_i such that $b_i|_{c,s^i} = \mathcal{B}^{c,s^i}_{i,\pi}\left(\hat{b}|_{\hat{c},\hat{s}^i}, z_i\right)$. By $\pi' > \pi$ and Assumption 5, Lemma 3 gives $b_i|_{c,s^i} \in \text{relint}\left(\text{co}\left(\left\{\mathcal{B}^{c,s^i}_{i,\pi'}\left(\hat{b}|_{\hat{c},\hat{s}^i}, z'_i\right)\right\}_{z'_i \in Z_i}\right)\right)$. As $\mathcal{M}\left(T^U_{i,\pi}\right) \subseteq \mathcal{M}\left(T^U_{i,\pi'}\right)$, we have $\hat{b}_i \in \mathcal{M}\left(T^U_{i,\pi'}\right)$, and hence $\left\{\mathcal{B}^{c,s^i}_{i,\pi'}\left(\hat{b}|_{\hat{c},\hat{s}^i}, z'_i\right)\right\}_{z'_i \in Z} \subseteq \mathcal{M}\left(T^U_{i,\pi'}\right)$. Therefore, $b_i \in \text{relint}\left(\mathcal{M}\left(T^U_{i,\pi'}\right)\right)$, and hence $b_i \in \text{int}\left(\mathcal{M}\left(T^U_{i,\pi'}\right)\right)$ because $\text{relint}\left(\mathcal{M}\left(T^U_{i,\pi'}\right)\right)$ has full dimension by Assumption 6. The result follows since this holds for each $b_i \in \text{ext}\left(\mathcal{M}\left(T^U_{i,\pi}\right)\right)$ and $\mathcal{M}\left(T^U_{i,\pi}\right)$ is compact.

C.5 Proof of Proposition 1

The argument is similar to the proof of Theorem 1, so we give only a sketch.

In the strategy profile constructed in the proof of Lemma 2, replace the off-path threat of pricing at $c^j_{j,t}$ (in every market $j \neq i$ and period t) with the threat of pricing at $c^j_{i,t}$. By Assumption 3, this continuation strategy holds every firm to its lowest continuation payoff in any cautious equilibrium. By Assumption 7, this continuation payoff depends on a firm's information set only through the pair $\left(c^i_{i,t}, s^i_{t-1}\right)$ (and thus does not depend on the information structure), as a firm's signals are not informative of its competitors' costs. Thus, as in the proof of Theorem 1, $\delta^*\left(\pi'\right) \geq \delta^*\left(\pi\right)$ if and only if $\max_{(c_i,s^i) \in C_i \times S^i} \sup_{b_i \in B_i(\pi')|_{c_i,s^i}} d_i\left(c_i, b_i\right) \geq \max_{(c_i,s^i) \in C_i \times S^i} \sup_{b_i \in B_i(\pi)|_{c_i,s^i}} d_i\left(c_i, b_i\right)$ (with the difference that now $b_i \in \Delta\left(S^{-i} \times C_{-i}\right)$ rather than $\Delta\left(S^{-i}\right)$), and this inequality follows from the same argument as in the proof of Theorem 1.

C.6 Proof of Proposition 2

Under the first-best action plan, a firm's future profit when the previous demand state in its home market was low and high, respectively, is given by

$$\begin{aligned} V_L &= \frac{\tilde{s}_L^2}{4} + \delta\left[\phi V_L + (1-\phi) V_H\right], \text{ and} \\ V_H &= \frac{\tilde{s}_H^2}{4} + \delta\left[\phi V_H + (1-\phi) V_L\right]. \end{aligned}$$

Solving for V_L and V_H gives

$$\begin{aligned} V_L &= \frac{1}{(1-\delta)(1+\delta-2\delta\phi)}\left[(1-\delta\phi)\frac{\tilde{s}_L^2}{4} + \delta(1-\phi)\frac{\tilde{s}_H^2}{4}\right], \\ V_H &= \frac{1}{(1-\delta)(1+\delta-2\delta\phi)}\left[(1-\delta\phi)\frac{\tilde{s}_H^2}{4} + \delta(1-\phi)\frac{\tilde{s}_L^2}{4}\right]. \end{aligned}$$

Note that V_L and V_H are increasing in δ and go to ∞ as $\delta \to 1$.

