# Performance Feedback in Long-Run Relationships: A Rate of Convergence Approach*

Takuo Sugaya
Stanford GSB

Alexander Wolitzky
MIT

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**Abstract**

In repeated principal-agent problems and games, witholding feedback about past performance from the agent/players relaxes incentive constraints and thereby expands the set of implementable outcomes. We investigate the value of withholding feedback by comparing equilibrium payoffs in repeated games with public signals and in repeated games where signals are drawn from the same distribution but are observed only by a principal/mediator. Under a pairwise identifiability condition, we find that the value of withholding feedback is small, in that inefficiency is of the same $1-\delta$ power order in both cases. Thus, while private strategies or monitoring (e.g., "review strategies") can outperform public ones for a fixed discount factor, they cannot accelerate the rate of convergence to the efficient payoff frontier when the folk theorem holds.

**Keywords:** repeated games, repeated agency, imperfect monitoring, feedback, review strategies, rate of convergence, folk theorem, martingales

**JEL codes:** C72, C73

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# 1 Introduction

The design of performance feedback rules is an important aspect of principal-agent relationships. While providing feedback can have many practical benefits, a well-known reason *not* to give feedback is that, by informing agents of their own and others' past performance, feedback lets agents game incentive schemes. In other words, withholding feedback pools agents' information sets, which relaxes incentive constraints and thereby expands the set of implementable outcomes. The goal of this paper is to systematically measure this benefit from withholding feedback in standard repeated agency problems and games.

To assess the value of withholding feedback, we consider a repeated game where, in each period, players take actions $a$, and a signal $y$ is drawn from a distribution $p(y|a)$, which we assume has non-moving support. We compare the equilibrium payoff sets in a version of the game with full feedback (or *public monitoring*), where the signal $y$ is publicly observed, and a version with no feedback, where the signal $y$ is observed only by a principal or mediator, who recommends actions to the players. We call these two versions of the game the *public game* and the *blind game*. By the revelation principle (Forges, 1986), for any discount factor $\delta$, the equilibrium payoff set is (weakly) larger in the blind game than in the public game. Our question is, how much larger?

For any fixed discount factor $\delta < 1$, this question is difficult to answer in any generality, because we lack an explicit characterization of the equilibrium payoff set in public games, and we know even less about the equilibrium payoff set in blind games.[1] We instead adopt a *rate of convergence* approach: under standard identification conditions that ensure that efficiency is attainable in the $\delta \to 1$ limit, how quickly does inefficiency vanish as $\delta \to 1$ in the most efficient equilibrium in the public game, as compared to that in the blind game?

Our main result is that inefficiency is of the same power order of $1 - \delta$ in both games. In this sense, the value of withholding feedback is small.

This result requires some unpacking. A key subtlety is that the order of inefficiency

[1] In a public game, the set of perfect public equilibrium payoffs admits a fixed-point characterization due to Spear and Srivastava (1987) (for agency problems) and Abreu, Pearce, and Stacchetti (1990) (for games). However, once we allow private strategies in games with public monitoring or consider blind games, the equilibrium payoff set at a fixed discount factor is intractable, as in repeated games with private monitoring (Kandori, 2002).

depends on the local geometry of the feasible payoff set. In a finite stage game, the boundary of the feasible payoff set is kinked, which implies inefficiency of power order $(1-\delta)^{1/2}$.[2] In a smooth stage game where the boundary of the feasible payoff set has positive quadratic curvature (as in Green and Porter, 1984; Sannikov, 2007, 2008; or Sadzik and Stacchetti, 2015), inefficiency is of order $1-\delta$.[3] Nonetheless, we show that, regardless of the local geometry of the feasible payoff set, inefficiency is of the same power order in the public game and the blind game.

Another subtlety is that inefficiency in the public and blind games can differ by a factor of $(-\log(1-\delta))^{1/2}$ when the boundary of the feasible payoff set is kinked. In contrast, inefficiency differs only by a constant factor (i.e., the rate of convergence is identical) when the boundary is smooth. Moreover, in a canonical class of smooth principal-agent models (similar to Sannikov, 2008), inefficiency in the public and blind games is identical up to a first-order approximation. Thus, while the value of withholding feedback in always "small" (i.e., no improvement in the power rate of convergence), it is somewhat less small in the kinked case (where there can be a log-factor improvement) than in the smooth case (where there is at most a constant-factor improvement, with no first-order improvement whatsoever in the canonical principal-agent model).

Whether the rate of convergence in $1-\delta$ is a good guide to the impact of feedback in practice must be assessed on a case-by-case basis, as the rate neglects higher-order terms that depend on details of the stage game and the monitoring structure, and these terms can outweigh the rate when $\delta$ is far from 1. However, the fact that the rate does depend on the curvature of the feasible payoff set and can differ between the public and blind games (by a log factor) provides some reassurance that it is a reasonably discerning measure, so our finding that it is largely unaffected by feedback has some economic significance.

The key force behind our results is that pooling information across periods—which is facilitated by withholding feedback—improves monitoring precision, but also necessitates larger rewards and punishments, which reduces the scope for providing incentives by transferring surplus over time rather than destroying it. As we show, these two effects essentially

[2] For public games, this was already observed by Hörner and Takahashi (2016).

[3] For a class of continuous-time principal-agent problems with public monitoring, this was already observed by Sannikov (2008).

cancel out. So, little is gained by withholding feedback.

To see the logic in more detail, consider first a finite stage game, where the boundary of the feasible payoff set is kinked. With public monitoring, Hörner and Takahashi (2016, henceforth HT) established that the rate of convergence toward a strictly individually rational payoff vector is $(1-\delta)^{1/2}$. This result builds on Fudenberg, Levine, and Maskin (1994, henceforth FLM), and similarly relies on *orthogonal enforcement*, where in every period continuation payoffs are transferred along translated tangent hyperplanes. In contrast, in the blind game, one could hope to exceed this rate by (for example) employing a *review strategy* (Radner, 1985; Abreu, Milgrom, and Pearce, 1991; Matsushima, 2004), which aggregates signals over $T$ periods—without providing feedback—before adjusting the players' continuation payoffs.[4]

It is therefore instructive to consider the possible advantage of review strategies over orthogonal enforcement. Heuristically, an efficient review strategy pools information for $T = O\left((1-\delta)^{-1}\right)$ periods—during which the players take constant actions—and then applies a penalty if the number of "good signals" over these periods falls short of a cutoff. Call the number of standard deviations by which the number of good signals falls short of its mean the *score*. Since the number of good signals, normalized by $T^{-1/2}$, is approximately normally distributed, for any cutoff score $z$ the probability that a single signal is pivotal is $O\left(T^{-1/2}\phi(z)\right) = O\left((1-\delta)^{1/2}\phi(z)\right)$.[5] As stage game payoffs are $O(1-\delta)$, incentive compatibility requires that $z$ is at most $O\left((-\log(1-\delta))^{1/2}\right)$. Thus, the cutoff score can increase only slowly as $\delta$ increases, or else the pivot probability decreases very quickly, which violates incentive compatibility. In particular, when $z = O\left((-\log(1-\delta))^{1/2}\right)$, the review strategy's "false positive rate" (and hence its minimum inefficiency) is $\Phi(-z) = O\left(((1-\delta)/(-\log(1-\delta)))^{1/2}\right)$.[6] Review strategies thus yield only a log-factor improvement over orthogonal enforcement when the boundary of the feasible payoff set is kinked. Moreover, no other strategies further improve the rate of convergence.

Next consider the case where the boundary of the feasible payoff set is smooth. Orthogo-

[4] Indeed, HT observed that "It is certainly possible that regarding imperfect monitoring, allowing equilibria in private strategies could accelerate the rate of convergence beyond the results that we have derived... This is left for future research." The current paper resolves this question.

[5] Here and throughout the paper, $\phi$ and $\Phi$ denote the standard normal pdf and cdf, respectively.

[6] This follows from the standard normal Mills ratio approximation: $\Phi(-z) \approx \phi(z)/z$ for $z \gg 0$.

nal enforcement is now more efficient than in the kinked case, as small payoff transfers along translated tangent hyperplanes are more efficient with a smooth boundary. In particular, we show that if the *order of curvature* of the boundary of the feasible payoff set at an exposed point is $\beta \in [1, 2]$, then inefficiency under orthogonal enforcement is $O\left((1-\delta)^{\beta/2}\right)$. For instance, in the positive quadratic curvature case (where $\beta = 2$), inefficiency under orthogonal enforcement is $O(1-\delta)$. However, since review strategies involve infrequent, large continuation payoff movements, their efficiency is the same whether the boundary is kinked or smooth: i.e., inefficiency under review strategies remains of power order $(1-\delta)^{1/2}$. Thus, for any $\beta > 1$ (i.e., whenever the boundary is smooth), orthogonal enforcement outperforms review strategies, and in fact attains the fastest possible rate of convergence to efficiency.

Methodologically, we develop a new technique for bounding equilibrium payoffs in repeated games with private monitoring. The starting point is that continuation payoff rewards or punishments incur an efficiency loss related to the curvature of the boundary of the feasible payoff set, while at the same time providing incentives that are proportional to a likelihood ratio difference $(p(y|a) - p(y|a'))/p(y|a)$. We observe that the likelihood ratio difference is a martingale increment (i.e., the expected likelihood ratio difference under $p(\cdot|a)$ equals 0), so we can apply results from the large deviations theory for martingales to bound the cumulative likelihood ratio difference over any number of periods. This in turn connects the inefficiency and "incentive strength" of any strategy profile, so that any equilibrium where players do not take myopic best responses must incur a certain amount of inefficiency, regardless of whether signals are public or private.

**Relation to the literature.** Our finding that the value of withholding feedback is small contrasts with two important strands of prior literature, which both find that this value is large. These strands share the feature that continuation value transfers (i.e., orthogonal enforcement) are impossible. This feature reduces efficiency under public monitoring, and thereby generates a large value of withholding feedback.

First, Holmström and Milgrom (1987) study a dynamic principal-agent model where the agent exerts effort over $T$ periods, but consumption occurs only at the end of the game. The value of withholding feedback is large: without feedback, first-best profits can be approximated as $T \to \infty$ using a review strategy that resembles the "penalty contract" of Mirrlees (1975); with feedback, optimal contracts are linear in the count of signal realizations, and profits are bounded away from the first best for all $T$. The key difference from our setup is that Holmström and Milgrom's model is not a repeated game (as consumption only occurs once), so there is no way to improve efficiency by transferring continuation payoffs over time.[7]

Second, several papers study principal-agent problems or games that, while repeated, do not permit orthogonal enforcement. Abreu, Milgrom, and Pearce (1991) restrict attention to strongly symmetric equilibria, while Matsushima (2004) and Fuchs (2007) restrict attention to block belief-free equilibria. These classes of equilibria preclude orthogonal enforcement, and, consequently, these papers all find that efficiency is attainable as $\delta \to 1$ only when feedback is withheld.[8] Similarly, in Sannikov and Skrzypacz (2007), pairwise identifiability is violated, so deviations cannot be attributed, and hence orthogonal enforcement is impossible; while Rahman (2014) considers the same model with a mediator who can attribute deviations by randomizing the players' private action recommendations, which restores orthogonal enforcement. In Sannikov and Skrzypacz, the equilibrium set collapses to static Nash; in Rahman, a folk theorem holds.

In past work (Sugaya and Wolitzky, 2017, 2018), we showed that the value of withholding feedback (or "maintaining privacy") is large in some specific repeated and dynamic games when $\delta$ is small. For example, our 2018 paper examined how maintaining privacy can help sustain multi-market collusion. In contrast, the current paper shows that the value of privacy in repeated games is small when $\delta$ is close to 1.

We also relate to the broader literature on feedback in dynamic agency and games. We consider standard repeated games without payoff-relevant state variables, so feedback concerns only past performance, which is payoff-irrelevant in the continuation game. In contrast,

[7] Relatedly, a recent paper by Frick, Iijima, and Ishii (2023) considers a one-shot principal-agent model and studies the rate at which profits converge to the first best as the number of signal observations increases. They find that this rate is much faster for review strategies than for linear contracts.

[8] More precisely, Matsushima considers two-player games where signals are conditionally independent, so each player does not learn about the status of her review. This form of lack of feedback is essential for supporting efficiency in a belief-free equilibrium. Sugaya (2022) shows how mixed strategies can be used to prevent learning with conditionally dependent signals, yielding a general folk theorem under imperfect private monitoring.

most of the literature on feedback in dynamic agency involves dynamic (non-repeated) games with additional state variables, such as an agent's ability (Ederer, 2010; Smolin, 2021), other agents' progress in a tournament (Gershkov and Perry, 2009; Aoyagi, 2010; Ely et al., 2022), whether a project has been completed (Halac, Kartik, and Liu, 2017; Ely et al., 2023), or the evolution of an exogenous state variable (Ely and Szydlowski, 2020; Orlov, Skrzypacz, and Zryumov, 2020; Ball, 2023). An exception is Lizzeri, Meyer, and Persico (2002), who examine optimal two-period agency contracts with and without a "midterm review."

We also contribute to the literature on review strategies, introduced by Rubinstein (1979), Rubinstein and Yaari (1983), and Radner (1985), and developed by Abreu, Milgrom, and Pearce (1991) and Matsushima (2001, 2004). These papers all show that review strategies can support efficient outcomes in various settings when $\delta \to 1$ (or when there is no discounting at all). In contrast, we identify limitations of review strategies when $\delta < 1$, and show that review strategies cannot greatly outperform orthogonal enforcement when $\delta$ is close to 1.

Methodologically, the closest papers are HT, who show that inefficiency is $O\left((1-\delta)^{1/2}\right)$ in repeated finite games with public monitoring; and Sugaya and Wolitzky (2023, henceforth SW), who obtain bounds on the strength of players' equilibrium incentives in repeated finite games with arbitrary (e.g., private) monitoring.[9] Rather than bounding incentives, the current paper derives a tradeoff between incentives and efficiency (see, e.g., program (5) below), and uses it to characterize the rate of convergence. In addition, the arguments in SW are based on variance decomposition, while the current paper requires more precise estimates from martingale large deviations theory.

Finally, our exact characterization of first-order inefficiency in a smooth principal-agent model relates to Sannikov (2008) and Sadzik and Stacchetti (2015), who derived a similar characterization of equilibrium under public monitoring. Here, our main contribution is showing that withholding feedback leaves first-order inefficiency unchanged.

**Outline.** The paper is organized as follows. Section 2 describes the model. Section 3 establishes upper bounds on equilibrium efficiency without feedback. Section 4 establishes that, for any $\beta \in [1, 2]$, these bounds are attainable with feedback (excepting a log factor

[9]HT also consider the rate of convergence toward weakly individually rational payoff vectors, which they show can be strictly slower. We focus on strictly individually rational payoffs.

when $\beta = 1$). Combining these results implies that the gains from withholding feedback are small. Section 5 gives a stronger result for the canonical principal-agent problem. Section 6 concludes and discusses some extensions.

# 2 Preliminaries

A *stage game* $G = (I, A, u)$ consists of a finite set of players $I = \{1, \ldots, N\}$, a product set of actions $A = \times_{i \in I} A_i$, and a payoff function $u_i : A \to \mathbb{R}$ for each $i \in I$. We assume that each $A_i$ is a nonempty, compact metric space, and each $u_i$ is continuous.[10] By the Debreu-Fan-Glicksberg theorem, the stage game admits a Nash equilibrium in mixed actions.

We fix some basic notation: the sets of stage-game Nash and correlated equilibria are $\Sigma^{NE} \subseteq \times_{i \in I} \Delta(A_i)$ and $\Sigma^{CE} \subseteq \Delta(A)$; the feasible payoff set is $F = \text{co}\left(\{u(a)\}_{a \in A}\right) \subseteq \mathbb{R}^N$; the sets of stage-game Nash and correlated equilibrium payoffs are $V^{NE} = \{v : v = u(\alpha)$ for some $\alpha \in \Sigma^{NE}\}$ and $V^{CE} = \left\{v : v = u(\alpha) \text{ for some } \alpha \in \Sigma^{CE}\right\}$; the Euclidean metric and norm on $\mathbb{R}^N$ are $d(\cdot, \cdot)$ and $\|\cdot\|$; the set of unit vectors (or *directions*) in $\mathbb{R}^N$ is $\Lambda = \{\lambda \in \mathbb{R}^N : \|\lambda\| = 1\}$; the boundary of $F$ is bnd$(F)$; the set of exposed points of $F$ is exp$(F)$; and, for any $v \in$exp$(F)$, the set of exposing directions is $\Lambda_v = \{\lambda \in \Lambda : v = \text{argmax}_{w \in F} \lambda \cdot w\}$.[11]

A *monitoring structure* $(Y, p)$ consists of a set of possible signal realizations $Y$ and a family of conditional probability distributions $p(y|a)$. We assume either $Y$ is finite and $y$ is drawn according to a probability mass function $p(y|a)$, or $Y$ is a subset of a measurable space and $y$ is drawn according to a density $p(y|a)$: we use the same notation $p(y|a)$ for both cases. We also assume $p(y|a) > 0$ for all $y \in Y, a \in A$. This non-moving support assumption is crucial: in particular, it excludes perfect monitoring. We also assume that there exists a number $K > 0$ such that, for any $a \in A$, $i \in I$ and $a_i' \in A_i$, we have

$$\mathbb{E}^{y \sim p(\cdot|a)}\left[\exp\left(\theta \frac{p(y|a) - p(y|a_i', a_{-i})}{p(y|a)}\right)\right] \leq \exp\left(\frac{\theta^2 K}{2}\right) \qquad \text{for all } \theta \in \mathbb{R}. \tag{1}$$

[10] As is standard, we linearly extend the payoff functions $u_i$ to distributions $\alpha \in \Delta(A)$. Here and throughout, for any compact metric space $X$, $\Delta(X)$ denotes the set of Borel probability measures on $X$, endowed with the weak* topology.

[11] Recall that, by definition, $v \in \exp(F)$ iff $\Lambda_v$ is non-empty. An example at the end of Section 3.1 will clarify the necessity of considering payoff vectors that are not just extreme but exposed.

This condition says that the likelihood ratio difference between $p(\cdot|a)$ and $p(\cdot|a_i', a_{-i})$ has a *sub-Gaussian distribution*, where the number $K$ is called a *variance proxy*.[12] For example, (1) holds if $Y$ is finite, or if $Y \subseteq \mathbb{R}^n$ and $y = g(a) + \varepsilon$, where $g : A \to Y$ is a deterministic function with a bounded gradient, and $\varepsilon$ has a multivariate normal distribution with covariance matrix independent of $a$. Moreover, in either of these special cases, (1) holds with $K$ equal to the variance of the likelihood ratio difference $(p(y|a) - p(y|a_i', a_{-i}))/p(y|a)$. This quantity is called the $\chi^2$-*divergence* of $p(|a_i', a_{-i})$ from $p(y|a)$.

In a *repeated game with public monitoring* (Abreu, Pearce, and Stacchetti, 1990, henceforth APS; FLM), in each period $t \in \mathbb{N}$, each player $i$ takes an action $a_i$, and then a signal $y$ is drawn according to $p(y|(a_i)_i)$ and is publicly observed. A history for player $i$ at the beginning of period $t$ takes the form $h_i^t = (a_{i,t'}, y_{t'})_{t'=1}^{t-1}$. A strategy $\sigma_i$ for player $i$ maps histories $h_i^t$ to distributions over actions $a_{i,t}$. A strategy for player $i$ is *public* if it depends on $h_i^t$ only through its public component $y^t = (y_{t'})_{t'=1}^{t-1}$. Players choose strategies to maximize discounted expected payoffs, with common discount factor $\delta \in [0,1)$. A *perfect public equilibrium* (*PPE*) is a profile of public strategies that, beginning at any period $t$ and any public history $y^t$, forms a Nash equilibrium from that period on. We denote the repeated game with public monitoring with stage game $G$, monitoring structure $(Y, p)$, and discount factor $\delta$ by $\Gamma^P(\delta)$, and we denote the corresponding set of PPE payoff vectors by $E^P(\delta) \subseteq \mathbb{R}^N$.

In a *blind repeated game* (Sugaya and Wolitzky, 2017, 2023), the players are assisted by a mediator. In each period $t \in \mathbb{N}$, (i) the mediator privately recommends an action $r_i \in A_i$ to each player $i$, (ii) each player $i$ takes an action $a_i$, and (iii) a signal $y$ is drawn according to $p(y|(a_i)_i)$ and is observed only by the mediator. A history for the mediator at the beginning of period $t$ takes the form $h_0^t = \left((r_{i,t'})_i, y_{t'}\right)_{t'=1}^{t-1}$, while a history for player $i$ just before she takes an action in period $t$ takes the form $h_i^t = \left((r_{i,t'}, a_{i,t'})_{t'=1}^{t-1}, r_{i,t}\right)$. A strategy $\sigma_0$ for the mediator maps histories $h_0^t$ to distributions over recommendation profiles $(r_{i,t})_i$, while a strategy $\sigma_i$ for player $i$ maps histories $h_i^t$ to distributions over actions $a_{i,t}$. We denote the blind repeated game with stage game $G$, monitoring structure $(Y, p)$, and discount factor $\delta$ by $\Gamma^B(\delta)$, and we denote the corresponding set of Nash equilibrium payoff vectors by

[12] See, e.g., Buldygin and Kozachenko (2000).

$E^B(\delta) \subseteq \mathbb{R}^N$.[13]

By standard arguments (similar to Forges, 1986), any Nash equilibrium outcome $\mu \in \Delta((A \times Y)^\infty)$ (i.e., any equilibrium distribution over infinite paths of action profiles and signals) in $\Gamma^P(\delta)$ can also be implemented by a Nash equilibrium in $\Gamma^B(\delta)$ where the players follow the mediator's recommendations on path. In particular, $E^P(\delta) \subseteq E^B(\delta)$.

Our goal is to assess the value of withholding feedback from the players. The maximum such value is described by the set of payoffs that are attainable with the smallest possible amount of feedback—i.e., are attainable in $\Gamma^B(\delta)$—but are not attainable with the largest possible amount of feedback—i.e., are not attainable in $\Gamma^P(\delta)$. Since our main result is that the value of withholding feedback is small when $\delta$ is close to 1, there is no harm in "over-estimating" this value by restricting attention to PPE in $\Gamma^P(\delta)$, while admitting all Nash equilibria in $\Gamma^B(\delta)$. We thus estimate the value of withholding feedback by the size of the set $E^B(\delta) \backslash E^P(\delta)$.