Suppose firms observe only industry demand. Then, as a deviator can be held to her minmax payoff of 0 (as in Lemma 2), the first-best action plan is sequentially rational on the equilibrium path at $t > 0$ if and only if

$$V_L \geq \frac{\tilde{s}_L^2}{4} + (n-2)\max\left\{\frac{\tilde{s}_{\min}(\tilde{s}_{\min}-2c)}{4}, \frac{\tilde{s}_{\max}(\tilde{s}_{\max}-2c)}{8}\right\} + \frac{\tilde{s}_H(\tilde{s}_H-2c)}{4}$$

and

$$V_H \geq \frac{\tilde{s}_H^2}{4} + (n-2)\max\left\{\frac{\tilde{s}_{\min}(\tilde{s}_{\min}-2c)}{4}, \frac{\tilde{s}_{\max}(\tilde{s}_{\max}-2c)}{8}\right\} + \frac{\tilde{s}_L(\tilde{s}_L-2c)}{4},$$

or equivalently

$$\delta\left[\phi V_L + (1-\phi) V_H\right] \geq (n-2)\max\left\{\frac{\tilde{s}_{\min}(\tilde{s}_{\min}-2c)}{4}, \frac{\tilde{s}_{\max}(\tilde{s}_{\max}-2c)}{8}\right\} + \frac{\tilde{s}_H(\tilde{s}_H-2c)}{4} \tag{5}$$

and

$$\delta\left[\phi V_H + (1-\phi) V_L\right] \geq (n-2)\max\left\{\frac{\tilde{s}_{\min}(\tilde{s}_{\min}-2c)}{4}, \frac{\tilde{s}_{\max}(\tilde{s}_{\max}-2c)}{8}\right\} + \frac{\tilde{s}_L(\tilde{s}_L-2c)}{4}. \tag{6}$$

Note that (5) implies (6) if $\phi > 1/2$, while (6) implies (5) if $\phi < 1/2$. Furthermore, noting that setting price $\bar{s}/2$ in all markets is the most tempting deviation at $t = 0$, sequential rationality holds at $t = 0$ if and only if

$$\delta\left[\frac{1}{2}V_L + \frac{1}{2}V_H\right] \geq (n-1)\frac{\bar{s}(\bar{s}-2c)}{4}. \tag{7}$$

Hence, first-best industry profits are sustainable if and only if (5), (6), and (7) hold. Finally, note that the left-hand sides of (5), (6), and (7) are increasing in δ (and go to 0 and ∞ as

$\delta \to 0$ and 1), and let δ^* be the cutoff value of δ such that one of (5), (6), and (7) holds with equality while the other two are satisfied.

Next, suppose firms observe all prices and sales. In this case, the first-best action plan is sustainable if and only if

$$\delta\left[\phi V_L + (1-\phi) V_H\right] \geq (n-2)\left(\frac{\tilde{s}_L\left(\tilde{s}_L - 2c\right)}{8} + \frac{\tilde{s}_H\left(\tilde{s}_H - 2c\right)}{8}\right) + \frac{\tilde{s}_H\left(\tilde{s}_H - 2c\right)}{4}, \tag{8}$$

$$\delta\left[\phi V_H + (1-\phi) V_L\right] \geq (n-2)\left(\frac{\tilde{s}_L\left(\tilde{s}_L - 2c\right)}{8} + \frac{\tilde{s}_H\left(\tilde{s}_H - 2c\right)}{8}\right) + \frac{\tilde{s}_L\left(\tilde{s}_L - 2c\right)}{4}, \tag{9}$$

and (7) hold. However, as $\bar{s} = \left(\tilde{s}_L + \tilde{s}_H\right)/2$, Jensen's inequality implies that

$$\frac{\bar{s}\left(\bar{s} - 2c\right)}{4} < \frac{\tilde{s}_L\left(\tilde{s}_L - 2c\right)}{8} + \frac{\tilde{s}_H\left(\tilde{s}_H - 2c\right)}{8},$$

so adding (8) and (9) and dividing by 2 yields (7). Hence, first-best industry profits are sustainable if and only if (8) and (9) hold.

Finally, as $\delta \neq 1/2$ implies that $\tilde{s}_{\min} < \tilde{s}_{\max}$, the right-hand side of (8) (resp., (9)) is strictly greater than the right-hand side of (5) (resp., (6)). Hence, letting δ^{**} be the cutoff value of δ such that (8) or (9) holds with equality while the other is satisfied, we have $\delta^* < \delta^{**}$.