**Remark 1** *The model is easily adapted to allow a player with commitment power (such as the principal in a standard principal-agent model) or one or more players with perfectly observed actions (such as a principal who offers contracts each period in a relational contracting model). A player with commitment power is treated like any other player, except that no incentive constraints are imposed on her strategy. For example, in a principal-agent model,* $\Sigma^{NE}$ *is the set of mixed action profiles where the agent does not have a profitable deviation. Moreover, it suffices to impose non-moving support only for the agent, so that* $\operatorname{supp} p(\cdot|a) = \operatorname{supp} p(\cdot|a')$ *for all* $a, a'$ *that agree on the principal's action. Similarly, to extend our results to the case where some players' actions are perfectly observed, let* $I^* \subseteq I$ *be the set of players with observable actions, and assume that deviations by players* $i \in I \backslash I^*$ *do not affect the support of* $p$*, so that* $\operatorname{supp} p(\cdot|a) = \operatorname{supp} p(\cdot|a_i', a_{-i})$ *for all* $a \in A$*,* $i \in I \backslash I^*$*, and* $a_i' \in A_i$*. Then Theorem 1 below applies for any* $v \in \exp(F)$ *that cannot be attained by an action profile distribution* $\alpha$ *such that* $g_i(s_i, \alpha) = 0$ *for each player* $i \in I \backslash I^*$ *and each manipulation* $s_i$*, and Theorem 2 applies verbatim.*

[13] Note that a player's payoff in the blind game is not measurable with respect to her own information. The blind game may thus withhold feedback from the players to an unrealistic extent—but this only strengthens our finding that withholding feedback has limited value.

# 3 Maximum Efficiency without Feedback

## 3.1 Main Result

Our first theorem gives an upper bound for the rate of convergence of $E^B(\delta)$ toward an exposed point $v \in \exp(F)$ that is not attainable as a static correlated equilibrium. The bound depends on the order of curvature of the boundary of $F$ at $v$.

**Definition 1** *Fix an exposed point $v \in \exp(F)$. For $\beta \geq 1$, the boundary of $F$ has* max-curvature of order at least $\beta$ at $v$ *if, for all $\lambda \in \Lambda_v$, there exists $\eta > 0$ such that*

$$\lambda \cdot (v - w) \geq \eta d(v, w)^{\beta} \qquad \text{for all } w \in \operatorname{bnd}(F).$$

*The boundary of $F$ has* max-curvature of order $\beta$ at $v$ *if*

$$\beta = \inf\left\{\tilde{\beta} : \operatorname{bnd}(F) \text{ has } \max\text{-curvature of order at least } \tilde{\beta} \text{ at } v\right\}.$$

This definition says that moving away from $v$ in $F$ entails an efficiency loss of order at least $\beta$, relative to Pareto weights $\lambda$. (Or, heuristically, $\operatorname{bnd}(F)$ is approximated by a power function of degree $\beta$ at $v$.) To understand the definition, the key cases to consider are $\beta = 1$, $\beta = 2$, and the limit case $\beta = \infty$.

- The $\beta = 1$ case arises when the stage game $G$ is *finite*, as in APS, FLM, HT, or SW. Here, $F$ is the convex hull of a finite collection of points, so the boundary of $F$ is kinked at every extreme point. This implies a first-order loss from moving away from any extreme point.

- The $\beta = 2$ case arises when the boundary of $F$ has *positive quadratic curvature*. This is the typical case in smooth games or agency models with continuous actions, such as Green and Porter (1984), Sannikov (2007, 2008), or Sadzik and Stacchetti (2015), as well as the principal-agent model we consider in Section 5. More generally, if $\beta \leq 2$ then the boundary of $F$ has *non-zero quadratic curvature*: its curvature is positive but finite if $\beta = 2$, and is infinite if $\beta < 2$.

- The $\beta = \infty$ case arises when the boundary of $F$ is *linear* at $v$. This is the case in repeated games with transferable utility, as in Athey and Bagwell (2001), Levin (2003), or Goldlücke and Kranz (2012).

- Finally, to appreciate the role of the "max" in the definition, suppose that $N = 2$, $(0, 0) \in F$, and the local boundary of $F$ at $(0, 0)$ is given by $f(x) = -x$ if $x < 0$ and $f(x) = x^2$ if $x \geq 0$. Then the max-curvature of $\text{bnd}(F)$ at $(0, 0)$ is 2.

Most of our insights can be obtained when $\beta \in \{1, 2\}$. We cover the cases where $\beta \in (1, 2)$ and $\beta > 2$ for completeness.

The following is our central technical result.

**Theorem 1** *Fix an exposed point $v \in \exp(F) \backslash V^{CE}$ at which* $\text{bnd}(F)$ *has* max-*curvature of order $\beta$, and a direction $\lambda \in \Lambda_v$. Then there exists $c > 0$ such that*

$$\lambda \cdot (v - w) \geq c\zeta(\delta) \qquad \textit{for all } \delta < 1 \textit{ and } w \in E^B(\delta), \tag{2}$$

*where*

$$\zeta(\delta) = \begin{cases} \left(\frac{1-\delta}{\max\{-\log(1-\delta),1\}}\right)^{1/2} & \textit{if } \beta = 1, \\ (1-\delta)^{\max\{\beta/2, \beta-1\}} & \textit{if } \beta > 1. \end{cases}$$

The key implications of Theorem 1 are as follows:

- For Pareto weights where welfare is maximized at a point where $\text{bnd}(F)$ is kinked (i.e., $\beta = 1$), equilibrium inefficiency in the blind game is at least $O\left(((1-\delta)/(-\log(1-\delta)))^{1/2}\right)$.

- For Pareto weights where welfare is maximized at a point where $\text{bnd}(F)$ has positive quadratic curvature (i.e., $\beta = 2$), equilibrium inefficiency in the blind game is at least $O(1-\delta)$.

We will see that both of these bounds—as well as the $(1-\delta)^{\beta/2}$ bound for $\beta \in (1, 2)$—are tight. Moreover, the bound in the kinked case remains tight up to log-factor slack for the public game, while the bound in the $\beta \in (1, 2]$ case remains tight up to constant-factor slack for the public game. These results imply that the gains from withholding feedback are small.

In contrast, for Pareto weights where welfare is maximized at a point where $\text{bnd}(F)$ is approximately linear, Theorem 1 allows inefficiency much smaller than $1-\delta$. This bound is tight in the $\beta \to \infty$ limit, since in some games with linear Pareto frontiers efficiency is exactly achieved at some $\delta < 1$ (see, e.g., Athey and Bagwell, 2001). We conjecture that the $(1-\delta)^{\beta-1}$ bound given by Theorem 1 is in fact tight for any $\beta > 2$—in that there exists some game and $v \in \exp(F) \setminus V^{CE}$ with max-curvature of order $\beta$ that can be approached at rate $(1-\delta)^{\beta-1}$—but we have not proved this.

A simple intuition for Theorem 1 relies on comparing orthogonal enforcement and review strategies, as described in the introduction. That argument explains why review strategies improve efficiency by a log factor in the kinked case and yield no improvement in the smooth case. Of course, the proof of Theorem 1 must account for arbitrary strategies. We outline the proof in the next subsection. The basic logic is that if a repeated game Nash equilibrium gives payoffs close to $v \in \exp(F)$, then the stage game payoff must be close to $v$ almost all the time along the equilibrium path of play. Since signals have full support, this implies that payoffs remain close to $v$ almost all the time even after low-probability (but on-path) signal realizations. This in turn implies that, on average, equilibrium continuation play does not vary much with the signal realizations. But then, if $v \notin V^{CE}$, we can conclude that $\delta$ must be so high that even small variations in continuation play can provide strong incentives.[14]

We mention a couple technical aspects of the statement of Theorem 1. First, generically, the condition $v \in \exp(F) \setminus V^{CE}$ is equivalent to $v \in \exp(F) \setminus V^{NE}$: since $v$ is extremal, the distinction only matters in the non-generic case where $v$ is attained at two different pure action profiles. Second, the condition $\lambda \in \Lambda_v$ (i.e., $v = \text{argmax}_{w \in F} \lambda \cdot w$) cannot be weakened

[14]This logic is the same as that of Theorem 6.5 of FLM (who credit Madrigale, 1986), which says that an extremal non-static Nash payoff vector $v$ cannot be exactly attained for any $\delta < 1$ under full-support monitoring. FLM state this result for PPE, but the same argument works for Nash. Theorem 1 can be viewed as a quantitative version of this result.

to $v \in \operatorname{argmax}_{w \in F} \lambda \cdot w$. To see this, consider the stage game

| | $L$ | $R$ |
|---|---|---|
| $C$ | $1,1$ | $0,1$ |
| $D1$ | $2,0$ | $-2,0$ |
| $D2$ | $-2,0$ | $2,0$ |

Here the point $v = (1,1)$ is exposed and is not attainable as a static CE; $\operatorname{bnd}(F)$ has curvature of order 1 (i.e., a kink) at $v$; and $v \in \operatorname{argmax}_{w \in F} \lambda \cdot w$ for $\lambda = (0,1)$. But, the point $w = (0.5, 1)$ is attained by the static NE $\left(C, \frac{1}{2}L + \frac{1}{2}R\right)$ (so $w \in E^B(\delta)$ for all $\delta \in [0,1)$) and satisfies $\lambda \cdot w = \lambda \cdot v$, so the conclusion of Theorem 1 fails.

## 3.2 Proof Sketch for Theorem 1

We sketch the proof of Theorem 1, deferring the details to the appendix. Fix any $v \in \exp(F) \setminus V^{CE}$ and $\lambda \in \Lambda_v$. We wish to derive a lower bound for $\lambda \cdot (v - w)$—the inefficiency of $w$ in direction $\lambda$—which holds for any $w \in E^B(\delta)$.

We introduce some notation. Note that any outcome $\mu \in \Delta((A \times Y)^\infty)$ defines a marginal distribution over period-$t$ action profiles, $\alpha_t^\mu \in \Delta(A)$, as well as the *occupation measure* $\alpha^\mu \in \Delta(A)$, defined as

$$\alpha^\mu = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \alpha_t^\mu.$$

Note also that ex ante payoffs under $\mu$ are determined by $\alpha^\mu$, as, by linearity of $u$,

$$(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u(\alpha_t^\mu) = u\left((1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \alpha_t^\mu\right) = u(\alpha^\mu).$$

Thus, letting $\mathcal{M}^B(\delta)$ be the set of Nash equilibrium outcomes $\mu$ in the blind game $\Gamma^B(\delta)$, we wish to derive a lower bound for

$$\inf_{\mu \in \mathcal{M}^B(\delta)} \lambda \cdot (v - u(\alpha^\mu)).$$

Next, for each player $i$, let $S_i$ denote the set of functions $s_i : A_i \to A_i$, which we call *manipulations.* For any $i \in I$, $\alpha \in \Delta(A)$, and $s_i \in S_i$, define the *deviation gain*

$$g_i(s_i, \alpha) = \sum_{a \in A} \alpha(a) \left( u_i(s_i(a_i), a_{-i}) - u_i(a) \right).$$

The interpretation is: if the recommended action profile $a$ is drawn according to $\alpha$ and player $i$ takes $s_i(a_i)$ when recommended $a_i$ rather than obeying the recommendation, her expected payoff gain is $g_i(s_i, \alpha)$. Finally, for any complete history of play $h = (a_t, y_t)_{t=1}^{\infty}$ and any player $i$ and manipulation $s_i$, let

$$\hat{u}_{i,t}(h) = u_i(a_t) - v_i \qquad \text{and} \qquad \ell_{i,t}(s_i, h) = \frac{p(y_t|a_t) - p(y_t|s_i(a_{i,t}), a_{-i,t})}{p(y_t|a_t)}.$$

That is, $\hat{u}_t(h)$ is the difference between player $i$'s realized period $t$ payoff at history $h$ and $v_i$—which we will call player $i$'s period $t$ *reward* at history $h$—and $\ell_t(h)$ is the realized likelihood ratio difference of the period $t$ signal $y_t$ at the period $t$ action profile $a_t$, as compared to the action profile $(s_i(a_{i,t}), a_{-i,t})$ that results when player $i$ manipulates according to $s_i$.

A simple necessary condition for an outcome $\mu$ to be consistent with equilibrium play (see Lemma 6 in the appendix) is that, for each player $i$, manipulation $s_i$, and period $t$, we have

$$g_i(s_i, \alpha_t^{\mu}) \leq \mathbb{E}^{\mu}\left[\ell_{i,t}(s_i, h) \sum_{t'=t+1}^{\infty} \delta^{t'-t} \hat{u}_{i,t'}(h)\right]. \tag{3}$$

This inequality holds because, if it were violated, player $i$ could gain by obeying her recommendation in every period other than $t$, while manipulating according to $s_i$ in period $t$. Given this inequality, since $\mathrm{bnd}(F)$ has max-curvature of order $\beta$ at $v$, we have

$$\begin{aligned} &\inf_{\mu \in \mathcal{M}^B(\delta)} \lambda \cdot (v - u(\alpha^{\mu})) \\ \geq \; &\inf_{\mu \in \Delta((A \times Y)^{\infty})} \sup_{i \in I, s_i \in S_i} \inf_{\substack{(\hat{u}_{i,t}(h))_{t,h} \in [-\bar{u}_i, \bar{u}_i] \\ \text{s.t. (3)}}} \mathbb{E}^{\mu}\left[(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \eta \left|\hat{u}_{i,t}(h)\right|^{\beta}\right], \end{aligned} \tag{4}$$

where $\bar{u}_i$ is the range of $u_i$. Intuitively, the program (4) minimizes the maximum over players $i$ and manipulations $s_i$ of the $\beta^{th}$ moment of the deviation of player $i$'s stage game payoff from $v_i$, subject to the incentive constraint (3).

To prove the theorem, it remains to bound (4) as a function of $\delta$ and $\beta$. To do so, consider the inner problem where $\mu$ is fixed and $(i, s_i) \in \text{argmax}_{i,s_i} g_i(s_i, \alpha^\mu)$. Let $(1-\delta)\delta^{t-1}\xi_t$ denote the Lagrange multiplier on the period $t$ incentive constraint, and form the Lagrangian

$$\sup_{(\xi_t)_t \geq 0} \inf_{(\hat{u}_t(h))_{t,h} \in [-\bar{u},\bar{u}]} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \mathbb{E}^\mu \left[ \eta \left| \hat{u}_t(h) \right|^\beta + \xi_t \left( g_t^\mu - \ell_t(h) \sum_{t'=t+1}^{\infty} \delta^{t'-t} \hat{u}_{t'}(h) \right) \right], \quad (5)$$

where we have simplified notation by letting $\hat{u}_t(h) = \hat{u}_{i,t}(h)$, $g_t^\mu = g_i(s_i, \alpha_t^\mu)$, and $\ell_t(h) = \ell_{i,t}(s_i, h)$. The Lagrangian expresses a tradeoff between efficiency and incentives: to maximize efficiency, the reward $\hat{u}_t(h)$ must minimize the sum of the inefficiency resulting from the curvature of bnd$(F)$ (i.e., $\eta \left| \hat{u}_t(h) \right|^\beta$) and an incentive cost in each earlier period $\tilde{t} < t$ (i.e., $-\xi_{\tilde{t}} \ell_{\tilde{t}}(h) \hat{u}_t(h)$). Moreover, if we take $\xi_t$ to be constant across periods (i.e., $\xi_t = \xi \ \forall t$), we can reverse the order of summation between $t$ and $t'$ (and also note that $\hat{u}_1(h) = 0$ for all $h$ at the optimum) to rewrite the Lagrangian as

$$\sup_{\xi \geq 0} \inf_{(\hat{u}_t(h))_{t \geq 2,h} \in [-\bar{u},\bar{u}]} (1-\delta) \sum_{t=2}^{\infty} \delta^{t-1} \mathbb{E}^\mu \left[ \eta \left| \hat{u}_t(h) \right|^\beta - \xi \mathcal{L}_{t-1}(h) \hat{u}_t(h) \right] + \xi g^\mu,$$

where $g^\mu = g_i(s_i, \alpha^\mu) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} g_t^\mu$. Thus, to bound the Lagrangian, it suffices to bound the probability that $|\mathcal{L}_t(h)|$ is large.[15] Since $\mathcal{L}_t$ is a martingale with sub-Gaussian increments (by (1)), the required bounds can be taken from large deviations theory (e.g., Buldygin and Kozachenko, 2000; see Lemma 9 in the appendix). Intuitively, these bounds reflect the fact that sequences of signals with large cumulative likelihood ratio differences—which are highly informative when they occur—also occur with low equilibrium probability, and hence do not provide a large amount of information on average. These bounds imply that the value of the Lagrangian—and hence inefficiency under outcome $\mu$—cannot be too much smaller than $g^\mu$. Finally, since $v \in \exp(F) \setminus V^{CE}$, if $u(\alpha^\mu)$ is close to $v$ then $g^\mu = \max_{i,s_i} g_i(s_i, \alpha^\mu)$ is not too small (see Lemma 4 in the appendix), which yields the desired efficiency bound.

[15] In the proof, we take $\xi_t = \xi$ for all $t$ less than some cutoff $T$, and $\xi_t = 0$ thereafter. We then need only bound $|\mathcal{L}_t|$ for $t \leq T$.

## 3.3 Tightness of the Efficiency Bound in the Kinked Case

We will see in the next section that inefficiency of order $(1-\delta)^{\beta/2}$ can be attained when $\beta \in [1,2]$ under public monitoring. This implies that the lower bound on inefficiency given in Theorem 1 cannot be improved when $\beta \in (1,2]$ (i.e., in the smooth, non-zero curvature case). Here we show that, when $\beta = 1$ (i.e., in the kinked case), inefficiency of order $\left((1-\delta)/-\log(1-\delta)\right)^{1/2}$ can be obtained in the blind game. This shows that the lower bound on inefficiency in Theorem 1 also cannot be improved when $\beta = 1$. Consequently, withholding feedback can accelerate the rate of convergence by at most a factor of $\left(-\log(1-\delta)\right)^{-1/2}$ in the kinked case.

We consider a one-sided prisoners' dilemma, where the stage game is

| | $L$ | $R$ |
|---|---|---|
| $C$ | $2,2$ | $0,0$ |
| $D$ | $3,0$ | $1,1$ |

and the monitoring structure is given by $Y = \{0,1\}$ and

$$p(y=1|a) = \begin{cases} 1/2 & \text{if } a_1 = C, \\ 1/4 & \text{if } a_1 = D. \end{cases}$$

We investigate the possibility of attaining payoffs close to $(2,2)$. Note that the signal does not depend on player 2's action, but this is no obstacle to attaining payoffs close to $(2,2)$, because at action profile $(C,L)$ player 2 is taking a static best response.

**Proposition 1** *In the one-sided prisoner's dilemma above, there exists $c > 0$ such that, for any sufficiently large $\delta < 1$, there exists $v \in E^B(\delta)$ satisfying*

$$v_1 = v_2 > 2 - c\left(\frac{1-\delta}{-\log(1-\delta)}\right)^{1/2}.$$

**Proof.** We sketch the proof, providing the details in the online appendix. Consider a review strategy where the game is divided into blocks of $T$ consecutive periods. We take $T = \lfloor \rho/(1-\delta) \rfloor$, where $\rho > 0$ is a small number to be determined: note that $\rho \approx 1-\delta^T$

when $\delta \approx 1$. In the first block, the players are prescribed $(C, L)$ in every period. At the end of the first block—as well any subsequent block where $(C, L)$ is prescribed—the mediator records the summary statistic

$$E = \mathbf{1}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(2y_t - 1) \leq -z\right\},$$

where $z > 0$ is a large number to be determined. (Here periods are numbered from the start of the block.) If $E = 0$, the players "pass the review," and $(C, L)$ is prescribed in the next block. If $E = 1$, then with some probability $q \in [0, 1]$ (which also remains to be determined), the players fail the review and $(D, R)$ is prescribed forever. With the complementary probability $1 - q$, the players pass the review anyway, and $(C, L)$ is prescribed in the next block.

We show that the parameters $\rho$, $z$, and $q$ can be chosen so that this strategy profile is an equilibrium that yields payoff $v > 2 - c\left((1-\delta)/(-\log(1-\delta))\right)^{1/2}$ for each player.[16]

Let $p$ be the probability that $E = 1$ when player 1 takes $C$ throughout a block; let $p_1$ be the probability that $E = 1$ when player 1 takes $D$ once and takes $C$ $T - 1$ times; and let $p_T$ be the probability that $E = 1$ when player 1 takes $D$ throughout. Observe that $v$ is given by

$$v = \left(1 - \delta^T\right)2 + \delta^T(1 - pq)v \quad \Longleftrightarrow \quad v = 2 - \frac{\delta^T pqv}{1 - \delta^T}. \tag{PK}$$

At the same time, the incentive conditions that player 1 prefers to take $C$ throughout a block where $(C, L)$ is prescribed, rather than taking $D$ in period 1 only, or always taking $D$, are

$$1 - \delta \leq \delta^T(p_1 - p)qv \quad \text{and} \tag{IC$_1$}$$

$$1 - \delta^T \leq \delta^T(p_T - p)qv. \tag{IC$_T$}$$

Conditions (IC$_1$) and (IC$_T$) are obviously necessary for the review strategy to be an equilibrium; moreover, as shown by Matsushima (2004, p. 846), they are also sufficient.[17] It thus suffices to find $\rho$, $z$, and $q$ that satisfy (PK), (IC$_1$), and (IC$_T$) with $v > 2 -$

[16] We write $v$ instead of $v_i$ here, since the players' payoffs are the same.

[17] Matsushima considered repeated games with two players who receive conditionally independent signals. Conditional independence implies that a player does not learn about her opponents signals during a review block, just as players do not learn about the mediator's signals in $\Gamma^B$. The same argument thus applies here.

$c\left(\left(1-\delta\right)/\left(-\log\left(1-\delta\right)\right)\right)^{1/2}$.

Since the random variable $2y_t - 1$ has zero mean and unit variance when player 1 takes $C$, when $\delta \approx 1$ the central limit theorem implies that the test statistic $\sum_{t=1}^{T}\left(2y_t-1\right)/\sqrt{T}$ is approximately $N\left(0,1\right)$, so that $p \approx \Phi\left(-z\right)$ and $p_1 - p \approx \left(\frac{3}{4}-\frac{1}{2}\right)\frac{\phi(-z)}{\sqrt{T}} = \frac{\phi(-z)}{4\sqrt{T}}$. Therefore, the smallest value for $q$ that satisfies ($\text{IC}_1$) is approximately $\frac{1-\delta}{\delta^T}\frac{4\sqrt{T}}{\phi(-z)v}$. For this value to be less than 1 when $v \approx 2$, we must have $\frac{1-\delta}{\delta^T}\frac{2\sqrt{T}}{\phi(-z)} \leq 1$. Since $\frac{1-\delta}{\delta^T}\sqrt{T} \approx \sqrt{1-\delta}$ and $\phi\left(-z\right) = \exp\left(-z^2/2\right)/\sqrt{2\pi}$, it follows that $z \leq c_0\sqrt{-\log\left(1-\delta\right)}$ for some constant $c_0$. At the same time, by (PK) and ($\text{IC}_1$), we have $v \approx 2 - \frac{\delta^T}{1-\delta^T}\Phi\left(-z\right)qv$ and $1-\delta \approx \delta^T\frac{\phi(-z)}{4\sqrt{T}}qv$, and hence

$$v \approx 2 - 4\frac{\left(1-\delta\right)\sqrt{T}}{1-\delta^T}\frac{\Phi\left(-z\right)}{\phi\left(-z\right)} \approx 2 - \frac{4\sqrt{1-\delta}}{\sqrt{\rho}z},$$

where the second approximation follows because, when $\rho$ is small and $z$ is large, $\frac{\sqrt{(1-\delta)T}}{1-\delta^T} \approx \frac{\sqrt{\rho}}{\rho} = \frac{1}{\sqrt{\rho}}$ and $\frac{\Phi(-z)}{\phi(-z)} \approx \frac{1}{z}$. Taking the largest possible value for $z$ for which $q \leq 1$—i.e., $z = c_0\sqrt{-\log\left(1-\delta\right)}$—now gives the desired bound for $v$. Finally, with this value for $z$ we have $p_T \approx 1$ and $p \approx 0$, so when $\rho \approx 0$, $q \approx 1$, and $v \approx 2$, ($\text{IC}_T$) holds, as the LHS is close to 0 and the RHS is close to 2. The constructed strategies therefore form an equilibrium. ■

# 4 Attainable Efficiency with Feedback

We now ask whether the maximum efficiency levels identified in Theorem 1 can be attained under public monitoring. To this end, denote the set of feasible and strictly individually rational payoffs by $F^* = \left\{v \in F : v_i > \underline{v}_i := \min_{\alpha_{-i}\in\times_{j\neq i}\Delta(A_j)}\max_{a_i\in A_i} u_i\left(a_i,\alpha_{-i}\right)\ \forall i\right\}$. For $v \in \text{bnd}\left(F^*\right)$, define $\Lambda_v^* = \left\{\lambda \in \Lambda : v \in \text{argmax}_{w\in F^*}\lambda \cdot w\right\}$.

**Definition 2** *Fix a boundary point $v \in \text{bnd}\left(F^*\right)$. For $\beta \geq 1$, the boundary of $F^*$ has* min-curvature of order at most $\beta$ *at $v$ if, for all $\lambda \in \Lambda_v^*$, there exists $k > 0$ such that*

$$\lambda \cdot \left(v - w\right) < kd\left(v,w\right)^{\beta} \qquad \textit{for all } w \in \text{bnd}\left(F^*\right).$$

*The boundary of* $F^*$ *has* min-curvature of order $\beta$ at $v$ *if*

$$\beta = \sup\left\{\tilde{\beta} : \text{bnd}\left(F^*\right) \text{ } \textit{has} \text{ } \min\textit{-curvature of order at most } \tilde{\beta} \textit{ at } v\right\}.$$

Definition 2 is a converse of Definition 1, suitably adjusted to apply to all boundary points rather than only exposed points. It says that moving away from $v$ along the boundary of $F^*$ entails an efficiency loss of order at most $\beta$, relative to Pareto weights $\lambda$. For example, if $N = 2$, $(0,0) \in F^*$, and the local boundary of $F^*$ at $(0,0)$ is given by $f(x) = -x$ if $x < 0$ and $f(x) = x^2$ if $x \geq 0$, then the min-curvature of $\text{bnd}(F^*)$ at $(0,0)$ is 1. Note that, at any exposed point $v \in \text{bnd}(F^*)$, the min-curvature of $\text{bnd}(F^*)$ is at least 1 and at most the max-curvature.

The following assumption generalizes standard identification conditions for the public-monitoring folk theorem to the case where action sets can be infinite.

**Assumption 1** There exists $\bar{x} > 0$ such that the following conditions hold:

**i.** For each $i$, there exists a minmax profile against $i$, $\alpha^i \in \times_{j \neq i} \Delta(A_j) \times A_i$, and $x_j : Y \to [-\bar{x}, \bar{x}]$ for each $j \neq i$, such that

$$a_j \in \underset{a'_j}{\text{argmax}}\, u_j\left(a'_j, \alpha_{-j}\right) + \mathbb{E}\left[x_j(y)\,|a'_j, \alpha_{-j}\right] \text{ for all } j \neq i \text{ and } a_j \in \text{supp}(\alpha_j). \quad (6)$$

**ii.** For each $a \in A$, $c \in \{-1, +1\}$, and $(i, j)$ with $i \neq j$, there exists $x_i : Y \to [-\bar{x}, \bar{x}]$ such that

$$a_i \in \underset{a'_i}{\text{argmax}}\, u_i\left(a'_i, \alpha_{-i}\right) + \mathbb{E}\left[x_i(y)\,|a'_i, \alpha_{-i}\right] \quad \text{and} \quad (7)$$

$$a_j \in \underset{a'_j}{\text{argmax}}\, \mathbb{E}\left[c x_i(y)\,|a'_j, a_{-j}\right]. \quad (8)$$

Intuitively, Assumption 1(i) says that, when payoff transfers of magnitude at most $\bar{x}$ are available, players $-i$ can be incentivized to minmax player $i$, and Assumption 1(ii) says that player $i$ can be incentivized to take $a_i$ via transfers from player $j$ without affecting player $j$'s incentive to take $a_j$. These conditions are similar to, e.g., assumptions (A1)–(A3) of Kandori and Matsushima (1998).[18]

Under Assumption 1, we examine the rate of convergence of $E^P(\delta)$ toward a strictly individually rational payoff vector $v \in \mathrm{bnd}(F^*)$. For finite games (where $\beta = 1$), HT show that this rate equals $(1-\delta)^{1/2}$. Thus, withholding feedback can accelerate the rate of convergence by at most a factor of $(-\log(1-\delta))^{-1/2}$ in finite games. We now show that whenever the boundary of $F^*$ has non-zero curvature (i.e., $\beta \leq 2$), the rate is $(1-\delta)^{\beta/2}$. (We discuss the zero curvature case below.) Thus, withholding feedback cannot accelerate the rate of convergence in smooth games with non-zero curvature.

Our result requires the standard assumption that $\dim F^* = N$ and further excludes payoff vectors where some player obtains her maximum feasible payoff.[19]

**Theorem 2** *Assume that Assumption 1 holds and that* $\dim F^* = N$*, and fix any* $v \in \mathrm{bnd}(F^*)$*, satisfying* $v_i < \max_a u_i(a)$ *for all* $i$*, at which* $\mathrm{bnd}(F^*)$ *has* min-*curvature of order* $\beta \geq 1$*. Then there exists* $c > 0$ *such that* $d\left(v, E^P(\delta)\right) \leq c(1-\delta)^{\min\{\beta,2\}/2}$ *for any sufficiently large* $\delta < 1$*.*

Theorem 2 builds on FLM, HT, and SW. As these authors showed, a level of inefficiency relative to an exposed point $v$ and a direction $\lambda \in \Lambda_v$ is attainable under public monitoring if it equals the distance in direction $\lambda$ between $v$ and a self-generating ball $B \subseteq F$. We must thus find a self-generating ball $B \subseteq F$ at distance $O\left((1-\delta)^{\min\{\beta,2\}/2}\right)$ to $v$ in direction $\lambda \in \Lambda_v$. To this end, let $d = d(B, v)$ be the desired distance, and (without loss) let $u = v - d\lambda$ be the closest point to $v$ in $B$. (See Figure 1.) Consider decomposing $u$ into an instantaneous payoff $v$ and continuation payoffs $(w(y))_y$ that lie on the translated tangent hyperplane $H$ with normal vector $\lambda$ passing through the point $\mathbb{E}[w(y)] = v - ((1-\delta)/\delta)\, d\lambda$. Under Assumption 1, the continuation payoffs $(w(y))_y$ can be chosen to enforce $v$ on $H \cap B$ if the diameter of $H \cap B$, which we denote by $x$, is of order $1-\delta$. At the same time, denoting

[18] The difference is that we allow $|A| = \infty$ and state Assumption 1 directly in terms of the existence of transfers $x$ that satisfy (6)–(8), while Kandori and Matsushima assume that $|A| < \infty$ and hence can state conditions in terms of the convex hull of the set of vectors of signal probabilities generated by different actions, which imply the existence of transfers $x$ satisfying (6)–(8) by the separating hyperplane theorem.

[19] To attain the same rate of convergence toward such "max points," one must show that, for $v \in \mathrm{argmax}_{v' \in F^*} \lambda \cdot v'$, as $\lambda$ approaches a coordinate direction $e_i$, $v$ must be implemented by action profiles where player $i$'s deviation gain vanishes. In finite games, HT show that this is possible under a genericity condition on payoffs. For a class of infinite games (i.e., the linear-concave games considered below), this is possible under a bounded cross-partial derivative condition.

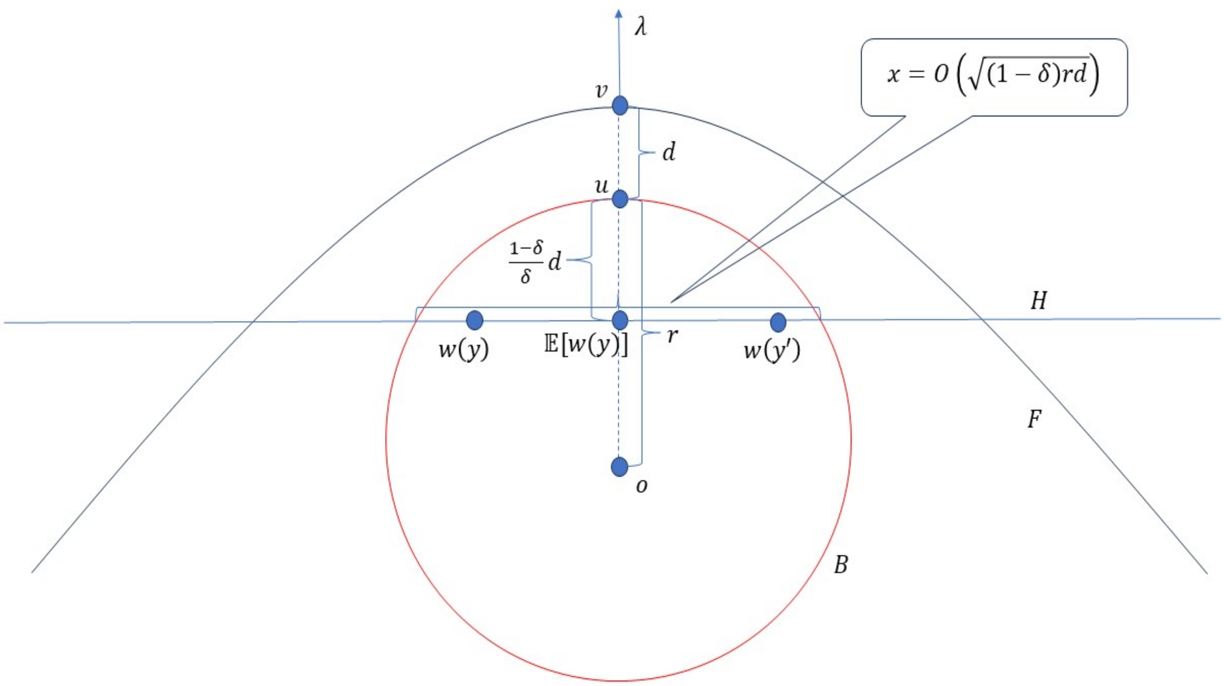


Figure 1: Self-Generating a Ball. To maximize efficiency, $r$ and $d$ must be chosen to minimize $d$ subject to the constraints that $B \subseteq F$ and $x$ is at least $O(1-\delta)$.

the radius of $B$ by $r$, the Pythagorean theorem gives $(x/2)^2 + (r - ((1-\delta)/\delta)\,d)^2 = r^2$, and hence $x = O\left(\sqrt{(1-\delta)\,rd}\right)$. It follows that the product $rd$ is of order $1-\delta$, and hence $r = O\left((1-\delta)^{1-\min\{\beta,2\}/2}\right)$. Finally, for a point $v$ where the (max-)curvature of $\operatorname{bnd}(F)$ equals $\beta$, a ball $B$ with radius $r = O\left((1-\delta)^{1-\min\{\beta,2\}/2}\right)$ and center $v-(r+d)\,\lambda$, where $d = O\left((1-\delta)^{\min\{\beta,2\}/2}\right)$, lies entirely within $F$. For example, if $\beta = 1$ then $r$ and $d$ are both $O\left((1-\delta)^{1/2}\right)$, and thus shrink at the same rate as $\delta \to 1$; while if $\beta \geq 2$ then $r = O(1)$ and $d = O(1-\delta)$, so $B$ simply shifts toward $v$ as $\delta \to 1$.[20]

In light of Theorem 1, when $\beta > 2$ one might hope to find conditions under which $d\left(v, E^P(\delta)\right) = O\left((1-\delta)^{\beta-1}\right)$. While this may be possible, we do not pursue such a result here. The difficulty is that the corresponding ball $B$ would have to have radius $r$ of at least $O\left((1-\delta)^{2-\beta}\right)$ (as $rd$ must be at least $O(1-\delta)$). While such a ball can satisfy the self-generation condition $B \subseteq F$ in a neighborhood of $v$, its radius explodes as $\delta \to 1$ (when $\beta > 2$), so it must violate self-generation at some point far from $v$. Therefore, any conditions that ensure that $d\left(v, E^P(\delta)\right)$ is less than $O(1-\delta)$ must involve the global geometry of the

[20]For the detailed argument for any $\beta$, see Lemma 17 in the online appendix.

feasible payoff set. Investigating such conditions is left for future work.

We finally mention a class of infinite games where Assumption 1(ii) holds.[21] Say that the game is *linear-concave* if (i) for each $i$, $A_i$ is a compact interval $\left[\underline{A}_i, \bar{A}_i\right] \subseteq \mathbb{R}$, and $u_i(a_i, a_{-i})$ is differentiable and concave in $a_i$ for every $a_{-i}$ with a bounded derivative: there exists $\kappa > 0$ such that $|\partial u_i(a_i, a_{-i}) / \partial a_i| \leq \kappa$ for all $i, a$; and (ii) the public signal is a $D$-dimensional real variable, $Y = \times_{d=1}^{D} Y^d \subseteq \mathbb{R}^D$, and $\mu^d(a) = \mathbb{E}\left[y^d | a\right]$ is a linear function of $a$ for each dimension $d$. In a linear-concave game, let $M^i(a) = \left(\frac{d}{da_i}\mu^d(\hat{a})\Big|_{\hat{a}=a}\right)_d$ be a $D$-dimensional vector representing the sensitivity of the mean public signal to player $i$'s action. Say that a linear-concave game satisfies *pairwise identifiability* if for any $a$ and $i \neq j$, $M^i(a) \neq 0$ and the spans of $M^i(a)$ and $M^j(a)$ intersect only at the origin.[22]

**Proposition 2** *In any linear-concave game satisfying pairwise identifiability, Assumption 1(ii) holds.*

## 4.1 Proof of Theorem 2

We recall a key definition and lemma from APS.

**Definition 3** *A bounded set $W \subseteq \mathbb{R}^N$ is* self-generating *if for all $\hat{v} \in W$, there exist $\alpha \in \times_i \Delta(A_i)$ and $w : Y \to \mathbb{R}^N$ satisfying*

**Promise keeping (PK)** $\hat{v} = (1-\delta) u(\alpha) + \delta \int_y w(y) p(y|\alpha) dy$.

**Incentive compatibility (IC)** $\mathrm{supp}(\alpha_i) \subseteq \mathrm{argmax}_{a_i} (1-\delta) u_i(a_i, \alpha_{-i}) + \delta \int_y w_i(y) p(y|a_i, \alpha_{-i}) dy$ *for all $i$.*

**Self-generation (SG)** $w(y) \in W$ *for all $y$.*

*When (PK), (IC), and (SG) hold, we say that the pair $(\alpha, w)$* decomposes *$\hat{v}$* on *$W$.*

**Lemma 1** *Any bounded, self-generating set $W$ is contained in $E^P(\delta)$.*

[21] If Assumption 1(ii) holds, then even if Assumption 1(i) fails a Nash-threat folk theorem still holds, i.e., Theorem 2 holds with $F^*$ replaced by the set of feasible payoffs that Pareto dominate a convex combination of static Nash payoffs.

[22] This condition is the same as Assumption 1 of Sannikov (2007).

It thus suffices to find a bounded, self-generating set $W$ such that $d(v, W) = O\left((1-\delta)^{\beta^*/2}\right)$, where $\beta^* = \min\{\beta, 2\}$. To do so, we first establish a sufficient condition for a ball $B$ to be self-generating. This condition builds on Fudenberg and Levine (1994) and SW.[23]

**Definition 4** *The* maximum score in direction $\lambda \in \Lambda$ with reward bound $\bar{x} > 0$ *is*

$$k(\lambda, \bar{x}) := \sup_{\alpha \in \times_i \Delta(A_i), x: Y \to \mathbb{R}^N} \lambda \cdot \left( u(\alpha) + \int_y x(y) p(y|\alpha) dy \right)$$

*subject to*

1. (IC): $\operatorname{supp}(\alpha_i) \subseteq \operatorname{argmax}_{a_i} u_i(a_i, \alpha_{-i}) + \int_y x_i(y) p(y|a_i, \alpha_{-i}) dy$ *for all* $i$.

2. Half-space decomposability with reward bound $\bar{x}$ (HS$\bar{x}$): $\lambda \cdot x(y) \leq 0$ *and* $\|x(y)\| \leq \bar{x}$ *for all* $y$.

**Lemma 2** *For any* $\bar{x} > \max_{u,u' \in F} \|u - u'\|$ *and* $\varepsilon > 0$*, if a ball* $B$ *of radius* $r$ *satisfies*

$$k(\lambda, \bar{x}) \geq \max_{v' \in B} \lambda \cdot v' + \varepsilon \quad \text{for all } \lambda \in \Lambda, \quad \text{and} \tag{9}$$

$$\bar{x}^2 \leq \frac{\delta}{1-\delta} \frac{\varepsilon r}{36}, \tag{10}$$

*then* $B$ *is self-generating.*

We then show that there exists $B$ with $d(v, B) = O\left((1-\delta)^{\beta^*/2}\right)$ that satisfies the sufficient condition for self-generation just given.

**Lemma 3** *There exist* $\bar{x} > \max_{u,u' \in F} \|u - u'\|$, $c > 0$ *and* $\bar{\delta} < 1$ *such that, for any* $\delta > \bar{\delta}$*, there exist* $\varepsilon > 0$ *and a ball* $B$ *of radius* $r$ *satisfying (9), (10), and* $d(v, B) \leq c(1-\delta)^{\beta^*/2}$.

The proof of Lemma 3 uses the assumptions that $\dim F^* = N$; $v_i \in (\underline{v}_i, \max_a u_i(a))$ for all $i$; $\operatorname{bnd}(F^*)$ has min-curvature of order $\beta \geq 1$ at $v$; and Assumption 1 holds. The logic is similar to that accompanying Figure 1.

The proofs of Lemmas 2 and 3 are deferred to the online appendix. Given these lemmas, taking $\bar{x}$, $c$, and $\bar{\delta}$ as in Lemma 3 establishes Theorem 2.

[23] Lemma 2 is similar to Lemma 6 of SW, but is simpler because the monitoring structure varies together with $\delta$ in SW but is fixed in the current paper, so less control over the relationship between $\delta$ and the reward bound $\bar{x}$ is required.

# 5 A Stronger Result for the Principal-Agent Problem

In this section, we establish that withholding feedback in a canonical repeated principal-agent problem leaves unchanged not only the rate of convergence to efficiency (i.e., the order of inefficiency in $1-\delta$), but also the exact level of first-order inefficiency (i.e., the constant multiplying $1-\delta$). This stronger result also has the virtue of identifying the precise features of the stage game and the monitoring structure that determine the level of first-order inefficiency.

We consider a canonical repeated principal-agent problem in discrete time. In each period $t$, an agent chooses an effort level $a$ from a compact interval $A = \left[0, \bar{A}\right]$, and a signal $y$ is then drawn according to a pmf or pdf $p\left(y|a\right)$. We assume that $p\left(y|a\right)$ is twice continuously differentiable in $a$, with first and second derivatives $p_a\left(y|a\right)$ and $p_{aa}\left(y|a\right)$. A *contract* specifies, for each period $t$, a recommended effort level $r_t \in A$ as a function of the history of past recommendations and signals $\left(r_{t'}, y_{t'}\right)_{t'=1}^{t-1}$, as well as the agent's current consumption $c_t \geq 0$ as a function of $\left(r_{t'}, y_{t'}\right)_{t'=1}^{t}$. In the public game, the agent chooses her period $t$ action $a_t$ as a function of $\left(\left(r_{t'}, a_{t'}, y_{t'}\right)_{t=1}^{t-1}, r_t\right)$; in the blind game, she chooses $a_t$ as a function of $\left(\left(r_{t'}, a_{t'}\right)_{t=1}^{t-1}, r_t\right)$ only.[24] The agent's payoff in period $t$ is $u\left(c_t\right) - \psi\left(a_t\right)$, where the consumption utility $u$ is twice continuously differentiable on $\mathbb{R}_+$ with $u\left(0\right) = 0$, $u' > 0$, $u'' < 0$, $\lim_{c\to\infty} u'\left(c\right) = 0$, and

$$\sup_{c\in[0,\infty)} \frac{u''\left(c\right)}{\left(u'\left(c\right)\right)^3} < 0, \tag{11}$$

and the effort cost $\psi$ is twice continuously differentiable on $A$ with $\psi\left(0\right) = \psi'\left(0\right) = 0$ and $\psi'' > 0$. We discuss the role of condition (11) below. The principal's payoff in period $t$ is $a_t - c_t$. The parties have the same discount factor $\delta \in [0, 1)$.

For any effort level $a \in A$, the *score* of the signal $y$ is

$$\nu\left(y|a\right) = \frac{p_a\left(y|a\right)}{p\left(y|a\right)} \qquad \text{for all } y \in Y,$$

[24] If a contract in the public game is deterministic, in that $r_t$ is a deterministic function of $\left(r_{t'}, y_{t'}\right)_{t'=1}^{t-1}$, then we can equivalently define the agent's strategy in the public game as a function of $\left(a_{t'}, y_{t'}\right)_{t=1}^{t-1}$ only, as the history of signals $\left(y_{t'}\right)_{t=1}^{t-1}$ recursively determines the history of recommendations $\left(r_{t'}\right)_{t=1}^{t-1}$.

and the *Fisher information*—the variance of the score—is

$$\mathcal{I}(a) = \int_y \frac{p_a(y|a)^2}{p(y|a)} dy.$$

We require the following technical assumption, which implies that the Fisher information is finite, strictly positive, and Lipschitz continuous in $a$; the distribution of the score is sub-Gaussian; and a second-order condition holds.