C.7 Proof of Proposition 3

It suffices to show that $\frac{\partial^2}{\partial \rho^2}\left[\rho\left(\frac{1}{\psi(\rho)} - 1\right)\right] \leq 0$, $\frac{\partial^2}{\partial \rho \partial \phi}\left[\rho\left(\frac{1}{\psi(\rho)} - 1\right)\right] \leq 0$, and $\psi\left(1\right) = 1$. We prove this by deriving the formula for $\psi\left(\rho\right)$ in closed form.

Note that ψ is equal to the greatest probability that firm i can ever assign to the event that $s_{t-1}^j = s_H$, as in equilibrium firm j prices at $\tilde{s}_H/2$ in period t if and only if $s_{t-1}^j = s_H$. Thus, ψ may be computed using the fixed point formula

$$\begin{aligned}
& \psi\phi + (1-\psi)(1-\phi) \\
= \; & \Pr\left(s_t^j = s_H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi\right) \\
= \; & \Pr\left(z_{t,i}^1 = H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi\right) \Pr\left(s_t^j = s_H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi, z_{t,i}^1 = H\right) \\
& + \Pr\left(z_{t,i}^1 = L \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi\right) \Pr\left(s_t^j = s_H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi, z_{t,i}^1 = L\right) \\
= \; & \Pr\left(z_{t,i}^1 = H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi\right) \psi + \Pr\left(s_t^j = s_H \middle| \Pr\left(s_{t-1}^j = s_H\right) = \psi\right) \frac{1-\rho}{2},
\end{aligned}$$

where the second equality follows by the law of total probability, and the third equality follows by reversing the order of conditioning on the event $z_{t,i}^1 = L$ in the second term. This formula is equivalent to

$$\frac{1+\rho}{2}\left[\psi\phi + (1-\psi)(1-\phi)\right] = \Pr\left(z_{t,i}^1 = H \middle| \Pr\left(s_{t-1}^j = H\right) = \psi\right)\psi.$$

Observing that

$$\Pr\left(z_{t,i}^{1}=H\middle|\Pr\left(s_{t-1}^{j}=H\right)=\psi\right)=\psi\left[\phi\frac{1+\rho}{2}+\left(1-\phi\right)\frac{1-\rho}{2}\right]+\left(1-\psi\right)\left[\phi\frac{1-\rho}{2}+\left(1-\phi\right)\frac{1+\rho}{2}\right],$$

we can solve the resulting quadratic equation for ψ, obtaining the formula

$$\psi\left(\rho\right)=\frac{\left(1+\rho\right)\left(2\phi-1\right)-\phi+\sqrt{\phi^{2}-\left(1+\rho\right)\left(1-\rho\right)\left(2\phi-1\right)}}{2\rho\left(2\phi-1\right)}.$$

One can now directly check that $\frac{\partial^2}{\partial\rho^2}\left[\rho\left(\frac{1}{\psi(\rho)}-1\right)\right]\leq 0$, $\frac{\partial^2}{\partial\rho\partial\phi}\left[\rho\left(\frac{1}{\psi(\rho)}-1\right)\right]\leq 0$, and $\psi\left(1\right)=1$.

C.8 Proof of Proposition 5

We show that, for any $\delta\in\Delta:=\left(0.148,0.149\right)$, market segmentation is not sustainable with mediated perfect monitoring but is sustainable with mediated private monitoring.

C.8.1 Impossibility under Perfect Monitoring

Suppose toward a contradiction that there exists a Nash equilibrium that implements market segmentation under mediated perfect monitoring. Fix such an equilibrium, and let p be the minimum price in the support of firm 1's strategy in period 0. Note that firm 1 gains $1-p$ from pricing at 1 rather p, so firm 1's per-period continuation payoff after pricing at p in period 0 must be at least $\mu\left(p\right):=\left(\left(1-\delta\right)/\delta\right)\left(1-p\right)$. In particular, there is at least one period t in which firm 1's expected payoff is at least $\mu\left(p\right)$. Let G be the cumulative distribution function of firm 1's period t price, conditional on the (publicly observable) event that firm 1 prices at p in period 0, and let $G^{-}\left(s\right)=\lim_{p\uparrow s}G\left(s\right)$. Note that $\int_0^1 s dG\left(s\right)\geq\mu\left(p\right)$.