**Assumption 2** The following hold:

**i.** For all $a \in A$, there exists $\Delta > 0$ such that

$$\int_y \frac{\max_{\tilde{a} \in [a, a+\Delta]} p_a(y|\tilde{a})^2}{p(y|a)} dy < \infty.$$

**ii.** $\mathcal{I}(a)$ is strictly positive and Lipschitz continuous on $A$.

**iii.** The score $\nu(y|a)$ is sub-Gaussian with variance proxy $\mathcal{I}(a)$:

$$\int_y \exp(\theta \nu(y|a)) p(y|a) dy \leq \exp\left(\frac{\theta^2 \mathcal{I}(a)}{2}\right) \qquad \text{for all } \theta \in \mathbb{R}.$$

**iv.** There exists $\hat{K}$ such that, for all $a, \hat{a} \in A$, we have

$$\int_y \nu(y|a) p_{aa}(y|\hat{a}) dy \leq 0 \qquad \text{and} \tag{12}$$

$$\int_y \frac{p_{aa}(y|\hat{a})^2}{p(y|a)} dy \leq \hat{K}. \tag{13}$$

For example, Assumption 2 is satisfied if $Y$ is finite, or if $Y \subseteq \mathbb{R}^n$ and $y = g(a) + \varepsilon$ for a deterministic function $g : A \to Y$ and multivariate normal noise $\varepsilon$ with covariance independent of $a$. Note that Assumption 2(iii) strengthens our maintained sub-Gaussianity assumption, (1).

For any $w \in \left[-\psi\left(\bar{A}\right), \bar{u}\right)$, where $\bar{u} = \lim_{c \to \infty} u(c) \in \mathbb{R}_+ \cup \{\infty\}$, let $\bar{F}(w)$ be the first-best payoff for the principal when the agent's payoff equals $w$. This is given by

$$\bar{F}(w) = \max_{a \in A} a - u^{-1}(w + \psi(a)).$$

Let $\bar{a}(w)$ be the maximizer (which is unique, as the maximand is strictly concave), and let $\bar{c}(w) = u^{-1}(w + \psi(\bar{a}(w)))$ be the corresponding consumption for the agent. Note that $\bar{F}$ is twice continuously differentiable, and $\bar{a}$ and $\bar{c}$ are continuously differentiable. In addition, since $\psi'(0) = 0$ and $u' > 0$, we have $\bar{a}(w) > 0$ for all $w \in \left[-\psi\left(\bar{A}\right), \bar{u}\right)$.

Finally, let $F_\delta^B(w)$ (resp., $F_\delta^P(w)$) denote the maximum payoff for the principal over all $v \in E^B(\delta)$ (resp., $v \in E^P(\delta)$) where the agent's payoff is $w$. That is, $F_\delta^B(w)$ is the principal's second-best payoff in the blind game, while $F_\delta^P(w)$ is her second-best payoff in the public game. Recall that $E^B(\delta) \supseteq E^P(\delta)$, so $F_\delta^B(w) \geq F_\delta^P(w)$. Nonetheless, we show that $F_\delta^B(w)$ and $F_\delta^P(w)$ agree up to a first-order approximation as $\delta \to 1$.

**Theorem 3** *For any $\delta < 1$ and $w \in (0, \bar{u})$, we have*

$$\begin{aligned} F_\delta^B(w) &= \bar{F}(w) + \frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2} + o(1-\delta) \qquad \textit{and} \\ F_\delta^P(w) &= \bar{F}(w) + \frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2} + o(1-\delta), \end{aligned}$$

*where in each equation $o(1-\delta)$ stands for a (different) function satisfying* $\lim_{\delta \to 1} o(1-\delta)/(1-\delta) = 0$.

Theorem 3 shows that, whether or not the agent receives feedback, the first-order inefficiency of an optimal contract is precisely

$$\frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2}. \tag{14}$$

When the agent receives feedback (i.e., in the public game), this result is similar to Theorem 5 of Sannikov (2008), but we consider a discrete-time game with a general monitoring structure, while Sannikov considered a continuous-time game with Brownian noise. Sannikov's theorem thus has the inverse volatility of a Brownian motion in place of the more general Fisher information $\mathcal{I}(\bar{a}(w))$ in (14).[25] More importantly, we show that the first-order inefficiency is exactly the same when the agent does not receive feedback. Thus, even when the principal can fully control the feedback received by the agent, she can do little better than to fully reveal the agent's past performance and utilize a public contract.

The intuition for Theorem 3 is that (14) is the first-order inefficiency that results when in every period $t$ the principal implements the first-best effort level $\bar{a}(w_t)$ by offering the corresponding consumption level $\bar{c}(w_t)$ and providing incentives entirely by varying the agent's continuation payoff $w_{t+1}$ while making it a martingale: $\mathbb{E}[w_{t+1}|w_t] = w_t$. For such a contract, martingale large deviations results imply that it is very unlikely that $w_t$ moves more than an $O(1-\delta)$ distance from $w_0$ within a timeframe that has more than an $O(1-\delta)$ payoff impact. Moreover, since $\bar{F}$ is smooth, the contract's first-order inefficiency is zero whenever $w_t$ is within an $O(1-\delta)$ distance from $w_0$. The contract is thus both approximately constant and approximately (i.e., first-order) optimal. Finally, when an approximately constant contract is approximately optimal, there is little loss from providing feedback, since the agent's past performance has little effect on the contract's future behavior. (In contrast, when $\bar{F}$ is kinked at $w_0$, this contract is not optimal up to a first-order approximation—due to a first-order loss at the kink—as we saw in the one-sided prisoner's dilemma in Section 3.3.)

The proof of Theorem 3 is facilitated by the principal's ability to commit to delivering any feasible promised continuation value for the agent. It may be possible to generalize Theorem 3 to a broader class of smooth games (e.g., smooth games with 1-dimensional actions and product structure monitoring, such as those considered in Sections 7.1 and 8 of Sannikov (2007)), but this would require constructing equilibria that attain specific continuation payoff vectors far from the initial target vector. This possibility is left for future research.

We finally comment on the role of condition (11). This condition implies that the second-order efficiency loss from varying the agent's utility is uniformly bounded away from zero. With CRRA utility $u(c) = c^{1-\gamma}/(1-\gamma)$, it holds iff $\gamma \geq 1/2$. Without this condition, review strategies with infrequent, large rewards may yield a first-order improvement over (14) if $u'(c)$ converges to 0 sufficiently slowly as $c \to \infty$.

[25] Another related result is Corollary 1 of Sadzik and Stacchetti (2015), which establishes the importance of the Fisher information in agency models with frequent actions for a class of monitoring structures that converge to Brownian noise.

# 6 Discussion

## 6.1 The Low-Discounting/Low-Monitoring Double Limit

The current paper focuses on the rate at which inefficiency vanishes as $\delta \to 1$, for a fixed monitoring structure. In contrast, in SW we showed that in the double limit where $\delta \to 1$ at the same time as monitoring precision degrades, whether a folk or anti-folk theorem holds depends on a ratio of discounting and monitoring precision. This double limit arises, for example, in the "frequent action limit" considered by Abreu, Milgrom and Pearce (1991), Fudenberg and Levine (2007), Sannikov and Skrzypacz (2010), and Sadzik and Stacchetti (2015), where signals are parameterized by an underlying continuous-time process, actions and signal observations occur simultaneously every $\Delta$ units of time, and the analysis concerns the $\Delta \to 0$ limit.

The results of the current paper immediately extend to the low-discounting/low-monitoring double limit. To see this, maintain the assumption that the monitoring structure is sub-Gaussian with variance proxy $K$, but now view $K$ as a variable that varies simultaneously with the discount factor. Since $K$ proxies for the variance of the likelihood ratio difference, a higher value for $K$ corresponds to more precise monitoring. Thus, the low-discounting/low-monitoring double limit arises when $K \to 0$ and $\delta \to 1$ simultaneously. In the standard frequent action limit, discounting and monitoring vanish at the same rate, so that $(1-\delta)/K$ remains constant.

From this more general perspective, it can be shown (by nearly the same proof) that Theorem 1 holds verbatim with $(1-\delta)/K$ in place of $1-\delta$. Conversely, Theorem 2 also holds with $(1-\delta)/K$ in place of $1-\delta$, under the condition that $\bar{x}$ in Assumption 1 can be taken to be of order $K^{-1/2}$. For example, this condition holds with finite signals with $p(y|a)$ bounded away from zero, or with Gaussian signals.

## 6.2 Summary and Directions for Future Research

This paper has taken a rate-of-convergence approach to studying the value of withholding feedback in standard repeated agency problems and games with patient players. The main result is that this value is “small”: (i) in finite-action settings where the feasible payoff set is kinked, withholding feedback accelerates convergence to efficiency by at most a log factor; (ii) in smooth settings, withholding feedback improves efficiency by at most a constant factor; and (iii) in a canonical smooth repeated principal-agent setting, withholding feedback leaves efficiency unchanged up to a first-order approximation. The key economic force underlying these results is that, while pooling information across many periods leads to more precise monitoring, it also entails larger rewards and punishments, which reduces the scope for providing incentives by transferring continuation value rather than destroying it.

A basic lesson of our analysis is that the value of withholding feedback is very different in a one-off production process that unfolds gradually over time (as in Holmström and Milgrom, 1987) as compared to a genuinely repeated interaction. Since continuation payoff transfers are impossible in one-shot interactions, the monitoring benefit of withholding feedback dominates, so withholding feedback can be very valuable. But in repeated interactions, this benefit is offset by the cost of using larger rewards and punishments, which limit continuation payoff transfers.

We mention some possible extensions of our results. First, as discussed in Section 4, further analyzing rates of convergence toward exposed points with curvature of order $\beta > 2$ is a challenging open question, which involves non-local geometric properties of the feasible payoff set. Second, as discussed in Section 5, it would be interesting to generalize Theorem 3 to games. Third, it would also be interesting to relax the assumption that the likelihood ratio difference is sub-Gaussian. This might result in a faster rate of convergence, because rare but highly informative signals would become more common, and such signals become more useful as $\delta$ increases. Fourth, the rate of convergence when discounting and monitoring vary simultaneously could be studied in detail. For example, it remains to analyze the rate of convergence in the frequent-action limit in the case where different actions of player 1 generate signals of player 2’s action of very unequal precision.

More broadly, we believe that the rate of convergence to efficiency as discounting vanishes can be a useful lens for analyzing other questions about long-run economic relationships, besides the impact of feedback. This may be particularly true in settings with private monitoring, where analyzing equilibrium payoffs for a fixed discount factor is typically intractable.

# Appendix

## A Proof of Theorem 1

We first bound a player's deviation gain at any $\alpha \in \Delta(A)$ that attains payoffs close to $v$.

**Lemma 4** *There exist $\varepsilon > 0$ and $\gamma > 0$ such that, for all $\alpha \in \Delta(A)$ satisfying $\lambda \cdot (v - u(\alpha)) < \varepsilon$, there exist a player $i$ and a manipulation $s_i$ such that $g_i(\alpha, s_i) > \gamma$.*

**Proof.** Since $v \in \exp(F) \setminus V^{CE}$, for all $\alpha \in \Delta(A)$ such that $v = u(\alpha)$, there exist $i$ and $s_i$ such that $g_i(s_i, \alpha) > 0$. Let

$$\gamma = \frac{1}{2} \inf_{\alpha \in \Delta(A): v = u(\alpha)} \sup_{i, s_i} g_i(s_i, \alpha).$$

Note that $\gamma > 0$. To see this, note that $g_i(\mathrm{Id}, \alpha) = 0$ for all $i, \alpha$, so $\gamma \geq 0$, and suppose toward a contradiction that there exists a sequence $\alpha^n$ such that $v = u(\alpha^n)$ for all $n$ and $\sup_{i, s_i} g_i(s_i, \alpha^n) \to 0$. Since $\Delta(A)$ is weak*-compact by Alaoglu's theorem, taking a subsequence if necessary, $\alpha^n \to \alpha \in \Delta(A)$. Moreover, since each $u_i$ is continuous, $u(\alpha) = v$; and since each $A_i$ is compact, by the maximum theorem, $\sup_{s_i} g_i(s_i, \alpha) = \lim_n \sup_{s_i} g_i(s_i, \alpha^n) = 0$ for all $i$, contradicting $v \notin V^{CE}$.

Now suppose that for all $\varepsilon > 0$ there exists $\alpha^\varepsilon \in \Delta(A)$ satisfying $\lambda \cdot (v - u(\alpha^\varepsilon)) < \varepsilon$ and $g_i(s_i, \alpha^\varepsilon) < \gamma$ for all $i, s_i$. Taking a subsequence if necessary, $\alpha^\varepsilon \to \alpha \in \Delta(A)$. Moreover, we have $u(\alpha) = \lim_\varepsilon u(\alpha^\varepsilon) = v$ (since $u(\alpha^\varepsilon) \in F$ and $v \in \exp(F)$), and $\sup_{s_i} g_i(s_i, \alpha) = \lim_\varepsilon \sup_{s_i} g_i(s_i, \alpha^\varepsilon) \leq \gamma$ for all $i$ (by the maximum theorem), so $\sup_{i, s_i} g_i(s_i, \alpha) \leq \gamma$, contradicting the definition of $\gamma$. ■

Fix such $\varepsilon$ and $\gamma$. Next, for any outcome $\mu$ and period $T$, define the occupation measure over the first $T$ periods by $\alpha^{\mu, T} = \left((1-\delta) / \left(1 - \delta^T\right)\right) \sum_{t=1}^{T} \delta^{t-1} \alpha_t^\mu$, and define $T(\delta) = \lceil (\log 2) / (-\log \delta) \rceil$. We first bound $\lambda \cdot (v - u(\alpha^\mu))$ for any $\mu$ where all player's deviation gains over the first $T(\delta)$ periods are small.

**Lemma 5** *For any outcome $\mu$ where $g_i\left(s_i, \alpha^{\mu, T(\delta)}\right) \leq \gamma$ for all players $i$ and manipulations $s_i$, we have $\lambda \cdot (v - u(\alpha^\mu)) \geq \varepsilon / 2$.*

**Proof.** Since $\delta^T \le 1/2$ by construction, we have

$$\begin{aligned}\lambda\cdot(v-u(\alpha^\mu)) &= (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\lambda\cdot(v-u(\alpha_t^\mu))\\ &\ge (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\lambda\cdot(v-u(\alpha_t^\mu)) = \left(1-\delta^T\right)\lambda\cdot\left(v-u\left(\alpha^T\right)\right)\ge\frac{\lambda}{2}\cdot\left(v-u\left(\alpha^T\right)\right).\end{aligned}$$

By construction of $(\varepsilon,\gamma)$, if $\lambda\cdot\left(v-u\left(\alpha^T\right)\right)<\varepsilon$ then $\sup_{i,s_i} g_i\left(\alpha^T,s_i\right)>\gamma$. Hence, $\sup_{i,s_i} g_i\left(\alpha^T,s_i\right)\le\gamma$ implies $\lambda\cdot\left(v-u\left(\alpha^T\right)\right)\ge\varepsilon$, as desired. ■

We next establish the incentive constraint, (3).

**Lemma 6** *For any equilibrium outcome $\mu\in\mathcal{M}^B(\delta)$, player $i$, manipulation $s_i$, and period $t$, we have $g_i(s_i,\alpha_t^\mu)\le\mathbb{E}^\mu\left[\ell_{i,t}(s_i,h)\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}(h)\right]$.*

**Proof.** For any sequence of action profiles $(a_t)_{t=1}^{\infty}$ and any period $t$, let $w_t(h)=\sum_{t'=t}^{\infty}\delta^{t'-t}u_i(a_{t'})$. Since $\mu$ is an equilibrium outcome, for every $t\in\mathbb{N}$ we have

$$g_i(\alpha_t^\mu,s_i)\le\int_{h^t,a_t,y_t}(p(y_t|a_t)-p(y_t|s_i(a_{i,t}),a_{-i,t}))\,\delta\mathbb{E}\left[w_{t+1}(h)\,|h^t,a_t,y_t\right]d\mu\left(h^t,a_t\right)dy_t.$$

This holds because, if she follows her recommendation in every period $t'\neq t$ while manipulating according to $s_i$ in period $t$, player $i$ obtains an expected continuation payoff of $\int_{h^t,a_t,y_t}p(y_t|s_i(a_{i,t}),a_{-i,t})\,\mathbb{E}[w_{t+1}(h)\,|h^t,a_t,y_t]\,d\mu(h^t,a_t)\,dy_t$ in period $t+1$, and this deviation must be unprofitable. The lemma follows as

$$\begin{aligned}&\int_{h^t,a_t,y_t}(p(y_t|a_t)-p(y_t|s_i(a_{i,t}),a_{-i,t}))\,\delta\mathbb{E}\left[w_{t+1}(h)\,|h^t,a_t,y_t\right]d\mu\left(h^t,a_t\right)dy_t\\ =&\int_{h^t,a_t,y_t}p(y_t|a_t)\,\ell_{i,t}(s_i,h)\,\delta\mathbb{E}\left[w_{t+1}(h)\,|h^t,a_t,y_t\right]d\mu\left(h^t,a_t\right)dy_t\\ =&\int_h\ell_{i,t}(s_i,h)\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}(h)\,d\mu(h),\end{aligned}$$

where the last line follows by iterated expectation. ■

We now come to our key lemma, which bounds (5)—and hence $\lambda\cdot(v-u(\alpha^\mu))$—for any $\mu$ where some player's deviation gain over the first $T(\delta)$ periods is large.

**Lemma 7** *There exists $\bar{c} > 0$ such that, for any outcome $\mu$, player $i$, and manipulation $s_i$, and discount factor $\delta < 1$ satisfying $g_i\left(s_i, \alpha^{\mu,T(\delta)}\right) > \gamma$, we have*

$$\sup_{(\xi_t)_t \geq 0} \inf_{(\hat{u}_t(h))_{t,h} \in [-\bar{u},\bar{u}]} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \mathbb{E}^{\mu} \left[ \eta \left| \hat{u}_t(h) \right|^{\beta} + \xi_t \left( g_t^{\mu} - \ell_t(h) \sum_{t'=t+1}^{\infty} \delta^{t'-t} \hat{u}_{t'}(h) \right) \right] \geq \bar{c}\zeta(\delta).$$

Together, Lemmas 5, 6, and 7 imply that $\lambda \cdot (v - u(\alpha^{\mu})) \geq \max\{\varepsilon/2, \bar{c}\zeta(\delta)\} \geq \max\{\varepsilon/2, \bar{c}\}\zeta(\delta)$ for all $\delta < 1$ and $\mu \in \mathcal{M}^B(\delta)$. Theorem 1 therefore holds with $c = \min\{\varepsilon/2, \bar{c}\}$.

It thus remains to prove Lemma 7. To this end, let $\xi_t = \xi \geq 0$ if $t \leq T(\delta)$, and $\xi_t = 0$ otherwise. Letting $T = T(\delta)$ to ease notation, we then have

$$\begin{aligned}
(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \xi_t \ell_t(h) \sum_{t'=t+1}^{\infty} \delta^{t'-t} \hat{u}_{t'}(h) &= (1-\delta)\xi \sum_{t=1}^{T} \ell_t(h) \sum_{t'=t+1}^{\infty} \delta^{t'-1} \hat{u}_{t'}(h) \\
&= (1-\delta)\xi \sum_{t=2}^{\infty} \delta^{t-1} \mathcal{L}_{\min\{t-1,T\}} \hat{u}_t(h), \qquad \text{and} \\
(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \xi_t g_t^{\mu} &= \xi(1-\delta) \sum_{t=1}^{T} \delta^{t-1} g_t^{\mu} = \xi\left(1-\delta^T\right) g_i\left(s_i, \alpha^{\mu,T}\right) \geq \frac{\xi\gamma}{2}.
\end{aligned}$$

In total, we see that (5) is no less than

$$\sup_{\xi \geq 0} \left( \frac{\xi\gamma}{2} + \inf_{(\hat{u}_t(h))_{t\geq 2,h} \in [-\bar{u},\bar{u}]} (1-\delta) \sum_{t=2}^{\infty} \delta^{t-1} \mathbb{E}^{\mu} \left[ \eta \left| \hat{u}_t(h) \right|^{\beta} - \xi \mathcal{L}_{\min\{t-1,T\}} \hat{u}_t(h) \right] \right). \tag{15}$$

The following lemma thus establishes Lemma 7.

**Lemma 8** *For each $\beta \geq 1$, there exists $\bar{c} > 0$ such that, for any $\mu$ and $\delta$, the value of (15) is no less than $\bar{c}\zeta(\delta)$.*

In turn, Lemma 8 relies on the following large deviations bound for martingales.