Next, let

$$d\left(p\right)=\max_{p'}\left(p'-c\right)\left(1-G^{-}\left(p'\right)\right)$$

be firm 2's maximum deviation gain from entering firm 1's market when firm 1's price is distributed according to G, where $c=c_2^1=0.7$. As firm 2's maximum deviation gain from entering firm 1's market in period 0 is at least $p-c$, firm 2's per-period equilibrium payoff is 1, and firm 2's minmax payoff is 0, we see that a necessary condition for firm 2's strategy to be optimal is

$$\max\left\{p-c,d\left(p\right)\right\}\leq\frac{\delta}{1-\delta}.$$

We will now derive a lower bound on $d\left(p\right)$, which will yield a range of discount factors over which this inequality cannot be satisfied.

Define the function $x\left(\mu\right)$ to be the solution to

$$\left(1-c\right)x\left[\frac{1}{s}\left(-c+\left(s-c_j\right)\ln\left(s-c\right)\right)\right]_{s=(1-c)x+c}^{s=1}+x=\mu$$

if $\mu>c$, and define $x\left(\mu\right)=0$ if $\mu\leq c$.

Lemma 7 $d(p) \geq (1-c)\, x(\mu)$.

Proof. The lemma is trivial if $\mu \leq c$. So suppose that $\mu > c$. Note that the solution to the following minmax problem gives a lower bound on $d(p)$:

$$\min_{G} \left(\max_{p'} (p'-c) \left(1 - G^{-}(p')\right) \right)$$

subject to

$$\int_0^1 s dG(s) \geq \mu.$$

Let $P_G = \arg\max_{p'} (p'-c)(1-G^{-}(p'))$.

We claim that, for any distribution G that solves the minmax problem, there is a number $p(G)$ such that P_G is given by the interval $[p(G), 1]$. (A solution G exists because $\max_{p'} (p'-c)(1-G^{-}(p'))$ is continuous in G in the weak topology.)

To see this, first note that $\int_0^1 s dG(s) = \mu$ for any distribution G that solves the minmax problem, as if $\int_0^1 s dG(s) > \mu$ then shifting mass to 0 decreases the value of the objective.

Next, note that if $\sup\{p \in P_G\} < 1$, then shifting a sufficiently small mass from $[0, \sup\{p \in P_G\}]$ to 1 increases $\int_0^1 s dG(s)$ but does not affect the value of the objective. This is a contradiction, so $\sup\{p : p \in P_G\} = 1$, and therefore $G^{-}(1) < 1$.

Finally, suppose toward a contradiction that P_G is not a convex set. Note that $(p'-c)(1-G^{-}(p'))$ is upper semi-continuous in p', so P_G is closed; hence, if P_G is not convex then there exists a maximal open interval $(p_0, p_1) \subseteq (\inf\{p \in P_G\}, \sup\{p \in P_G\}) \setminus P_G$. Note that G must put positive mass on the half-open interval $[p_0, p_1)$; otherwise, we would have $(p_1 - c)(1 - G^{-}(p_1)) > (p_0 - c)(1 - G^{-}(p_0))$ and hence $p_0 \notin P_G$, contradicting the maximality of (p_0, p_1). But then shifting a sufficiently small mass from $[p_0, p_1)$ to p_1 would increase $\int_0^1 s dG(s)$ without affecting the value of the objective, a contradiction. This completes the proof of the claim.

Now, as $P_G = [p(G), 1]$, we have, for all $p' \geq p(G)$, $(p'-c)(1-G^{-}(p')) = (1-c)(1-G^{-}(1))$. This equation is inconsistent with G having an atom below 1, so we obtain the formula

$$G(p') = 1 - \frac{(1-c)(1-G^{-}(1))}{p'-c}$$

for all $p' \geq p(G)$. Moreover, we must have $G(p(G)) = 0$, as otherwise shifting mass from $[0, p(G)]$ to $p(G)$ would again increase $\int_0^1 s dG(s)$ without affecting the objective of the minmax problem. We thus have

$$1 - \frac{(1-c)(1-G^{-}(1))}{p(G)-c} = 0,$$

or equivalently $p(G) = (1-c)(1-G^{-}(1)) + c$. Finally, we may solve for $G^{-}(1)$ according to the equation

$$(1-c)\left(1-G^{-}(1)\right) \int_{(1-c)(1-G^{-}(1))+c}^{1} \underbrace{\frac{s}{(s-c)^2}}_{dG(s)} ds + \left(1-G^{-}(1)\right) = \mu,$$

or equivalently

$$(1-c)\left(1-G^{-}(1)\right)\left[-\frac{1}{-s}\left(-c+(s-c)\ln(s-c)\right)\right]_{(1-c)(1-G^{-}(1))+c}^{1}+\left(1-G^{-}(1)\right)=\mu.$$