**Lemma 9** *Let $(X_t)_{t\geq 1}$ be a sequence of martingale increments adapted to a filtration $(H_t)_{t\geq 0}$, so that $\mathbb{E}[X_t|H_{t-1}] = 0$, and let $(\omega_t)_{t\geq 1}$ be a stochastic process adapted to the same filtration satisfying $\mathbb{E}[\exp(\theta X_t)|H_{t-1}] \leq \exp\left(\theta^2\omega_t/2\right)$ for all $t \geq 1$ and $\theta \in \mathbb{R}$. Let $S_T = \sum_{t=1}^{T} X_t$ and $W_T = \sum_{t=1}^{T} \omega_t$. For all $T \geq 1$, we have $\mathbb{E}[\exp(\theta S_T)] \leq \exp\left(\theta^2 W_T/2\right)$, and hence (i) $\Pr(|S_T| \geq x) \leq 2\exp\left(-x^2/(2W_T)\right)$ for all $x \geq 0$, and (ii) $\mathbb{E}[|S_T|^{\varphi}] \leq 2(\varphi W_T/e)^{\varphi/2}$ for all $\varphi \geq 0$.*

**Proof.** By iterated expectation,

$$\mathbb{E}\left[\exp\left(\theta S_T\right)\right] = \mathbb{E}\left[\exp\left(\theta S_{T-1}\right)\mathbb{E}\left[\exp\left(\theta X_T\right)|H_{T-1}\right]\right] \leq \mathbb{E}\left[\exp\left(\theta S_{T-1}\right)\right]\exp\left(\theta^2\omega_T/2\right).$$

Recursively applying the same argument gives $\mathbb{E}\left[\exp\left(\theta S_T\right)\right] \leq \exp\left(\theta^2 W_T/2\right)$. Applying the Chernoff bound then gives (i) and (ii): see, e.g., Lemmas 1.3 and 1.4 of Buldygin and Kozachenko (2000). ■

**Proof of Lemma 8.** We consider separately the cases where $\beta = 1$ and $\beta > 1$.

**Case 1:** When $\beta = 1$, the minimand in (15) is linear in $\hat{u}_t(h)$. Minimizing over $\hat{u}_t(h) \in [-\bar{u}, \bar{u}]$, we see that (15) equals

$$\sup_{\xi \geq 0}\left(\frac{\xi\gamma}{2} + (1-\delta)\sum_{t=2}^{\infty}\delta^{t-1}\int_{h:\xi\left|\Sigma_{t'=1}^{\min\{t-1,T\}}\ell_{t'}(h)\right|\geq\eta}\left(\eta - \xi\left|\mathcal{L}_{\min\{t-1,T\}}\right|\right)\bar{u}d\mu(h)\right). \tag{16}$$

Note that $\mathbb{E}\left[\ell_t|\left(a_{t'}, y_{t'}\right)_{t'=1}^{t-1}, a_t\right] = 0$ and $\mathbb{E}\left[\exp\left(\theta\ell_t\right)|\left(a_{t'}, y_{t'}\right)_{t'=1}^{t-1}, a_t\right] \leq \exp\left(\theta^2 K/2\right)$ for all $t$, $h$, and $\theta$, by (1). Hence, by Lemma 9, $\mathcal{L}_{\min\{t-1,T\}}$ is sub-Gaussian with variance proxy $K\min\{t-1,T\}$, and thus satisfies

$$\Pr\left(\xi\left|\mathcal{L}_{\min\{t-1,T\}}\right| \geq x\right) \leq 2\left(\exp\left(-\frac{x^2}{2\xi^2 K\min\{t-1,T\}}\right)\right).$$

We thus have

$$\begin{aligned}
&\int_{h:\xi\mathcal{L}_{\min\{t-1,T\}}\geq\eta}\left(\eta - \xi\left|\mathcal{L}_{\min\{t-1,T\}}\right|\right)d\mu(h) \\
=\ & \Pr^{\mu}\left(\xi\left|\mathcal{L}_{\min\{t-1,T\}}\right| \geq \eta\right)\eta - \mathbb{E}^{\mu}\left[\mathbf{1}\left\{\xi\left|\mathcal{L}_{\min\{t-1,T\}}\right| \geq \eta\right\}\left|\xi\mathcal{L}_{\min\{t-1,T\}}\right|\right] \\
=\ & -\int_{x\geq\eta}\Pr^{\mu}\left(\xi\left|\mathcal{L}_{\min\{t-1,T\}}\right| \geq x\right)dx \geq -2\int_{x\geq\eta}\exp\left(-\frac{x^2}{2K\xi^2\min\{t-1,T\}}\right)dx,
\end{aligned}$$

where the second equality is by integration by parts. Now note that

$$
\begin{aligned}
\int_{x\geq\eta}\exp\left(-\frac{x^2}{2K\xi^2\min\{t-1,T\}}\right)dx &= \sqrt{2\xi^2K}\sqrt{\min\{t-1,T\}}\int_{y\geq\frac{\eta}{\sqrt{2\xi^2K}\sqrt{\min\{t-1,T\}}}}\exp\left(-y^2\right)dy \\
&\leq \frac{2\xi^2K}{\eta}\min\{t-1,T\}\exp\left(-\frac{\eta^2}{2\xi^2K\min\{t-1,T\}}\right) \\
&\leq \frac{2\xi^2KT}{\eta}\exp\left(-\frac{\eta^2}{2\xi^2KT}\right), \qquad (17)
\end{aligned}
$$

where the first inequality uses the Mills ratio inequality $\phi(-x)/\Phi(-x)\geq x$ for $x\geq 0$. Hence, (16) is no less than

$$
\sup_{\xi\geq 0}\left(\frac{\xi\gamma}{2}-\frac{4\bar{u}\xi^2K}{\eta}(1-\delta)\sum_{t=2}^{\infty}\delta^{t-1}T\exp\left(-\frac{\eta^2}{2\xi^2KT}\right)\right)\geq\sup_{\xi\geq 0}\xi\left(\frac{\gamma}{2}-\frac{4\bar{u}\xi KT}{\eta}\exp\left(-\frac{\eta^2}{2\xi^2KT}\right)\right).
$$

Finally, letting

$$
\xi^*=\eta\left(KT\max\left\{\log\left(2^8KT\bar{u}^2\gamma^{-2}\right),1\right\}\right)^{-1/2},
$$

we have

$$
\begin{aligned}
&\sup_{\xi\geq 0}\xi\left(\frac{\gamma}{2}-\frac{4\bar{u}\xi KT}{\eta}\exp\left(-\frac{\eta^2}{2\xi^2KT}\right)\right) \\
\geq\;& \xi^*\left(\frac{\gamma}{2}-4\bar{u}\sqrt{KT}\left(\max\left\{\log\left(2^8KT\bar{u}^2\gamma^{-2}\right),1\right\}\right)^{-1/2}\exp\left(-\frac{1}{2}\max\left\{\log\left(2^8KT\bar{u}^2\gamma^{-2}\right),1\right\}\right)\right) \\
\geq\;& \xi^*\left(\frac{\gamma}{2}-4\bar{u}\sqrt{KT}\exp\left(-\frac{1}{2}\log\left(2^8KT\bar{u}^2\gamma^{-2}\right)\right)\right) \\
=\;& \frac{\xi^*\gamma}{4} \\
=\;& \frac{\eta\gamma}{4}\left(KT\max\left\{\log\left(2^8KT\bar{u}^2\gamma^{-2}\right),1\right\}\right)^{-1/2} \\
\geq\;& \frac{\eta\gamma}{4}\left((4K\log 2)\max\left\{\log\left(2^9(\log 2)K\bar{u}^2\gamma^{-2}\right),1\right\}\right)^{-1/2}\zeta(\delta),
\end{aligned}
$$

where the last inequality follows because $T\leq\lceil(\log 2)/(1-\delta)\rceil\leq 2(\log 2)/(1-\delta)$ and $\max\{-\log xy,1\}\leq 2\max\{-\log x,1\}\max\{-\log y,1\}$ for all $x,y\in\mathbb{R}$. This is a constant multiple of $\zeta(\delta)$, as desired.

**Case 2:** When $\beta > 1$, the minimand in (15) is convex in $\hat{u}_t(h)$. Relaxing the constraint $\hat{u}_t(h) \in [-\bar{u}, \bar{u}]$ and minimizing over $\hat{u}_t(h) \in \mathbb{R}$ gives

$$\hat{u}_t(h) = \left(\frac{\xi}{\eta\beta}\right)^{\frac{1}{\beta-1}} \operatorname{sign}\left(\mathcal{L}_{\min\{t-1,T\}}\right) \left|\mathcal{L}_{\min\{t-1,T\}}\right|^{\frac{1}{\beta-1}} \qquad \text{for all } t \geq 2.$$

Hence, substituting for $\hat{u}_t(h)$, (15) is no less than

$$\sup_{\xi \geq 0} \left( \frac{\xi\gamma}{2} - \xi^{\frac{\beta}{\beta-1}} \left(\frac{1}{\eta}\frac{\beta^{\beta-1}-1}{\beta^\beta}\right)^{\frac{1}{\beta-1}} (1-\delta) \sum_{t=2}^{\infty} \delta^{t-1} \mathbb{E}\left[\left|\mathcal{L}_{\min\{t-1,T\}}\right|^{\frac{\beta}{\beta-1}}\right] \right). \tag{18}$$

By Lemma 9, $\mathcal{L}_{\min\{t-1,T\}}$ is sub-Gaussian with variance proxy $K \min\{t-1, T\} \leq K(t-1)$, and thus satisfies

$$\begin{aligned} \mathbb{E}^\mu\left[\left|\mathcal{L}_{\min\{t-1,T\}}\right|^{\frac{\beta}{\beta-1}}\right] &\leq 2\left(\frac{\beta}{e(\beta-1)}\right)^{\frac{\beta}{2(\beta-1)}} (K(t-1))^{\frac{\beta}{2(\beta-1)}} \\ &\leq 2\left(\frac{\beta}{e(\beta-1)}\right)^{\frac{\beta}{2(\beta-1)}} K^{\frac{\beta}{2(\beta-1)}} (t-1)^{\max\left\{\frac{\beta}{2(\beta-1)},1\right\}}. \end{aligned}$$

Next, for any $\vartheta \geq 1$, we let $k(\vartheta) \geq 1$ satisfy

$$\sum_{t=1}^{\infty} \delta^t t^\vartheta \leq \frac{k(\vartheta)}{(1-\delta)^{\vartheta+1}} \qquad \text{for all } \delta. \tag{19}$$

(The existence of such $k(\vartheta)$ follows from the standard fact that $\sum_{t=1}^{\infty} \delta^t t^\vartheta = \Gamma(\vartheta+1)(1-\delta)^{-(\vartheta+1)} + O\left((1-\delta)^{-\vartheta}\right)$: see, e.g., Wood, 1992, eqn. (6.4).) With this definition, we have

$$\begin{aligned} &(1-\delta) \sum_{t=2}^{\infty} \delta^{t-1} \mathbb{E}\left[\left|\mathcal{L}_{\min\{t-1,T\}}\right|^{\frac{\beta}{\beta-1}}\right] \\ \leq\ & 2\left(\frac{\beta}{e(\beta-1)}\right)^{\frac{\beta}{2(\beta-1)}} K^{\frac{\beta}{2(\beta-1)}} k\left(\max\left\{\tfrac{\beta}{2(\beta-1)},1\right\}\right)(1-\delta)^{-\max\left\{\frac{\beta}{2(\beta-1)},1\right\}} \\ \leq\ & 2\left(\frac{\beta}{e(\beta-1)}\right)^{\frac{\beta}{2(\beta-1)}} K^{\frac{\beta}{2(\beta-1)}} k\left(\max\left\{\tfrac{\beta}{2(\beta-1)},1\right\}\right)(1-\delta)^{-\max\left\{\frac{\beta}{2(\beta-1)},1\right\}}. \end{aligned}$$

Thus, (18) is no less than

$$\sup_{\xi\geq 0}\xi\left(\frac{\gamma}{2}-2\left(\frac{\beta}{e\left(\beta-1\right)}\right)^{\frac{\beta}{2(\beta-1)}}\left(\frac{\beta^{\beta-1}-1}{\eta\beta^{\beta}}\right)^{\frac{1}{\beta-1}}K^{\frac{\beta}{2(\beta-1)}}k\left(\max\left\{\tfrac{\beta}{2(\beta-1)},1\right\}\right)\left(\frac{\xi}{\left(1-\delta\right)^{\max\{\beta/2,\beta-1\}}}\right)^{\frac{1}{\beta-1}}\right).$$

Since the coefficient of $\left(\xi/\left(1-\delta\right)^{\max\{\beta/2,(\beta-1)\}}\right)^{\frac{1}{\beta-1}}$ is independent of $\delta$, there exists $\hat{c}>0$ such that if $\xi=4\hat{c}\left(1-\delta\right)^{\max\{\beta/2,\beta-1\}}$ then the resulting value is no less than $\xi\gamma/4$, which is again a constant multiple of $\zeta\left(\delta\right)$. ■

# B Proof of Theorem 3

We first show that first-order inefficiency in the blind game is no less than (14). Fix $\delta<1$, $w\in(0,\bar{u})$, and a Nash equilibrium in the blind game where the agent's payoff is $w$. Let $\mu\in\Delta\left(\left(A\times Y\right)^{\infty}\right)$ and $\alpha\in\Delta\left(A\right)$ be the corresponding outcome and occupation measure. Let $\hat{u}_t=u\left(c_t\right)-\psi\left(a_t\right)-w$.

By feasibility, the principal's payoff is at most

$$\left(1-\delta\right)\sum_{t=1}^{\infty}\delta^{t-1}\mathbb{E}^{\mu}\left[\bar{F}\left(w+\hat{u}_t\right)\right].$$

At the same time, incentive compatibility requires that

$$\mathbb{E}^{\alpha_t}\left[\frac{\psi'\left(a\right)^2}{\mathcal{I}\left(a\right)}\right]\leq\mathbb{E}^{\mu}\left[\frac{\psi'\left(a_t\right)\nu_t\left(y_t|a_t\right)}{\mathcal{I}_t\left(a_t\right)}\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}\right]\qquad\text{for all } t. \tag{20}$$

(See Lemma 22 in the online appendix). Intuitively, if (20) were violated, the agent could profitably deviate by slightly reducing her effort in period $t$. Finally, promise keeping implies that

$$\left(1-\delta\right)\sum_{t=1}^{\infty}\delta^{t-1}\mathbb{E}^{\mu}\left[\hat{u}_t\right]=0. \tag{21}$$

The following is the key lemma.

**Lemma 10** *There exist $c, \varepsilon > 0$ such that, for any sufficiently large $\delta < 1$, we have*

$$\begin{aligned}
&\max_{\substack{(\hat{u}_t(h))_{t,h} \geq -\left(w + h\left(\bar{A}\right)\right) \\ \text{s.t. (20) and (21)}}} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \mathbb{E}^{\mu} \left[\bar{F}\left(w + \hat{u}_t\left(h\right)\right)\right] \\
\leq \quad & \bar{F}\left(w\right) + \frac{1-\delta}{\delta} \mathbb{E}^{\alpha} \left[\frac{\psi'\left(a\right)^2}{\mathcal{I}\left(a\right)}\right] \frac{\bar{F}''\left(w\right)}{2} + c\left(1-\delta\right)^{1+\varepsilon}. \qquad (22)
\end{aligned}$$

We sketch the proof of Lemma 10, relegating the details to the online appendix. Subtracting $\bar{F}(w)$ from both sides of (22), multiplying both sides by $2\delta / \left(\left(1-\delta\right) \bar{F}''\left(w\right)\right)$, and taking a second-order Taylor approximation of the LHS (where the first-order term is zero by (21)) gives

$$\inf_{(\hat{u}_t(h))_{t,h} \geq -\left(w + \psi\left(\bar{A}\right)\right) \text{ s.t. (20)}} \sum_{t=1}^{\infty} \delta^t \mathbb{E}^{\mu} \left[\hat{u}_t\left(h\right)^2\right].$$

To establish Lemma 10, it suffices to show that the value of this program exceeds $\mathbb{E}^{\alpha}\left[\psi'\left(a\right)^2 / \mathcal{I}\left(a\right)\right]$. To see this, take a common Lagrange multiplier of $2\left(1-\delta\right)\delta^{t-1}$ on (20) for each $t$. Then, by weak duality, the value of the program is no less than

$$\inf_{(\hat{u}_t(h))_{t,h}} \sum_{t=1}^{\infty} \mathbb{E}^{\mu} \left[\delta^t \hat{u}_t\left(h\right)^2 - 2\frac{\psi'\left(a_t\right) \nu\left(y_t | a_t\right)}{\mathcal{I}\left(a_t\right)} \frac{1-\delta}{\delta} \sum_{t'=t+1}^{\infty} \delta^{t'} \hat{u}_{t'}\left(h\right)\right] + 2\mathbb{E}^{\alpha}\left[\frac{\psi'\left(a\right)^2}{\mathcal{I}\left(a\right)}\right]. \qquad (23)$$

Taking the first-order condition for $\hat{u}_t\left(h\right)$ in (23) gives

$$\hat{u}_t\left(h\right) = \frac{1-\delta}{\delta} \sum_{t'=1}^{t-1} \frac{\psi'\left(a_{t'}\right) \nu\left(y_{t'} | a_{t'}\right)}{\mathcal{I}\left(a_{t'}\right)} \qquad \text{for all } t \geq 2 \text{ and } h,$$

and substituting this equation into (23) (together with $\hat{u}_1\left(h\right) = 0$ for all $h$), gives

$$-\left(\frac{1-\delta}{\delta}\right)^2 \sum_{t=2}^{\infty} \delta^t \mathbb{E}^{\mu} \left[\left(\sum_{t'=1}^{t-1} \frac{\psi'\left(a_{t'}\right) \nu\left(y_{t'} | a_{t'}\right)}{\mathcal{I}\left(a_{t'}\right)}\right)^2\right] + 2\mathbb{E}^{\alpha}\left[\frac{\psi'\left(a\right)^2}{\mathcal{I}\left(a\right)}\right] = \mathbb{E}^{\alpha}\left[\frac{\psi'\left(a\right)^2}{\mathcal{I}\left(a\right)}\right],$$

where the equality follows because, since $\left(\psi_t\left(a_t\right) v_t\left(y_t | a_t\right) / \mathcal{I}_t\left(a_t\right)\right)_t$ is a sequence of martingale increments, we have

$$\begin{aligned}
\mathbb{E}^{\mu}\left[\left(\sum_{t'=1}^{t-1}\frac{\psi'(a_{t'})\,\nu(y_{t'}|a_{t'})}{\mathcal{I}(a_{t'})}\right)^2\right] &= \sum_{t'=1}^{t-1}\mathbb{E}^{\mu}\left[\left(\frac{\psi'(a_{t'})\,\nu(y_{t'}|a_{t'})}{\mathcal{I}(a_{t'})}\right)^2\right] \\
&= \sum_{t'=1}^{t-1}\mathbb{E}^{\alpha_{t'}}\left[\left(\frac{\psi'(a)}{\mathcal{I}(a)}\right)^2\mathbb{E}\left[\nu(y|a)\right]\right] = \sum_{t'=1}^{t-1}\mathbb{E}^{\alpha_{t'}}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right],
\end{aligned}$$

and hence

$$\begin{aligned}
&\left(\frac{1-\delta}{\delta}\right)^2\sum_{t=2}^{\infty}\delta^t\mathbb{E}^{\mu}\left[\left(\sum_{t'=1}^{t-1}\frac{\psi'(a_{t'})\,\nu(y_{t'}|a_{t'})}{\mathcal{I}(a_{t'})}\right)^2\right] = \left(\frac{1-\delta}{\delta}\right)^2\sum_{t=2}^{\infty}\delta^t\sum_{t'=1}^{t-1}\mathbb{E}^{\alpha_{t'}}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] \\
= \; &(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\mathbb{E}^{\alpha_t}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] = \mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]. \qquad (24)
\end{aligned}$$

At the same time, since $\bar{F}$ is strictly concave, there exists $\varepsilon_1 > 0$ such that the principal's payoff is at most $\bar{F}(w) - \varepsilon_1\mathbb{E}^{\alpha}\left[(a-\bar{a}(w))^2\right]$. (See Lemma 19 in the online appendix.) So, together with Lemma 10, the following lemma establishes that first-order inefficiency in the blind game is no less than (14).

**Lemma 11** *There exist $\hat{c}, \hat{\varepsilon} > 0$ such that, for any sufficiently large $\delta < 1$, we have*

$$\begin{aligned}
&\max_{\alpha}\min\left\{\frac{1-\delta}{\delta}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]\frac{\bar{F}''(w)}{2} + c(1-\delta)^{1+\varepsilon}, -\varepsilon_1\mathbb{E}^{\alpha}\left[(a-\bar{a}(w))^2\right]\right\} \\
\leq \; &\frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2} + \hat{c}(1-\delta)^{1+\hat{\varepsilon}}.
\end{aligned}$$

We now show that first-order inefficiency in the public game is no more than (14). The proof is constructive. As a first step, it is helpful to first construct a static contract that induces a target effort level $\bar{a} \in A$. In particular, if the agent is rewarded with a utility of $(\psi'(\bar{a})/\mathcal{I}(\bar{a}))\,\nu(y|\bar{a})$ following any signal realization $y$, she chooses $a$ to maximize

$$\int_y \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a})\,p(y|a)\,dy - \psi(a).$$

The solution is $a = \bar{a}$, because $a = \bar{a}$ satisfies the first-order condition

$$\int_y \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})} \nu(y|\bar{a})\, p_a(y|a)\, dy = \psi'(a),$$

and the second-order condition holds by (12). Moreover, the expected reward equals zero.

Heuristically, the repeated game equilibrium is constructed by using the above reward to adjust the agent's continuation payoff $w_t$ after each period $t$ (so the agent's continuation payoff is a martingale), while targeting effort level $\bar{a}(w_t)$ in each period $t$. This heuristic requires two adjustments, however. First, if the score $\nu(y|a)$ is unbounded, we must truncate the reward for extreme scores, and then further adjust the reward so the agent's first-order condition is exactly satisfied. Second, it is convenient to target zero effort once the agent's continuation payoff $w_t$ strays too far from its initial value $w$.

Formally, fix any $\varepsilon > 0$. Recursively, given the agent's promised continuation payoff $w_t(h^t)$ at history $h^t = (r_{t'}, y_{t'})_{t'=1}^{t-1}$, we define the recommended period $t$ action $r_t(h^t)$ and consumption level $c_t(h^t)$ (which is independent of the period $t$ signal $y_t$), as well as the next period continuation payoff $w_{t+1}(h^t, y_t)$, as follows. First, say that a history $h^t$ is *regular* if $|w_t(h^t) - w| \leq (1-\delta)^{1/2-\varepsilon}$, and *irregular* otherwise. At a regular history, define $r_t(h^t) = \bar{a}(w_t(h^t))$, $c_t(h^t) = \bar{c}(w_t(h^t))$, and

$$w_{t+1}\left(h^t, y_t\right) = w_t\left(h^t\right) + \frac{1-\delta}{\delta} x_{r_t(h^t)}(y),$$

where, for each $\bar{a} \in A$, $x_{\bar{a}}(y)$ is an adjusted version of $(\psi'(\bar{a})/\mathcal{I}(\bar{a}))\,\nu(y|\bar{a})$ (constructed in Lemma 25 in the online appendix) that satisfies

$$\begin{aligned}
\bar{a} &\in \underset{a}{\operatorname{argmax}} \int_y x_{\bar{a}}(y)\, p(y|a)\, dy - \psi(a), \\
\int_y x_{\bar{a}}(y)\, p(y|\bar{a})\, dy &= 0, && (25) \\
\int_y x_{\bar{a}}(y)^2\, p(y|\bar{a})\, dy &= \frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})} + O(1-\delta), \qquad \text{and} && (26) \\
|x_{\bar{a}}(y)| &\leq (1-\delta)^{-1/4}.
\end{aligned}$$

At an irregular history, define $r_t(h^t) = 0$, $c_t(h^t) = u^{-1}(w_t(h^t))$, and $w_{t+1}(h^t, y_t) = w_t(h^t)$ for all $y_t$. Note that the initial history $h^1$ is regular, and that if a history $h^t$ is irregular, then so is every subsequent history. Note also that, by construction, $|w_{t+1}(h^t, y_t) - w_t(h^t)| = O(1-\delta)^{3/4}$ for every regular history $h^t$ and signal $y_t$. Since $|w_t(h^t) - w| \leq (1-\delta)^{1/2-\varepsilon}$ for every regular history $h^t$, this implies that, for sufficiently large $\delta < 1$, we have $w_t(h^t) \in [0, 2w]$ for *every* history $h^t$.

The proof is completed by the following lemma, which shows that the first-order inefficiency of this equilibrium is no more than (14).

**Lemma 12** *There exist $\tilde{c}, \tilde{\varepsilon} > 0$ such that, for any sufficiently large $\delta < 1$, the principal's payoff in the above equilibrium is no less than*

$$\bar{F}(w) + \frac{1-\delta}{\delta} \frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))} \frac{\bar{F}''(w)}{2} - \tilde{c}(1-\delta)^{1+\tilde{\varepsilon}}.$$

Intuitively, since $w_t(h^t)$ is a martingale with volatility of order $(1-\delta)^2$ (by (25) and (26)), it is very unlikely that $w_t(h^t)$ moves more than a $O(1-\delta)$ distance away from $w$ within a timeframe that has more than an $O(1-\delta)$ payoff impact. Consequently, the principal's payoff is almost entirely determined by her payoff at regular histories, and thus equals

$$\mathbb{E}^{\mu}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\bar{F}\left(w_t\left(h^t\right)\right)\right] + o(1-\delta),$$

where $\mu$ is the equilibrium outcome. Taking a second-order Taylor expansion around $w_t(h^t) = w$ and ignoring the remainder, this equals

$$\bar{F}(w) + \mathbb{E}^{\mu}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\frac{\bar{F}''(w)}{2}\left(w_t\left(h^t\right) - w\right)^2\right].$$

Since

$$w_t\left(h^t\right) = w + \frac{1-\delta}{\delta}\sum_{t'=1}^{t-1} x_{a_{t'}}(y_{t'}) \qquad \text{for all regular histories } h^t$$

and (26) holds, the same calculation as for (24) gives

$$\mathbb{E}^{\mu}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\frac{\bar{F}''(w)}{2}\left(w_t\left(h^t\right)-w\right)^2\right]=\frac{1-\delta}{\delta}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)}{\mathcal{I}(a)}\right]\frac{\bar{F}''(w)}{2}+o(1-\delta),$$

where $\alpha$ is the equilibrium occupation measure. Finally, since $w_t\left(h^t\right)$ is close to $w$ with high probability under $\alpha$, we have $\mathbb{E}^{\alpha}\left[\psi'(a)/\mathcal{I}(a)\right]=\psi'(\bar{a}(w))/\mathcal{I}(\bar{a}(w))+o(1-\delta)$, completing the proof.

# References


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# Online Appendix

## C Proof of Proposition 1

Consider the strategies described in the text, with $z = \sqrt{-\log(1-\delta)}$ (i.e., $c_0 = 1$). Define

$$v = 2 - \frac{1-\delta}{1-\delta^T}\frac{p}{p_1 - p} \qquad \text{and} \qquad q = \left(\frac{2\delta^T}{1-\delta}(p_1 - p) - \frac{\delta^T}{1-\delta^T}p\right)^{-1}. \tag{27}$$

With $v$ and $q$ so defined, (PK) and (IC$_1$) hold with equality. We show that

$$\lim_{\delta\to 1}\frac{\frac{1-\delta}{1-\delta^T}\frac{p}{p_1-p}}{\sqrt{\frac{1-\delta}{-\log(1-\delta)}}} < \frac{5\sqrt{\rho}e^{\rho}}{e^{\rho}-1} \text{ for all } \rho > 0, \tag{28}$$

$$\lim_{\rho\to 0}\lim_{\delta\to 1}\frac{2\delta^T(p_1-p)}{1-\delta} - \frac{\delta^T p}{1-\delta^T} > 1, \qquad \text{and} \tag{29}$$

$$\lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T q(p_T - p)}{1-\delta^T} > 1. \tag{30}$$

Given these results, the proof is completed by first taking $\rho > 0$ and $\bar{\delta}_1 > 0$ such that the inequalities in (29) and (30) hold for $\rho$ and all $\delta > \bar{\delta}_1$, then taking $\bar{\delta}_2 > 0$ such that the inequality in (28) holds for $\rho$ for all $\delta > \bar{\delta}_2$, and finally taking $c = 5\sqrt{\rho}e^{\rho}/(e^{\rho}-1)$ and $\bar{\delta} = \max\{\bar{\delta}_1, \bar{\delta}_2\}$.

We now establish (28)–(30). Let $k \in \mathbb{N}$ be the unique integer satisfying $k \in [\frac{\sqrt{T}}{2}\left(\sqrt{T} - z\right) - 1, \frac{\sqrt{T}}{2}\left(\sqrt{T} - z\right))$. Note that

$$\begin{aligned} p &= \Pr\left(\sum_{t=2}^{T} y_t < k\right) + \frac{1}{2}\Pr\left(\sum_{t=2}^{T} y_t = k\right) < \Pr\left(\sum_{t=2}^{T} y_t \le k\right), \qquad \text{and} \\ p_1 - p &= \frac{1}{4}\Pr\left(\sum_{t=2}^{T} y_t = k\right) = \frac{(T-1)!}{k!(T-1-k)!}\left(\frac{1}{2}\right)^{T+1} \ge \frac{(T)!}{k!(T-k)!}\left(\frac{1}{2}\right)^{T+2}, \end{aligned} \tag{31}$$

where the last inequality holds because $k \le T/2$.

We first establish (28). Recall that the $y_t$ are independent Bernoulli random variables. As shown by Zhu, Li, and Hayashi (2022, Theorem 2.1),

$$\frac{\Pr\left(\sum_{t=2}^{T} y_t \le k\right)}{\Pr\left(\sum_{t=2}^{T} y_t = k\right)} \le k + 1 - \frac{T}{2} + \sqrt{\left(k - 1 - \frac{T}{2}\right)^2 + 2k}.$$

Since

$$\frac{p}{p_1 - p} < 4\frac{\Pr\left(\sum_{t=2}^{T} y_t \leq k\right)}{\Pr\left(\sum_{t=2}^{T} y_t = k\right)} \qquad \text{and} \qquad \lim_{\delta\to 1} \frac{\frac{1-\delta}{1-\delta^T}\left(k+1-\frac{T}{2}+\sqrt{\left(k-1-\frac{T}{2}\right)^2+2k}\right)}{\sqrt{\frac{1-\delta}{-\log(1-\delta)}}} = \frac{\sqrt{\rho}e^{\rho}}{e^{\rho}-1},$$

where the second line follows by l'Hopital's rule, we have

$$\lim_{\delta\to 1} \frac{\frac{1-\delta}{1-\delta^T}\frac{p}{p_1-p}}{\sqrt{\frac{1-\delta}{-\log(1-\delta)}}} \leq \lim_{\delta\to 1} \frac{\frac{1-\delta}{1-\delta^T}4\frac{\Pr\left(\sum_{t=2}^{T} y_t \leq k\right)}{\Pr\left(\sum_{t=2}^{T} y_t = k\right)}}{\sqrt{\frac{1-\delta}{-\log(1-\delta)}}} = \frac{4\sqrt{\rho}e^{\rho}}{e^{\rho}-1} < \frac{5\sqrt{\rho}e^{\rho}}{e^{\rho}-1},$$

which establishes (28).

We next establish (29). Applying Stirling's formula to (31), we have

$$p_1 - p \geq \frac{\sqrt{2\pi(T-1)}}{4e^2\sqrt{k(T-1-k)}}\left(\frac{T-1}{2k}\right)^k\left(\frac{T-1}{2(T-1-k)}\right)^{T-1-k}. \tag{32}$$

Therefore,

$$\lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T(p_1-p)}{1-\delta} \geq \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T}{1-\delta}\frac{\sqrt{2\pi(T-1)}}{4e^2\sqrt{k(T-1-k)}}\left(\frac{T-1}{2k}\right)^k\left(\frac{T-1}{2(T-1-k)}\right)^{T-1-k} = \infty.$$

On the other hand,

$$\lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T p}{1-\delta^T} \leq \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T}{1-\delta^T} = \lim_{\rho\to 0}\frac{e^{-\rho}}{1-e^{-\rho}} = \infty,$$

which establishes (29).

Finally, we establish (30). We will show that $\lim_{\delta\to 1} p = 0$ and $\lim_{\delta\to 1} p_T = 1$. Hence, for sufficiently large $\delta$, $p_T - p \geq 1/2$. This implies (30), as we have

$$\begin{aligned}
& \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T q(p_T-p)}{1-\delta^T} \\
= \; & \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{\delta^T}{1-\delta^T}\frac{1}{\delta^T}\frac{p_T-p}{2\frac{p_1-p}{1-\delta}-\frac{p}{1-\delta^T}} \qquad \text{by (27)} \\
\geq \; & \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{1}{1-\delta^T}\frac{\frac{1}{2}}{2\frac{p_1-p}{1-\delta}} \\
\geq \; & \lim_{\rho\to 0}\lim_{\delta\to 1}\frac{1-\delta}{1-\delta^T}\left(\frac{\sqrt{e(T-1)}}{2\pi\sqrt{k(T-1-k)}}\left(\frac{T-1}{2k}\right)^k\left(\frac{T-1}{2(T-1-k)}\right)^{T-1-k}\right)^{-1} = \infty,
\end{aligned}$$

where the second inequality follows by applying Stirling's formula to (31).

It remains to show that $\lim_{\delta\to 1} p = 0$ and $\lim_{\delta\to 1} p_T = 1$. Recall that $2y_t - 1$ has zero mean and unit variance when player 1 takes $C$. Thus, by the Berry–Esseen theorem, there exists an absolute constant $C_0$ such that

$$\begin{aligned} p &= \Pr^{\text{player 1 takes } C}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(2y_t-1) \le -\sqrt{-\log(1-\delta)}\right) \\ &\le \Phi\left(-\sqrt{-\log(1-\delta)}\right) + C_0\left(\frac{\mathbb{E}^{\text{player 1 takes } C}\left[|2y_t-1|^3\right]}{\sqrt{T}}\right) \overset{\delta\to 1}{\longrightarrow} 0. \end{aligned}$$

On the other hand, $(4y_t - 1)/\sqrt{3}$ has zero mean and unit variance when player 1 takes $D$. Thus, again by Berry–Esseen,

$$\begin{aligned} p_T &= \Pr^{\text{player 1 takes } D}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(2y_t-1) \le -\sqrt{-\log(1-\delta)}\right) \\ &= \Pr^{\text{player 1 takes } D}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{4y_t-1}{\sqrt{3}} \le \frac{\sqrt{T}-2\sqrt{-\log(1-\delta)}}{\sqrt{3}}\right) \\ &\ge \Phi\left(\frac{\sqrt{T}-2\sqrt{-\log(1-\delta)}}{\sqrt{3}}\right) + C_0\left(\frac{\mathbb{E}^{\text{player 1 takes } D}\left[\left|(4y_t-1)/\sqrt{3}\right|^3\right]}{\sqrt{T}}\right) \overset{\delta\to 1}{\longrightarrow} 1, \end{aligned}$$

completing the proof.

# D Proof of Lemma 2

The proof is similar to (but simpler than) the proof of Lemma 6 of SW. To show that $B$ is self-generating, it suffices to show that the extreme points of any ball $B' \subseteq B$ of radius $r/2$ are decomposable on $B'$.

**Lemma 13 (SW, Lemma 10)** *Suppose that for any ball $B' \subseteq B$ with radius $r/2$ and any direction $\lambda \in \Lambda$, the point $\hat{v} = \operatorname{argmax}_{v' \in B'} \lambda \cdot v'$ is decomposable on $B'$. Then $B$ is self-generating.*

We thus fix a ball $B' \subseteq B$ of radius $r/2$ and a direction $\lambda \in \Lambda$, and let $\hat{v} = \operatorname{argmax}_{v' \in B'} \lambda \cdot v'$. We construct $(\alpha, w)$ that decompose $\hat{v}$ on $B'$.

Since $k(\lambda, \bar{x}) \ge \max_{v' \in B} \lambda \cdot v' + \varepsilon$ by hypothesis, there exist $\alpha$ and $x : Y \to \mathbb{R}^N$ satisfying (IC), (HS$\bar{x}$), and

$$\lambda \cdot \left(u(\alpha) + \int_y x(y)\, p(y|\alpha)\, dy\right) \ge \max_{v' \in B} \lambda \cdot v' + \varepsilon/2 \ge \max_{v' \in B'} \lambda \cdot v' + \varepsilon/2. \tag{33}$$

To construct $w$, for each $y$, let

$$w(y) = \hat{v} + \frac{1-\delta}{\delta}\left(\hat{v} - u(\alpha) + x(y) - \int_{y'} p(y'|\alpha)\, x(y')\, dy'\right).$$

We show that $(\alpha, w)$ decomposes $\hat{v}$ on $B'$ by verifying (PK), (IC), and (SG).

**(PK):** This holds by construction: we have $\int_y w(y)\, p(y|\alpha)\, dy = (1/\delta)(\hat{v} - (1-\delta)\, u(\alpha))$, and hence $(1-\delta)\, u(\alpha) + \delta \int_y w(y)\, p(y|\alpha)\, dy = \hat{v}$.

**(IC):** Setting aside the constant terms in $w(y)$, we see that an action $a_i$ maximizes $(1-\delta)\, u_i(a_i, \alpha_{-i}) + \delta \int_y w_i(y)\, p(y|a_i, \alpha_{-i})\, dy$ iff it maximizes $u_i(a_i, \alpha_{-i}) + \int_y x_i(y)\, p(y|a_i, \alpha_{-i})\, dy$, which follows from (IC).

**(SG):** We start with a simple geometric observation.

**Lemma 14 (SW, Lemma 11)** *For each $w \in \mathbb{R}^N$, we have $w \in B'$ if $\lambda \cdot (\hat{v} - w) \geq 0$ and*

$$d(\hat{v}, w) \leq \sqrt{(r/2)\, \lambda \cdot (\hat{v} - w)}. \tag{34}$$

We thus show that, for each $y$, $w(y)$ satisfies $\lambda \cdot (\hat{v} - w(y)) \geq 0$ and (34). Note that

$$\hat{v} - w(y) = \frac{1-\delta}{\delta}\left(u(\alpha) + \int_{y'} x(y')\, p(y'|\alpha)\, dy' - \hat{v} - x(y)\right).$$

By (HS$\bar{x}$) and (33), we have $\lambda \cdot (\hat{v} - w(y)) \geq (\delta/(1-\delta))\, \varepsilon/2$, and therefore

$$\sqrt{(r/2)\, \lambda \cdot (\hat{v} - w(y))} \geq \frac{1-\delta}{\delta}\sqrt{\frac{\delta}{1-\delta}\frac{\varepsilon r}{4}}. \tag{35}$$

Similarly, we have

$$\begin{aligned} d(\hat{v}, w) &\leq \frac{1-\delta}{\delta}\left(d(\hat{v}, u(\alpha)) + d\left(\left(\int_{y'} x(y')\, p(y'|\alpha)\, dy', x(y)\right)\right)\right) \\ &\leq \frac{1-\delta}{\delta}\left(\max_{u,u' \in F} \|u - u'\| + 2\bar{x}\right) \leq \frac{1-\delta}{\delta} 3\bar{x}. \end{aligned} \tag{36}$$

Comparing (35) and (36), we see that $w(y)$ satisfies (34) whenever $3\bar{x} \leq \sqrt{(\delta/(1-\delta))\, \varepsilon r/4}$, which holds by (10).

# E Proof of Lemma 3

Since $\dim F^* = N$ and $v_i < \max_a u_i(a)$ for all $i$, there exist $\bar{\eta} > 0$ and $\bar{F} \subseteq F^*$ such that $\{w \in F^* : d(v, w) \leq \bar{\eta}\} \subseteq \bar{F}$, $\dim \bar{F} = N$, and $\underline{v}_i < w_i < \max_a u_i(a)$ for all $i$ and $w \in \bar{F}$.

Fix any such $\left(\bar{\eta}, \bar{F}\right)$.

The following lemma is similar to Lemma 5 of HT or Lemma 7 and pp. 1750–1751 of SW.

**Lemma 15** *There exists $\bar{x} > \bar{u}$ such that $k\left(\lambda, \bar{x}\right) \geq \max_{v' \in \bar{F}} \lambda \cdot v'$ for all $\lambda \in \Lambda$.*

**Proof.** Let $\hat{x} > 0$ satisfy the conditions of Assumption 1. For each $i$, since $\underline{v}_i < w_i < \max_a u_i\left(a\right)$ for all $w \in \bar{F}$, there exist $\underline{\lambda}_i > -1$ and $\bar{\lambda}_i < 1$ such that (i) for all $\lambda \in \Lambda$ with $\lambda_i \leq \underline{\lambda}_i$, we have $\lambda \cdot u\left(\alpha\right) - \sum_{n \neq i} \left|\lambda_n\right| \hat{x} \geq \max_{w \in \bar{F}} \lambda \cdot w$ for all $\alpha$ satisfying $u_i\left(\alpha\right) = \underline{v}_i$; and (ii) for all $\lambda \in \Lambda$ with $\lambda_i \geq \bar{\lambda}_i$, we have $\lambda \cdot u\left(\alpha\right) - \sum_{n \neq i} \left|\lambda_n\right| \hat{x} \geq \max_{w \in \bar{F}} \lambda \cdot w$ for all $\alpha$ satisfying $u_i\left(\alpha\right) = \max_a u_i\left(a\right)$. Given such $\left(\underline{\lambda}_i, \bar{\lambda}_i\right)_i$, we define

$$\bar{x} = \sqrt{N}\left(2N + \max_i \frac{\max\{\left|\underline{\lambda}_i\right|, \bar{\lambda}_i\}}{\sqrt{\left(1 - \max\{\left|\underline{\lambda}_i\right|, \bar{\lambda}_i\}^2\right)/\left(N-1\right)}}\right)\hat{x}.$$

For each $\lambda$, we now construct $\left(\alpha, x\left(y\right)\right)$ such that $\lambda \cdot \left(u\left(\alpha\right) + \mathbb{E}\left[x\left(y\right)|\alpha\right]\right) \geq \max_{w \in \bar{F}} \lambda \cdot w$ and (IC) and (HS$\bar{\text{x}}$) hold. To do so, fix any $i \in \operatorname{argmax}\left|\lambda_i\right|$, and consider three cases.

(i) $\lambda_i \leq \underline{\lambda}_i$. In this case, take a minmax profile $\alpha^i$ and $\left(\hat{x}_j\left(y\right)\right)_{j \neq i}$ that satisfy Assumption 1(i). Define $x_i\left(y\right) = \sum_{n \neq i} \left|\lambda_n\right| \hat{x}/\lambda_i$ and $x_j\left(y\right) = \hat{x}_j\left(y\right)$ for all $y$. (IC) holds for $j \neq i$ given Assumption 1(i), and (IC) holds for $i$ since player $i$ takes a best response in $\alpha^i$ and $x_i\left(y\right)$ is independent of $y$. (HS$\bar{\text{x}}$) holds with $\bar{x} \geq 2N\hat{x}$ given Assumption 1(i). Finally, $\lambda \cdot u\left(\alpha^i\right) + \mathbb{E}\left[\lambda \cdot x\left(y\right)\right] \geq \lambda \cdot u\left(\alpha^i\right) - \sum_{n \neq i} \left|\lambda_n\right| \hat{x} \geq \max_{w \in \bar{F}} \lambda \cdot w$.

(ii) $\lambda_i \geq \underline{\lambda}_i$. In this case, take $\bar{a}^i \in \max u_i\left(a\right)$ and $\left(\hat{x}_j\left(y\right)\right)_{j \neq i}$ that satisfies (7) for player $j$. Define $x_i\left(y\right) = \sum_{n \neq i} \left|\lambda_n\right| \hat{x}/\lambda_i$ and $x_j\left(y\right) = \hat{x}_j\left(y\right)$ for all $y$. The argument is now the same as case (ii).

(iii) Otherwise, there exists $n \neq i$ such that $\left|\lambda_i\right| / \left|\lambda_n\right| \leq \max\{\left|\underline{\lambda}_i\right|, \bar{\lambda}_i\}/\sqrt{\left(1 - \max\{\left|\underline{\lambda}_i\right|, \bar{\lambda}_i\}^2\right)/\left(N-1\right)}$. Fix such $n$. Next, fix $a \in A$ such that $\lambda \cdot u\left(a\right) \geq \max_{w \in \bar{F}} \lambda \cdot w$. For player $i$, take $x^i\left(y\right)$ such that Assumption 1(ii) holds for $a$, $\left(i, n\right)$, and $c = \operatorname{sign}\left(-\lambda_n/\lambda_i\right)$. For player $j \neq i$, take $x^j\left(y\right)$ such that Assumption 1(ii) holds for $a$, $\left(j, i\right)$, and $c = \operatorname{sign}\left(-\lambda_j/\lambda_i\right)$. We then define $x_i\left(y\right) = x^i\left(y\right) - \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} x^j\left(y\right)$, $x_n\left(y\right) = x^n\left(y\right) - \frac{\lambda_i}{\lambda_n} x^i\left(y\right)$, and $x_j\left(y\right) = x^j\left(y\right)$ for $j \neq i, n$. Then, (IC) holds for player $i$ since $a_i \in \arg\max_{a_i'} u\left(a_i', a_{-i}\right) + \mathbb{E}\left[x^i\left(y\right)|a_i', a_{-i}\right]$ and $a_i \in \arg\max_{a_i'} \mathbb{E}\left[-\left(\lambda_j/\lambda_i\right) x^j\left(y\right)|a_i', a_{-i}\right]$ for all $j \neq i$ by Assumption 1(ii). (IC) holds for player $n$ since $a_n \in \arg\max_{a_n'} u\left(a_n', a_{-n}\right) + \mathbb{E}\left[x^n\left(y\right)|a_n', a_{-n}\right]$ and $a_n \in \arg\max_{a_n'} \mathbb{E}\left[-\left(\lambda_i/\lambda_n\right) x^i\left(y\right)|a_n', a_{-n}\right]$ by Assumption 1(ii). In addition, (IC) holds for player $j \neq i, n$ since $a_j \in \arg\max_{a_j'} u\left(a_j', a_{-j}\right) + \mathbb{E}\left[x^j\left(y\right)|a_j', a_{-j}\right]$ by Assumption 1(ii). Finally, (HS$\bar{\text{x}}$) holds since $\lambda \cdot x\left(y\right) = 0$ for all $y$ and $\left\|x\left(y\right)\right\| \leq \sqrt{N} \sum_j \left|x_j\left(y\right)\right| \leq \sqrt{N}\left(2N + \left|\lambda_i/\lambda_n\right|\right)\hat{x} \leq \bar{x}$. ■

By Lemma 15, it suffices to find $c > 0$ and $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, there exist

$\varepsilon > 0$ and a ball $B$ with radius $r > 0$ such that

$$\begin{aligned}
\max_{v' \in \bar{F}} \lambda' \cdot v' &\geq \max_{v' \in B} \lambda' \cdot v' + \varepsilon \quad \text{for all } \lambda' \in \Lambda, & (37)\\
r\varepsilon &\geq 36\bar{x}^2 (1-\delta), \quad \text{and} & (38)\\
d(v, B) &\leq c(1-\delta)^{\beta^*/2}. & (39)
\end{aligned}$$

If $\beta^* = 1$ then, as in Lemma 3 of HT, it suffices to take any $o \in \text{int}(\bar{F})$ and any $\ell > 0$ sufficiently large compared to $36\bar{x}^2$, let $r = (1-\delta)^{1/2}$, and take $B$ to have radius $r$ and center $(1-\ell r) v + \ell r o$.