Thus, $1-G^{-}(1)=x(\mu)$, and therefore $d(p)\geq(1-c)\,x(\mu)$. ■

We conclude that market segmentation is not sustainable in Nash equilibrium with mediated perfect monitoring if

$$\min_{p\in[0,1]}\max\left\{p-c,(1-c)\,x(\mu(p))\right\}>\frac{\delta}{1-\delta}. \tag{10}$$

Finally, it may be checked numerically that (10) holds when $c=0.7$ and $\delta\in(0.148,0.149)$.

C.8.2 Possibility under Private Monitoring

Consider the following strategy for the mediator (which, together with the hypothesis that the firms always obey the mediator's recommendations at every history, describes a complete strategy profile): First, choose an initial market 1 price $p_1^1\in\left\{\bar{p},\underline{p}\right\}$, uniformly at random. Then privately recommend p_1^1 to firm 1, and subsequently recommend alternation between $\bar{p}$ and $\underline{p}$ (so that, for example, firm 1 is recommended price $\underline{p}$ in period 2 if $p_1^1=\bar{p}$). Always recommend $p_2^2=1$ to firm 2. Finally, recommend $p=0$ in both markets if either firm ever fails to follow its recommendation.

We claim that this strategy profile (together with consistent beliefs) constitutes a perfect Bayesian equilibrium when $c=0.7$ and $\delta\in(0.148,0.149)$. Indeed, there are only two equilibrium conditions to check:

1. It is not profitable for firm 2 to enter market 1:

$$(1-\delta)\max\left\{\underline{p}-c,\frac{1}{2}(\bar{p}-c)\right\}\leq\delta\times(1).$$

   (Note that the left-hand side is firm 2's best deviation gain, as firm 2 always believes that $p_1^1=\bar{p}$ with probability 1/2.)

2. It is not profitable for firm 1 to increase its price from $\underline{p}$ to 1:

$$(1-\delta)\left(1-\underline{p}\right)\leq\delta\times\left(\frac{1}{1+\delta}\bar{p}+\frac{\delta}{1+\delta}\underline{p}\right).$$

It is straightforward to check that, when $\bar{p}=1$ and $\underline{p}=0.85$, both inequalities are satisfied for $c=0.7$ and $\delta\in(0.148,0.149)$.

This completes the proof of Proposition 5.

C.8.3 Asymmetric Demand and Heterogeneous Discounting

We establish the following corollary of Proposition 5:

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Corollary 1 *Suppose demand in market 1 is* 100 *rather than* 1*. Then, for* $c = 0.7$ *and every discount factor in an open interval* $\Delta = \Delta_1 \times \Delta_2 \ni (0.142, 0.949)$*, market segmentation is not sustainable with mediated perfect monitoring but is sustainable with mediated private monitoring.*

We sketch the proof, which is similar to the symmetric case.

With perfect monitoring, define p and $\mu(p; \delta_1)$ as in the symmetric case (where we have clarified that it is firm 1's discount factor on which μ depends). There must then exist a period t in which firm 1's expected payoff is at least $100\mu(p; \delta_1)$ (conditional on firm 1 pricing at p in period 0). By the same argument as in the symmetric case, firm 2's deviation gain in period t is at least $100(1-c)\,x(\mu(p; \delta_1))$. As firm 2's deviation in period 0 is at least $100(p-c)$, we see that market segmentation is not sustainable with mediated perfect monitoring if

$$100 \min_{p \in [0,1]} \max\{p - c, (1-c)\,x(\mu(p; \delta_1))\} > \frac{\delta_2}{1-\delta_2}.$$

This inequality holds with $c = 0.7$, $\delta_1 = 0.142$, and $\delta_2 = 0.949$, and therefore also holds for all (δ_1, δ_2) in a neighborhood of this point.