For the rest of the proof, we assume that $\beta^* > 1$. We first derive a geometric condition for $w \in F^*$, similar to Lemma 14.

**Lemma 16** *There exist $\lambda \in \Lambda_v^*$, $\rho > 0$, and $\kappa > 0$ such that, if $d(v, w) < \rho$ and $\kappa d(v, w)^{\beta^*} < \lambda \cdot (v - w)$, then $w \in \text{int}(F^*)$.*

**Proof.** Since $F^*$ is full-dimensional and has min-curvature of order at most $\beta$ at $v$, there exist $\bar{\varepsilon} > 0$ and $\kappa > 0$ such that, for all $w \in \text{bnd}(F^*)$ satisfying $d(v, w) < \bar{\varepsilon}$, we have $\lambda \cdot (v - w) < \kappa d(v, w)^\beta \leq \kappa d(v, w)^{\beta^*}$ for all $\lambda \in \Lambda_v^*$. Let $B_{\varepsilon'}(v) = \{w \in \mathbb{R}^N : d(v, w) = \varepsilon'\}$. Since $F^*$ is full-dimensional, there exists $\lambda \in \Lambda_v^*$, $\varepsilon' > 0$, and $t > 0$ such that $C := B_{\varepsilon'}(v) - t\lambda \subseteq F^*$. Fix such $\lambda$, $\varepsilon'$, and $t$, and let $\varepsilon = \min\{\bar{\varepsilon}, \varepsilon', t\}$.

Now fix any $\rho < \min\{\varepsilon, (t/2\kappa)^{1/\beta^*}\}$ and $d < \rho$, and let $G = \{w \in B_d(v) : \lambda \cdot (v - w) \geq 2\kappa d^{\beta^*}\}$. We wish to show that $G \subseteq F^*$ (and in particular $G \subseteq \text{int}(F^*)$, since $G \cap \text{bnd}(F^*) = \emptyset$).

To see this, let $W = B_d(v) \cap \text{bnd}(F^*)$, $H = \{w : \lambda \cdot (v - w) = \kappa d^{\beta^*}\}$, $H' = \{w : \lambda \cdot (v - w) = t\}$, and $D = C \cap H'$. Since $d < \rho < \min\{\varepsilon, (t/\kappa)^{1/\beta^*}\}$, $G$ lies in between $H$ and $H'$. In addition, the projection of $G$ onto $H$ is a subset of the projection of $W$ onto $H$, and the projection of $G$ onto $H'$ is a subset of $D$. Hence, we have $G \subseteq \text{co}(W \cup D)$. Finally, since $W \subseteq F^*$ and $D \subseteq C \subseteq F^*$, and $F^*$ is convex, we have $\text{co}(W \cup D) \subseteq F^*$, so $G \subseteq F^*$. ■

**Lemma 17** *There exist $\bar{c} > 0$, $\eta > 0$, and $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, there exists a ball $B \subseteq \bar{F}$ of radius $r = \eta(1-\delta)^{1-\beta^*/2}$ satisfying $d(v, B) = \bar{c}(1-\delta)^{\beta^*/2}$.*

**Proof.** Fix $\lambda \in \Lambda_v^*$, $\rho > 0$, and $\kappa > 0$ as in Lemma 16. Given $\bar{c}$ and $\eta$ to be determined, let $B$ be the ball with radius $r = \eta(1-\delta)^{1-\beta^*/2}$ and center $o = v - (r + d)\lambda$, where $d = \bar{c}(1-\delta)^{\beta^*/2}$, and take any $\hat{w} \in \partial B$. Let $x = \lambda \cdot (\hat{w} - o)$, so that $x\lambda$ is the projection of $\hat{w} - o$ on $\lambda$. Then,

$$\begin{aligned}
\|v - \hat{w}\|^2 &= \|v - o - x\lambda\|^2 + \|\hat{w} - o - x\lambda\|^2 = (r + d - x)^2 + r^2 - x^2, \qquad \text{and}\\
\lambda \cdot (v - \hat{w}) &= r + d - x.
\end{aligned}$$

Recall that, by construction, $\{w \in F^* : d(v,w) \leq \bar{\eta}\} \subseteq \bar{F}$. Since $d(v,w) \leq d(v,o) + d(o,w) \leq 2r + d$ for all $w \in B$, it suffices to show that $2r + d \leq \bar{\eta}$ and $B \subseteq F^*$. By Lemma 16, if $d(v,w) < \rho$ and $\kappa d(v,w)^{\beta^*} \leq \lambda \cdot (v - w)$ then $w \in F^*$. Since $x \in [-r,r]$, it suffices to find $\bar{c}$, $\eta$, and $\bar{\delta}$ such that, for all $\delta > \bar{\delta}$, we have

$$
\begin{aligned}
2r + d &\leq \bar{\eta}, && (40)\\
((r+d) - x)^2 + r^2 - x^2 &\leq \rho^2 \qquad \text{for all } x \in [-r,r], \qquad \text{and} && (41)\\
\max_{x \in [-r,r]} f(x, \delta, \beta^*) &\leq 0, && (42)
\end{aligned}
$$

where

$$f(x,\delta,\beta^*) := \kappa\left((r+d-x)^2 + r^2 - x^2\right)^{\beta^*/2} - (r+d-x).$$

We consider separately the cases where $\beta^* = 2$ and $\beta^* \in (1,2)$. First consider $\beta^* = 2$. Let $\hat{\eta} > 0$ be such that (41) holds whenever $\max\{r,d\} \leq \hat{\eta}$, and let any $\bar{c} = 1$ and $\eta = \min\{\hat{\eta}, \bar{\eta}/4, \kappa/4\}$, so that $r = \eta$ and $d = 1 - \delta$. For sufficiently large $\delta$, we have $2r + d \leq \bar{\eta}$ and $d \leq \hat{\eta}$, and hence (40) and (41) hold. In addition, since $f(x,\delta,2)$ is linear in $x$ when $\beta^* = 2$, (42) holds whenever $f(r,\delta,2) \leq 0$ and $f(-r,\delta,2) \leq 0$. In turn, these inequalities hold for sufficiently large $\delta$, since $f(r,\delta,2) = d(\kappa d - 1)$ and $\lim_{\delta \to 1} \kappa d - 1 < 0$, and $f(-r,\delta,\beta^*) = (2r+d)(\kappa(2r+d) - 1)$ and $\lim_{\delta \to 1} \kappa(2r+d) - 1 = 2\kappa\eta - 1 < 0$.

Next, consider $\beta^* \in (1,2)$. Let $\bar{c} = 2^{2/(2-\beta^*)}\kappa^{2/(2-\beta^*)}\beta^{*\beta^*/(2-\beta^*)}$ and $\eta = 1$, so that $r = (1-\delta)^{1-\beta^*/2}$ and $d = \bar{c}(1-\delta)^{\beta^*/2}$. Since $\max\{r,d\} \to 0$ as $\delta \to 1$, (40) and (41) hold for sufficiently large $\delta$. In addition, $f(x,\delta,\beta^*)$ is concave in $x$ and is maximized over $x \in [-r,r]$ at

$$x^* = \frac{2r^2 + 2dr + d^2 - (\kappa(r+d)\beta^*)^{\frac{2}{2-\beta^*}}}{2(r+d)}.$$

It thus suffices to show that $f(x^*,\delta,\beta^*) \leq 0$ for sufficiently large $\delta$. By algebra,

$$f(x^*,\delta,\beta^*) = -\frac{2r+d}{r+d}\frac{d}{2} + \left(\beta^{*\frac{\beta^*}{2-\beta^*}} - \frac{1}{2}\beta^{*\frac{2}{2-\beta^*}}\right)\kappa^{\frac{2}{2-\beta^*}}(r+d)^{\frac{\beta^*}{2-\beta^*}}.$$

Finally, since $r = (1-\delta)^{1-\beta^*/2} \geq \bar{c}(1-\delta)^{\beta^*/2} = d$ for sufficiently large $\delta$, we have

$$
\begin{aligned}
f(x^*,\delta,\beta^*) &\leq -\frac{d}{2} + 2^{\frac{\beta^*}{2-\beta^*}}\kappa^{\frac{2}{2-\beta^*}}\beta^{*\frac{\beta^*}{2-\beta^*}} r^{\frac{\beta^*}{2-\beta^*}}\\
&= -\frac{\bar{c}(1-\delta)^{\frac{\beta^*}{2}}}{2} + 2^{\frac{\beta^*}{2-\beta^*}}\kappa^{\frac{2}{2-\beta^*}}\beta^{*\frac{\beta^*}{2-\beta^*}}(1-\delta)^{\left(1-\frac{\beta^*}{2}\right)\frac{\beta^*}{2-\beta^*}}\\
&= (1-\delta)^{\frac{\beta^*}{2}}\left(-\frac{\bar{c}}{2} + 2^{\frac{\beta^*}{2-\beta^*}}\kappa^{\frac{2}{2-\beta^*}}\beta^{*\frac{\beta^*}{2-\beta^*}}\right) = 0.
\end{aligned}
$$

■

We now complete the proof of Lemma 3. Take $\bar{c}$, $\eta$, $\bar{\delta}$, $B$, and $r$ as in Lemma 17. Let $B'$ be the radial contraction of $B$ by a factor of $1 - 72\bar{x}^2 (1-\delta)^{\beta^*/2} / (\eta r)$, and define $\varepsilon = 72\bar{x}^2 (1-\delta)^{\beta^*/2} / \eta$ and $c = \bar{c} + 72\bar{x}^2/\eta$. Since $d(v, B) = \bar{c}(1-\delta)^{\beta^*/2}$, we have $d(v, B') = (\bar{c} + 72\bar{x}^2/\eta)(1-\delta)^{\beta^*/2} = c(1-\delta)^{\beta^*/2}$, so (39) holds. Moreover, denoting the radius of $B'$ by $r'$, we have

$$r'\varepsilon = \left(1 - \frac{72\bar{x}^2 (1-\delta)^{\beta^*/2}}{\eta^2 (1-\delta)^{1-\beta^*/2}}\right) \eta (1-\delta)^{1-\beta^*/2} \times \frac{72\bar{x}^2 (1-\delta)^{\beta^*/2}}{\eta} = \left(1 - \frac{72\bar{x}^2 (1-\delta)^{\beta^*-1}}{\eta^2}\right) 72\bar{x}^2 (1-\delta).$$

For sufficiently large $\delta$, this is greater than $36\bar{x}^2 (1-\delta)$, so (38) holds. Finally, since $B \subseteq \bar{F}$, for all $\lambda' \in \Lambda$ we have $\max_{v' \in \bar{F}} \lambda' \cdot v' \geq \max_{v' \in B} \lambda' \cdot v' = \max_{v' \in B'} \lambda' \cdot v' + 72\bar{x}^2 (1-\delta)^{\beta^*/2} / \eta = \max_{v' \in B'} \lambda' \cdot v' + \varepsilon$, so (37) holds.

# F Proof of Proposition 2

To define $\bar{x}$, we first observe that for each pair of players $i \neq j$ and each action profile $a$, we can take $\left(x_i^j (d; a)\right)_d$ such that (i) $\sum_d x_i^j (d; a) y^d$ has mean 0 and bounded Euclidean norm; (ii) rewards $\sum_d x_i^j (d; a) y^d$ induce player $i$ to take $a_i$ when her opponents take $a_{-i}$; and (iii) $\mathbb{E}\left[\sum_d x_i^j (d; a) y^d | a\right]$ is independent of player $j$'s action.

**Lemma 18** *There exists $\hat{x}$ such that, for each pair of players $i \neq j$ and action profile $a \in A$, there exist $\left(x_i^j (d; a)\right)_d$ such that $\mathbb{E}\left[\sum_d x_i^j (d; a) y^d | a\right] = 0$, $\frac{d}{da_i}\mathbb{E}\left[\sum_d x_i^j (d; a) y^d | a\right] = 1$, $\frac{d}{da_j}\mathbb{E}\left[\sum_d x_i^j (d; a) y^d | a\right] = 0$, and $\left|\sum_d x_i^j (d; a) y^d\right| \leq \hat{x}$ for all $y$.*

**Proof.** For each $a$ and $(i, j)$, let $f^{ij}(a)$ be the value of the program

$$\inf_{b \in \mathbb{R}^D} |b| \qquad \text{subject to}$$

$$\sum_d b_d \frac{d}{da_i} \mu(a_i, a_{-i}) = 1, \text{ or equivalently } bM_i(a) = 1,$$

$$\sum_d b_d \frac{d}{da_j} \mu(a_i, a_{-i}) = 0, \text{ or equivalently } bM_j(a) = 0.$$

(Here $b$ is a row vector while $M_i(a)$ and $M_j(a)$ are column vectors.)

Since $A \ni a$ is compact and $N$ is finite, it suffices to prove that, for each $(i, j)$, (i) $f^{ij}(a) < \infty$ for all $a$, and (ii) $f^{ij}(a)$ is upper-semicontinuous.

We first prove (i). As in Lemma 1 of Sannikov (2007), pairwise identifiability implies that the columns of $[M^i(a); M^j(a)]$ are linearly independent, so there exists $L(a)$ such that

$[M^i(a); M^j(a); L(a)]$ is a $D$-dimensional invertible matrix. For

$$Q(a) = [M_i(a); 0; 0] \left[M^i(a); M^j(a); L(a)\right]^{-1},$$

we have $Q(a) M^i(a) = M^i(a)$ and $Q(a) M^j(a) = 0$. Moreover, since $M^i(a)$ is non-degenerate, there exists $\bar{b}$ such that $\bar{b}M^i(a) = 1$. Since $b = \bar{b}Q(a)$ satisfies the constraints, we have $f^{ij}(a) < \infty$.

We next prove (ii). Fix any $a$ and $\eta_0$. There exists $b$ such that $|b| \leq f^{ij}(a) + \frac{\eta_0}{2}$ and $b$ satisfies $bM_i(a) = 1$ and $bM_j(a) = 0$. Take $L(a)$ as in the proof of (i). Taking $\eta_1 > 0$ sufficiently small, we can guarantee that $[M^i(a'); M^j(a'); L(a)]$ is a $D$-dimensional invertible matrix for each $a'$ with $|a - a'| \leq \eta_1$. Define a $D$-dimensional vector $\Delta_{a'}$ by

$$\Delta_{a'} = [b(M_i(a') - M_i(a)), b(M_j(a') - M_j(a)), 0] \left[M^i(a'); M^j(a'); L(a)\right]^{-1}.$$

By definition,

$$\begin{aligned}
(b + \Delta_{a'}) M_i(a') &= bM_i(a') - b(M_i(a') - M_i(a)) = bM_i(a) = 1, \\
(b + \Delta_{a'}) M_j(a') &= bM_j(a') - b(M_j(a') - M_j(a)) = bM_j(a) = 0.
\end{aligned}$$

Thus, $b - \Delta_{a'}$ satisfies the constraint for $a'$, and hence $f^{ij}(a') \leq |b| + |\Delta_{a'}|$. Since $\limsup_{\eta_1 \to 0} \sup_{a':|a-a'| \leq \eta_1} |\Delta_{a'}| = 0$, for sufficiently small $\eta_1 > 0$, we have $f^{ij}(a') \leq |b| + |\Delta_{a'}| \leq |b| + \frac{1}{2}\eta_0 \leq f^{ij}(a) + \eta_0$ for all $a'$ with $|a - a'| \leq \eta_1$, establishing upper-semicontinuity.
■

Given Lemma 18, Assumption 1(ii) holds with $\bar{x} = \bar{u}\hat{x}$. To see why, for any $i$ and $a$, let $\partial u_i = \frac{\partial}{\partial a_i'} u_i(a_i', a_{-i})\Big|_{a_i'=a_i}$ and $x_i(y) = -\partial u_i \sum_d x_i^j\left(d; a^\lambda\right) y^d$. Then,

$$\begin{aligned}
\frac{\partial}{\partial a_i'}\left(u_i(a_i', a_{-i}) + \mathbb{E}[x_i(y) | a_i', a_{-i}]\right)\Big|_{a_i'=a_i} &= 0 \qquad \text{and} \\
\frac{\partial}{\partial a_j'} \mathbb{E}\left[c x_i(y) | a_j', a_{-i}\right]\Big|_{a_j'=a_j} &= 0 \qquad \text{for all } j \neq i.
\end{aligned}$$

Since $u_i$ is concave in $a_i$, $\mathbb{E}[x_i(y) | a_i', a_{-i}]$ is linear in $a_i'$, and $\mathbb{E}\left[c x_i(y) | a_j', a_{-j}\right]$ is linear in $a_j'$, we have (7) and (8). Moreover, since $|\partial u_i| \leq \bar{u}$, we have $|x_i(y)| \leq \bar{u}\hat{x}$ for all $i, y$.

# G Omitted Details for the Proof of Theorem 3

We require some preliminary lemmas. The first two derive properties of the feasible payoff set.

**Lemma 19** *There exists $\varepsilon_1 > 0$ such that, for any $\alpha \in \Delta(A)$, we have*

$$\mathbb{E}^\alpha[a] - u^{-1}(w + \mathbb{E}^\alpha[\psi(a)]) \leq \bar{F}(w) - \varepsilon_1 \mathbb{E}^\alpha\left[(a - \bar{a}(w))^2\right].$$

**Proof.** Since $\psi \in C^2$ and $\psi'' > 0$, there exists $\varepsilon_1 > 0$ such that, for all $a \in A$, we have

$$\psi(a) - \psi(\bar{a}(w)) \geq \psi'(\bar{a}(w))(a - \bar{a}(w)) + \varepsilon_1 (a - \bar{a}(w))^2.$$

Thus, for any $\alpha \in \Delta(A)$, we have

$$\begin{aligned}
& \bar{F}(w) - \mathbb{E}^\alpha[a] + u^{-1}(w + \mathbb{E}^\alpha[\psi(a)]) \\
= \; & \bar{a}(w) - u^{-1}(w + \psi(\bar{a}(w))) - \mathbb{E}^\alpha[a] + u^{-1}(w + \mathbb{E}^\alpha[\psi(a)]) \\
\geq \; & \bar{a}(w) - \mathbb{E}^\alpha[a] + \frac{\mathbb{E}^\alpha[\psi(a)] - \psi(\bar{a}(w))}{u'(\bar{c}(w))} \\
\geq \; & \bar{a}(w) - \mathbb{E}^\alpha[a] + \frac{\psi'(\bar{a}(w))(\mathbb{E}^\alpha[a] - \bar{a}(w)) + \varepsilon_1 \mathbb{E}^\alpha\left[(a - \bar{a}(w))^2\right]}{u'(\bar{c}(w))} \geq \frac{\varepsilon_1 \mathbb{E}^\alpha\left[(a - \bar{a}(w))^2\right]}{u'(\bar{c}(w))},
\end{aligned}$$

where the first inequality is by Taylor expansion of $u^{-1}(w + \mathbb{E}^\alpha[\psi(a)]) - u^{-1}(w + \psi(\bar{a}(w)))$ around $w + \mathbb{E}^\alpha[\psi(a)] = w + \psi(\bar{a}(w)) = \bar{c}(w)$, the second inequality is by Taylor expansion of $\psi(a) - \psi(\bar{a}(w))$ around $a = \bar{a}(w)$, and the last equality is by $u'(\bar{c}(w)) = \psi'(\bar{a}(w))$ (by definition of $\bar{a}(w)$). ■

**Lemma 20** *There exists $\varepsilon_2 > 0$ such that $\bar{F}''(w) \leq -\varepsilon_2$ for all $w \geq -\psi(\bar{A})$.*

**Proof.** Differentiating the equality $u'(\bar{c}(w)) = \psi'(\bar{a}(w))$ with respect to $w$ yields

$$\bar{a}'(w) = \frac{u''(\bar{c}(w))}{u'(\bar{a}(w))(\psi''(\bar{a}(w)) - u''(\bar{c}(w)))}. \tag{43}$$

Since $\bar{F}(w) = \max_{a \in A} a - u^{-1}(w + \psi(a))$, by the envelope theorem we have $\bar{F}'(w) = -1/u'(\bar{c}(w))$, or equivalently $\bar{F}'(w) = -1/u'(w + \psi(\bar{a}(w)))$. Differentiating this equality respect to $w$ and substituting (43) yields

$$\bar{F}''(w) = \frac{u''(\bar{c}(w))\psi''(\bar{a}(w))}{u'(\bar{c}(w))^3(\psi''(\bar{a}(w)) - u''(\bar{c}(w)))}.$$

The lemma follows since $\psi'' > 0$, $u'' < 0$, and $u''(c)/u'(c)^3$ is bounded away from zero by (11). ■

The next lemma gives a key probability bound.