With private monitoring, when firm 1 alternates between prices $\underline{p}$ and $\bar{p}$ as in the symmetric case, the equilibrium conditions are

1. It is not profitable for firm 2 to enter market 1:

$$100(1-\delta_2)\max\left\{\underline{p} - c, \frac{1}{2}(\bar{p} - c)\right\} \le \delta_2 \times (1).$$

2. It is not profitable for firm 1 to increase its price from $\underline{p}$ to 1:

$$(1-\delta_1)\left(1 - \underline{p}\right) \le \delta_1 \times \left(\frac{1}{1+\delta_1}\bar{p} + \frac{\delta_1}{1+\delta_1}\underline{p}\right).$$

When $\bar{p} = 1$ and $\underline{p} = 0.85$, these inequalities hold (with strict inequality) with $c = 0.7$, $\delta_1 = 0.142$, and $\delta_2 = 0.949$, and therefore also hold for all (δ_1, δ_2) in a neighborhood.

C.9 Proof of Proposition 6

We first show that industry profits are maximized at a price vector (p_h, p_f) with either $p_h = \infty$ or $p_f = \infty$. To see this, suppose industry profits are maximized at (p_h, p_f), and note that the increase in industry profits from moving to (p_h, ∞) is given by

$$\begin{aligned} & \frac{(p_h)^{-\sigma}(p_h - c_h)}{(p_h)^{\gamma-\sigma}} - \frac{(p_h)^{-\sigma}(p_h - c_h)}{(p_h)^{\gamma-\sigma} + (p_f)^{\gamma-\sigma}} - \frac{(p_f)^{-\sigma}(p_f - c_f)}{(p_h)^{\gamma-\sigma} + (p_f)^{\gamma-\sigma}} \\ = \; & \frac{(p_f)^{\gamma-\sigma}}{(p_h)^{\gamma-\sigma} + (p_f)^{\gamma-\sigma}}\left(\frac{p_h - c_h}{(p_h)^{\gamma}} - \frac{p_f - c_f}{(p_f)^{\gamma}}\right). \end{aligned}$$

Hence, if $(p_h - c_h)/(p_h)^{\gamma} \ge (p_f - c_f)/(p_f)^{\gamma}$ then (p_h, ∞) is optimal. Symmetrically, if $(p_h - c_h)/(p_h)^{\gamma} \le (p_f - c_f)/(p_f)^{\gamma}$ then (∞, p_f) is optimal.

Now, since $c_h < c_f$, it must be that a price vector of the form (p_h, ∞) is optimal. Finally, the optimal price p_h is given by $\operatorname{argmax}_{p_h} s_h (p_h - c_h) / (p_h)^\gamma$, or $p_h = (\gamma / (\gamma - 1)) c_h$.

C.10 Proof of Lemma 6

Note that pricing at 0 or ∞ is not consistent with Nash equilibrium, since it is always possible to make a positive profit. So the first-order condition is necessary:

$$\begin{aligned} \frac{d}{dp_h} \frac{(p_h)^{-\sigma} (p_h - c_h)}{(p_h)^{\gamma-\sigma} + (p_f)^{\gamma-\sigma}} &= 0, \\ \frac{d}{dp_f} \frac{(p_f)^{-\sigma} (p_f - c_f)}{(p_h)^{\gamma-\sigma} + (p_h)^{\gamma-\sigma}} &= 0, \end{aligned}$$

or equivalently

$$\begin{aligned} (p_f)^{\gamma-\sigma} &= \frac{p_h - \gamma (p_h - c_h)}{\sigma (p_h - c_h) - p_h} (p_h)^{\gamma-\sigma}, \qquad (11) \\ (p_h)^{\gamma-\sigma} &= \frac{p_f - \gamma (p_f - c_f)}{\sigma (p_f - c_f) - p_f} (p_f)^{\gamma-\sigma}. \qquad (12) \end{aligned}$$

Given $(p_f)^{\gamma-\sigma} > 0$, there is a unique solution with $p_h \geq c_h$ for (11). To see why, for $p_h \in [c_h, \frac{\sigma}{\sigma-1} c_h)$, the right-hand side of (11) is negative; and $\lim_{p_h \downarrow \frac{\sigma}{\sigma-1} c_h} \frac{p_h - \gamma(p_h - c_h)}{\sigma(p_h - c_h) - p_h} (p_h)^{\gamma-\sigma} = \infty$. Since $\frac{p_h - \gamma(p_h - c_h)}{\sigma(p_h - c_h) - p_h} (p_h)^{\gamma-\sigma} \leq 0$ for each $p_h \geq \frac{\gamma}{\gamma-1} c_h$, we are left to show that $\frac{p_h - \gamma(p_h - c_h)}{\sigma(p_h - c_h) - p_h} (p_h)^{\gamma-\sigma}$ is decreasing in p_h for $p_h \in \left(\frac{\sigma}{\sigma-1} c_h, \frac{\gamma}{\gamma-1} c_h \right)$. This may be verified directly by differentiating $\frac{p_h - \gamma(p_h - c_h)}{\sigma(p_h - c_h) - p_h} (p_h)^{\gamma-\sigma}$ and using the assumptions that $\sigma > \gamma > 1$ and $(\sigma - 1)(\gamma - 1) > 1$.

Since $\frac{p_h - \gamma(p_h - c_h)}{\sigma(p_h - c_h) - p_h} (p_h)^{\gamma-\sigma}$ is decreasing in p_h for $p_h > \frac{\sigma}{\sigma-1} c_h$, the solution to (11) also satisfies the second-order condition. Moreover, the solution is increasing in p_f.

A symmetric argument shows that, given $(p_h)^{\gamma-\sigma}$, there is a unique best response p_f, and the best response is increasing in p_h.

The result now follows from the observation that $p_h \to \frac{\sigma}{\sigma-1} c_h$ as $p_f \to 0$, $p_f \to \frac{\sigma}{\sigma-1} c_f$ as $p_h \to 0$, $p_h \to \frac{\gamma}{\gamma-1} c_h$ as $p_f \to \infty$, and $p_f \to \frac{\gamma}{\gamma-1} c_f$ as $p_h \to \infty$.

C.11 Proof of Proposition 7

Let $p_h^* := (\gamma / (\gamma - 1))\, c_h$, and let

$$\begin{aligned}
v_h &: = \frac{p_h^* - c_h}{(p_h^*)^{\gamma}}, \\
v_h^{Nash} &: = \frac{\left(p_h^{Nash}\right)^{-\sigma} \left(p_h^{Nash} - c_h\right)}{\left(p_h^{Nash}\right)^{\gamma-\sigma} + \left(p_f^{Nash}\right)^{\gamma-\sigma}}, \\
v_f^{Nash} &: = \frac{\left(p_f^{Nash}\right)^{-\sigma} \left(p_f^{Nash} - c_h\right)}{\left(p_h^{Nash}\right)^{\gamma-\sigma} + \left(p_f^{Nash}\right)^{\gamma-\sigma}}, \text{ and} \\
v_f^{dev} &: = \max_{p_f} \frac{(p_f)^{-\sigma}}{(p_h^*)^{\gamma-\sigma} + (p_f)^{\gamma-\sigma}} (p_f - c_f),
\end{aligned}$$

be ($1/s_h$ times) the equilibrium payoff, punishment payoffs, and maximum deviation payoff, respectively. With notation as in the proof of Theorem 1, a necessary and sufficient condition for the home-market principle to be sustainable with Nash reversion is

$$\sup_{I_i^t \in \mathcal{I}_i(\pi)} \begin{array}{c} \mathbb{E}\left[s_{j,t} | I_i^t\right] v_f^{dev} + \mathbb{E}\left[\sum_{\tau \geq t+1} \delta^{\tau-t} s_{j,t} | I_i^t\right] v_f^{Nash} \\ -\mathbb{E}\left[\sum_{\tau \geq t+1} \delta^{\tau-t} s_{i,\tau} | I_i^t\right] \left(v_h - v_h^{Nash}\right) \end{array} \leq 0.$$

As in the proof of Theorem 1, the left-hand side of this inequality is larger when the convex hull of the set of beliefs $B_i(\pi)|_{s^i} = \left\{ b_i\left(I_i^t\right) : I_i^t|_{s_{t-1}^i} = s^i, I_i^t \in \mathcal{I}_i(\pi) \right\}$ is larger. Finally, the more informative is π, the larger is the convex hull of $B_i(\pi)$.