**Lemma 21** *For any $c > 0$ and $\vartheta \geq 0$, there exists $\bar{\delta} < 1$ such that, for any $\delta > \bar{\delta}$ and any sequence of non-negative random variables $(X_t)_{t\geq 1}$, where $X_t$ is distributed according to a cdf $G_t$ satisfying $1 - G_t(x) \leq 2\exp\left(-cx^2/\left((1-\delta)^2 t\right)\right)$ for all $t$, we have*

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}t\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta}dG_t(x) \leq (1-\delta)^2.$$

**Proof.** Let $T = (1-\delta)^{\frac{1}{2}-\varepsilon}$. It suffices to show that

$$\lim_{\delta\to 1}(1-\delta)^{-1}\sum_{t=1}^{T}\delta^{t-1}t\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta}dG_t(x) = \lim_{\delta\to 1}(1-\delta)^{-1}\sum_{t=T+1}^{\infty}\delta^{t-1}t\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta}dG_t(x) = 0.$$

For each $t$, we have

$$\begin{aligned}
& \int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta}dG_t(x) \\
= & \int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} \vartheta x^{\vartheta-1}(1-G_t(x))\,dx + (1-\delta)^{\vartheta\left(\frac{1}{2}-\varepsilon\right)}\left(1-G_t\left((1-\delta)^{\frac{1}{2}-\varepsilon}\right)\right) \\
\leq & \; 2\vartheta\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^2 t}\right)dx + 2(1-\delta)^{\vartheta\left(\frac{1}{2}-\varepsilon\right)}\exp\left(-\frac{c(1-\delta)^{1-2\varepsilon}}{(1-\delta)^2 t}\right),
\end{aligned}$$

where the equality is by integration by parts and $\lim_{x\to\infty} x^{\vartheta}(1-G_t(x)) = 0$, and the inequality is by $1-G_t(x) \leq 2\exp\left(-cx^2/\left((1-\delta)^2 t\right)\right)$. Note that $\sum_{t=1}^{T} t\delta^{t-1} \leq (1-\delta)^{-2}$ and, if $t \leq T$, then $(1-\delta)^{1-2\varepsilon}/\left((1-\delta)^2 t\right) \geq (1-\delta)^{-\varepsilon}$. Therefore

$$\lim_{\delta\to 1}(1-\delta)^{-1}\sum_{t=1}^{T} t\delta^{t-1}(1-\delta)^{\vartheta\left(\frac{1}{2}-\varepsilon\right)}\exp\left(-\frac{c(1-\delta)^{1-2\varepsilon}}{(1-\delta)^2 t}\right) = 0.$$

At the same time, since $\sum_{t=T+1}^{\infty} t\delta^{t-1} = \frac{\delta^{T+1}(1+T(1-\delta))}{\delta(1-\delta)^2} \leq (1-\delta)^{-2-\varepsilon}\delta^{T} \leq (1-\delta)^{-2-\varepsilon}\exp\left(-(1-\delta)^{-\varepsilon}\right)$, we have

$$\lim_{\delta\to 1}(1-\delta)^{-1}\sum_{t=T+1}^{\infty} t\delta^{t-1}(1-\delta)^{\vartheta\left(\frac{1}{2}-\varepsilon\right)}\exp\left(-\frac{c(1-\delta)^{1-2\varepsilon}}{(1-\delta)^2 t}\right) = 0.$$

It thus suffices to show that

$$\lim_{\delta\to1}(1-\delta)^{-1}\sum_{t=1}^{T}t\delta^{t-1}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^2\,t}\right)dx \;=\; 0, \qquad \text{and} \tag{44}$$

$$\lim_{\delta\to1}(1-\delta)^{-1}\sum_{t=T+1}^{\infty}t\delta^{t-1}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^2\,t}\right)dx \;=\; 0. \tag{45}$$

We first establish (44). Since $\sum_{t=1}^{T}t\delta^{t-1}\leq(1-\delta)^{-2}$ and $(1-\delta)^2\,T=(1-\delta)^{1-\varepsilon}$, it suffices to show that

$$\lim_{\delta\to1}(1-\delta)^{-3}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^{1-\varepsilon}}\right)dx=0.$$

If $\vartheta\leq1$ then

$$\begin{aligned}
& (1-\delta)^{-3}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^{1-\varepsilon}}\right)dx \\
\leq\;& (1-\delta)^{\left(\frac{1}{2}-\varepsilon\right)(\vartheta-1)-3}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}\exp\left(-\frac{cx^2}{(1-\delta)^{1-\varepsilon}}\right)dx \\
\leq\;& (1-\delta)^{\left(\frac{1}{2}-\varepsilon\right)(\vartheta-1)-3}\,\frac{\sqrt{\pi}}{c}\,(1-\delta)^{1/2}\exp\left(-c\,(1-\delta)^{-\varepsilon}\right)\xrightarrow{\delta\to1}0,
\end{aligned}$$

where the second inequality follows by the same calculation as (17). If instead $\vartheta>1$ then, for sufficiently large $\delta$, we have

$$\begin{aligned}
& (1-\delta)^{-3}\int_{x\geq(1-\delta)^{\frac{1}{2}-\varepsilon}}x^{\vartheta-1}\exp\left(-\frac{cx^2}{(1-\delta)^{1-\varepsilon}}\right)dx \\
=\;& \frac{1}{2}(1-\delta)^{-3}\int_{y\geq0}\left(y+(1-\delta)^{1-2\varepsilon}\right)^{\vartheta-1}\exp\left(-\frac{c\left(y+(1-\delta)^{1-2\varepsilon}\right)}{(1-\delta)^{1-\varepsilon}}\right)dy \\
\leq\;& \frac{1}{2}(1-\delta)^{-3+\vartheta(1-2\varepsilon)}\exp\left(-c\,(1-\delta)^{-\varepsilon}\right)\int_{y\geq0}\exp\left(\left(\frac{\vartheta-1}{(1-\delta)^{1-2\varepsilon}}-\frac{c}{(1-\delta)^{1-\varepsilon}}\right)y\right)dy \\
=\;& \frac{1}{2}(1-\delta)^{-3+\vartheta(1-2\varepsilon)}\exp\left(-c\,(1-\delta)^{-\varepsilon}\right)\left(-\frac{\vartheta-1}{(1-\delta)^{1-2\varepsilon}}+\frac{c}{(1-\delta)^{1-\varepsilon}}\right)^{-1}\xrightarrow{\delta\to1}0,
\end{aligned}$$

where the second follows by integration by substitution (setting $y = x^2 - (1-\delta)^{1-2\varepsilon}$), the third line follows because

$$\begin{aligned}\left(y + (1-\delta)^{1-2\varepsilon}\right)^{\vartheta-1} &= (1-\delta)^{(1-2\varepsilon)(\vartheta-1)} \exp\left((\vartheta-1)\log\left(\frac{y}{(1-\delta)^{1-2\varepsilon}} + 1\right)\right) \\ &\le (1-\delta)^{(1-2\varepsilon)(\vartheta-1)} \exp\left(\frac{(\vartheta-1)\, y}{(1-\delta)^{1-2\varepsilon}}\right),\end{aligned}$$

and the fourth line follows because $\frac{\vartheta-1}{(1-\delta)^{1-2\varepsilon}} - \frac{c}{(1-\delta)^{1-\varepsilon}} < 0$ for sufficiently large $\delta$.

We next establish (45). For any $t$, we have

$$\begin{aligned}\int_{x\ge 0} x^{\vartheta-1} \exp\left(-\frac{cx^2}{(1-\delta)^2 t}\right) dx &= \frac{1}{2}\left(\frac{(1-\delta)^2 t}{c}\right)^{\frac{\vartheta}{2}} \int_{y\ge 0} y^{\frac{\vartheta}{2}-1} \exp(-y)\, dy \\ &= \frac{1}{2}\left(\frac{(1-\delta)^2 t}{c}\right)^{\frac{\vartheta}{2}} \Gamma\left(\frac{\vartheta}{2}\right),\end{aligned}$$

where the first line follows by setting $y = cx^2/\left((1-\delta)^2 t\right)$, and the second line follows by the definition of the gamma function, $\Gamma$. Hence, there exist constants $c_1, c_2$ such that

$$\int_{x\ge 0} x^{\vartheta-1} \exp\left(-\frac{cx^2}{(1-\delta)^2 t}\right) dx \le c_1 (1-\delta)^{2c_2} t^{c_2} \qquad \text{for all } t.$$

We thus have

$$\begin{aligned}&(1-\delta)^{-1} \sum_{t=T+1}^{\infty} t\delta^{t-1} \int_{x\ge (1-\delta)^{\frac{1}{2}-\varepsilon}} x^{\vartheta-1} \exp\left(-\frac{cx^2}{(1-\delta)^2 t}\right) dx \\ \le\; &(1-\delta)^{-1} \sum_{t=T+1}^{\infty} t\delta^{t-1} \int_{x\ge 0} x^{\vartheta-1} \exp\left(-\frac{cx^2}{(1-\delta)^2 t}\right) dx \\ \le\; &c_1 (1-\delta)^{2c_2-1} \sum_{t=T+1}^{\infty} \delta^{t-1} t^{c_2+1} \\ \le\; &c_1 (1-\delta)^{2c_2-1} \delta^{T-1} (T+1)^{c_2+1} \sum_{t=1}^{\infty} \delta^t t^{c_2} \\ \le\; &c_1 (1-\delta)^{c_2-2} \delta^{T-1} (T+1)^{c_2+1} k(c_2) \xrightarrow{\delta\to 1} 0,\end{aligned}$$

where $k$ is defined in (19) and the limit follows recalling that $\delta^T \le \exp\left(-(1-\delta)^{-\varepsilon}\right)$. ■

We now establish inequality (20).

**Lemma 22** *We have*

$$\mathbb{E}^{\alpha_t}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] \le \mathbb{E}^{\mu}\left[\frac{\psi'(a_t)\,\nu(y_t|a_t)}{\mathcal{I}(a_t)}\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}\right] \qquad \textit{for all } t.$$

**Proof.** Fix any $t$ and $\varepsilon > 0$. Since $\psi'(a)/\mathcal{I}(a)$ is bounded (uniformly in $a$) given Assumption 2(ii), there exists $\tilde{\Delta} > 0$ such that, for all $\Delta < \tilde{\Delta}$, we have $(\psi'(a)/\mathcal{I}(a))\Delta \le \varepsilon$ for all $a \in A$. Fix $\bar{\Delta} < \tilde{\Delta}$ such that, for all $a \in A$, $\int_y \left(\max_{\tilde{a}\in[a,a+\bar{\Delta}]} p_a(y|\tilde{a})^2/p(y|a)\right) dy < \infty$. (Such $\bar{\Delta}$ exists by Assumption 2(i).) Now, for any $\Delta < \bar{\Delta}$, consider the manipulation where, whenever the agent is recommended action $a \ge \varepsilon$ in period $t$, she instead takes $a - (\psi'(a)/\mathcal{I}(a))\Delta$. This manipulation is unprofitable for the agent if and only if

$$\mathbb{E}^{\alpha_t}\left[\mathbf{1}\{a\ge\varepsilon\}\left(\psi(a)-\psi\left(a-\frac{\psi'(a)}{\mathcal{I}(a)}\Delta\right)\right)\right]$$
$$\le \mathbb{E}^{\mu}\left[\mathbf{1}\{a_t\ge\varepsilon\}\frac{p(y_t|a_t)-p\left(y_t|a_t-\frac{\psi'(a_t)}{\mathcal{I}(a_t)}\Delta\right)}{p(y|a_t)}\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}\right].$$

Since this holds for all $\Delta < \bar{\Delta}$, we have

$$\lim_{\Delta\to 0}\mathbb{E}^{\alpha_t}\left[\mathbf{1}\{a\ge\varepsilon\}\frac{\psi(a)-\psi\left(a-\frac{\psi'(a)}{\mathcal{I}(a)}\Delta\right)}{\Delta}\right]$$
$$\le \lim_{\Delta\to 0}\mathbb{E}^{\mu}\left[\mathbf{1}\{a_t\ge\varepsilon\}\frac{p(y_t|a_t)-p\left(y_t|a_t-\frac{\psi'(a_t)}{\mathcal{I}(a_t)}\Delta\right)}{\Delta p(y_t|a_t)}\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}\right]$$

In this inequality, the LHS is bounded because $\psi'(a)^2/\mathcal{I}(a)$ is bounded given Assumption 2(ii), and hence is equal to $\mathbb{E}^{\alpha_t}\left[\mathbf{1}\{a\ge\varepsilon\}\psi'(a)^2/\mathcal{I}(a)\right]$, by dominated convergence. As for the RHS, for each $t' \ge t+1$, by Cauchy-Schwarz the corresponding term in the sum is bounded by

$$\mathbb{E}^{\mu}\left[\mathbf{1}\{a_t\ge\varepsilon\}\left(\frac{p(y_t|a_t)-p\left(y_t|a_t-\frac{\psi'(a_t)}{\mathcal{I}(a_t)}\Delta\right)}{\Delta p(y|a_t)}\right)^2\right]^{1/2}\times\mathbb{E}^{\mu}\left[\hat{u}_{t'}^2\right]^{1/2}.$$

This is finite because $\int_y \left(\max_{\tilde{a}\in[a,a+\bar{\Delta}]} p_a(y|\tilde{a})^2/p(y|a)\right) dy < \infty$ (by Assumption 2(i)) and $\mathbb{E}^{\mu}[\hat{u}_{t'}^2] < \infty$ for all $t'$ (as otherwise the principal's expected payoff would equal $-\infty$ by Lemma 20). Hence, the entire RHS is bounded, and hence is equal to $\mathbb{E}^{\mu}\Big[\mathbf{1}\{a_t\ge\varepsilon\}(\psi'(a_t)\,\nu(y_t|a_t)/\mathcal{I}(a_t))$

by dominated convergence. In total, we have

$$\mathbb{E}^{\alpha_t}\left[\mathbf{1}\{a\geq\varepsilon\}\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]\leq\mathbb{E}^{\mu}\left[\mathbf{1}\{a_t\geq\varepsilon\}\frac{\psi'(a_t)\,\nu(y_t|a_t)}{\mathcal{I}(a_t)}\sum_{t'=t+1}^{\infty}\delta^{t'-t}\hat{u}_{t'}\right].$$

Since this holds for all $\varepsilon>0$, and $\psi'(a)^2/\mathcal{I}(a)$ and $\psi'(a_t)\,\nu(y_t|a_t)/\mathcal{I}(a_t)$ are continuous, taking $\varepsilon\to 0$ completes the proof. ■

Now we prove our key lemmas, Lemmas 10 and 11. These complete the proof that inefficiency is at least (14) in the blind game.

## G.1 Proof of Lemma 10

Multiplying both sides of (22) by $2\delta/((1-\delta)\bar{F}''(w))$ and using (21), it suffices to find $c,\varepsilon>0$ such that

$$\min_{(\hat{u}_t(h))_{t,h}\geq-\left(w+\psi\left(\bar{A}\right)\right)\text{ s.t. }(20)}\sum_{t=1}^{\infty}\delta^t\frac{2\mathbb{E}^{\mu}\left[\tilde{F}(\hat{u}_t(h))\right]}{\bar{F}''(w)}\geq\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]-c(1-\delta)^{\varepsilon},$$

where

$$\tilde{F}(\hat{u})=\bar{F}(w+\hat{u})-\bar{F}(w)-\bar{F}'(w)\,\hat{u}.$$

By weak duality, taking a Lagrange multiplier of $2(1-\delta)\,\delta^{t-1}$ on (20) for each $t$, the LHS is no less than

$$\min_{(\hat{u}_t(h))_{t,h}\geq-\left(w+\psi\left(\bar{A}\right)\right)}2\sum_{t=1}^{\infty}\delta^t\mathbb{E}^{\mu}\left[\frac{\tilde{F}(\hat{u}_t(h))}{\bar{F}''(w)}-\underbrace{\left(\frac{1-\delta}{\delta}\sum_{t'=1}^{t-1}\frac{\psi'(a_{t'})\,\nu(y_t|a_{t'})}{\mathcal{I}(a_{t'})}\right)}_{:=\Omega_t^{\delta}(h)}\hat{u}_t(h)\right]+2\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]. \tag{46}$$

It remains to bound (46). Since $\tilde{F}$ is concave, the first-order necessary and sufficient condition for $\hat{u}_t(h)$ is

$$\hat{u}_t(h)=\begin{cases}\tilde{F}'^{-1}\left(\bar{F}''(w)\,\Omega_t^{\delta}(h)\right) & \text{if }\tilde{F}'^{-1}\left(\bar{F}''(w)\,\Omega_t^{\delta}(h)\right)\geq-\left(w+\psi\left(\bar{A}\right)\right),\\ -\left(w+\psi\left(\bar{A}\right)\right) & \text{if }\tilde{F}'^{-1}\left(\bar{F}''(w)\,\Omega_t^{\delta}(h)\right)<-\left(w+\psi\left(\bar{A}\right)\right).\end{cases}$$

Now fix any $\varepsilon\in(0,1/8)$ and let

$$\bar{H}^t=\left\{h\in(A\times Y)^{\infty}:-(1-\delta)^{\frac{1}{2}-\varepsilon}\leq\Omega_t^{\delta}(h)\leq(1-\delta)^{\frac{1}{2}-\varepsilon}\right\}\qquad\text{for all }t. \tag{47}$$

We establish two further lemmas.

**Lemma 23** *There exists $\hat{c} > 0$ such that, for any sufficiently large $\delta < 1$, we have*

$$2\left(\frac{\tilde{F}\left(\hat{u}_t\left(h\right)\right)}{\bar{F}''\left(w\right)} - \Omega_t^{\delta}\left(h\right)\hat{u}_t\left(h\right)\right) \geq -\Omega_t^{\delta}\left(h\right)^2 - \hat{c}\left(1-\delta\right)^{1+\varepsilon} \qquad \text{for all } t \text{ and } h \in \bar{H}_t. \tag{48}$$

**Proof.** For sufficiently large $\delta$, we have $\hat{u}_t\left(h\right) = \tilde{F}'^{-1}\left(\bar{F}''\left(w\right)\Omega_t^{\delta}\left(h\right)\right)$ for all $t$ and $h \in \bar{H}^t$. Since $\tilde{F} \in C^2$ and $\Omega_t^{\delta}\left(h\right)$ is bounded (uniformly in $\delta$) for $h \in \bar{H}^t$, by Taylor expansion of $\tilde{F}'^{-1}$ and $\tilde{F} \circ \tilde{F}'^{-1}$ around 0, there exists $\hat{c} > 0$ such that, for any $\delta < 1$ and $h \in \bar{H}^t$, we have

$$\begin{aligned}
\left|\hat{u}_t\left(h\right) - \left(\tilde{F}'^{-1}\left(0\right) - \frac{\bar{F}''\left(w\right)}{\tilde{F}'' \circ \tilde{F}'^{-1}\left(0\right)}\Omega_t^{\delta}\left(h\right)\right)\right| &\leq \frac{\hat{c}}{3}\Omega_t^{\delta}\left(h\right)^2 \qquad \text{and} \\
\left|\frac{2\tilde{F}\left(\hat{u}_t\left(h\right)\right)}{\bar{F}''\left(w\right)} - \left(\begin{array}{c} \frac{2\tilde{F}\circ\tilde{F}'^{-1}(0)}{\bar{F}''(w)} - \frac{2\tilde{F}'\circ\tilde{F}'^{-1}(0)}{\tilde{F}''\circ\tilde{F}'^{-1}(0)}\Omega_t^{\delta}\left(h\right) \\ +\frac{\bar{F}''(w)}{\tilde{F}''\circ\tilde{F}'^{-1}(0)}\left(1 - \frac{\tilde{F}'\circ\tilde{F}'^{-1}(0)\times\tilde{F}'''\circ\tilde{F}'^{-1}(0)}{\left(\tilde{F}''\circ\tilde{F}'^{-1}(0)\right)^2}\right)\Omega_t^{\delta}\left(h\right)^2 \end{array}\right)\right| &\leq \frac{\hat{c}}{3}\left|\Omega_t^{\delta}\left(h\right)\right|^3.
\end{aligned}$$

Since $\tilde{F}'^{-1}\left(0\right) = \tilde{F}' \circ \tilde{F}'^{-1}\left(0\right) = 0$ and $\tilde{F}'' \circ \tilde{F}'^{-1}\left(0\right) = \bar{F}''\left(w\right)$ by definition of $\tilde{F}$, these inequalities simplify to

$$\begin{aligned}
\left|\hat{u}_t\left(h\right) - \Omega_t^{\delta}\left(h\right)\right| &\leq \frac{\hat{c}}{3}\Omega_t^{\delta}\left(h\right)^2 \qquad \text{and} \\
\left|\frac{2\tilde{F}\left(\hat{u}_t\left(h\right)\right)}{\bar{F}''\left(w\right)} - \Omega_t^{\delta}\left(h\right)^2\right| &\leq \frac{\hat{c}}{3}\left|\Omega_t^{\delta}\left(h\right)\right|^3.
\end{aligned}$$

Multiplying the first inequality by $\Omega_t^{\delta}\left(h\right)$ and applying the triangle inequality gives

$$\begin{aligned}
\left|\Omega_t^{\delta}\left(h\right)\hat{u}_t\left(h\right) - \Omega_t^{\delta}\left(h\right)^2\right| &\leq \frac{\hat{c}}{3}\Omega_t^{\delta}\left(h\right)^3 \qquad \text{and} \\
\left|\frac{2\tilde{F}\left(\hat{u}_t\left(h\right)\right)}{\bar{F}''\left(w\right)} - \Omega_t^{\delta}\left(h\right)\hat{u}_t\left(h\right)\right| &\leq \frac{2\hat{c}}{3}\left|\Omega_t^{\delta}\left(h\right)\right|^3.
\end{aligned}$$

We thus have

$$\begin{aligned}
2\left(\frac{\tilde{F}\left(\hat{u}_t\left(h\right)\right)}{\bar{F}''\left(w\right)} - \Omega_t^{\delta}\left(h\right)\hat{u}_t\left(h\right)\right) &\geq -\Omega_t^{\delta}\left(h\right)\hat{u}_t\left(h\right) - \frac{2\hat{c}}{3}\left|\Omega_t^{\delta}\left(h\right)\right|^3 \\
&\geq -\Omega_t^{\delta}\left(h\right)^2 - \hat{c}\left|\Omega_t^{\delta}\left(h\right)\right|^3 \\
&\geq -\Omega_t^{\delta}\left(h\right)^2 - \hat{c}\left(1-\delta\right)^{\frac{3}{2}-3\varepsilon} \\
&\geq -\Omega_t^{\delta}\left(h\right)^2 - \hat{c}\left(1-\delta\right)^{1+\varepsilon},
\end{aligned}$$

where the third line follows by (47), and the fourth line follows because $\varepsilon < 1/8$. ■

**Lemma 24** *For any sufficiently large $\delta < 1$, we have*

$$2\sum_{t=1}^{\infty}\delta^t \int_{h\notin \bar{H}^t}\left(\frac{\tilde{F}(\hat{u}_t(h))}{\bar{F}''(w)} - \Omega_t^{\delta}(h)\,\hat{u}_t(h)\right)d\mu(h) \geq -(1-\delta)^{\varepsilon}. \tag{49}$$

**Proof.** We first show that, for sufficiently large $\delta$,

$$\sum_{t=1}^{\infty}\delta^t \int_{h\notin \bar{H}^t}\Omega_t^{\delta}(h)^2\, d\mu(h) \leq (1-\delta)^{2\varepsilon}. \tag{50}$$

To see this, note that $\left(\psi'(a_t)\,\nu(y_t|a_t)/\mathcal{I}(a_t)\right)_t$ is a sequence of martingale increments where, for all $\theta > 0$,

$$\begin{aligned}
\mathbb{E}^{\mu}\left[\exp\left(\theta\frac{1-\delta}{\delta}\frac{\psi'(a_t)\,\nu(y_t|a_t)}{\mathcal{I}(a_t)}\right)|h^t\right] &\leq \max_{a\in A}\mathbb{E}^{y\sim p(y|a)}\left[\exp\left(\theta\frac{1-\delta}{\delta}\frac{\psi'(a)\,\nu(y|a)}{\mathcal{I}(a)}\right)\right] \\
&\leq \exp\left(\frac{\theta^2}{2}\left(\frac{1-\delta}{\delta}\right)^2\max_{a\in A}\frac{\psi'(a)^2}{\mathcal{I}(a)}\right),
\end{aligned}$$

where the second line follows from Assumption 2(iii), noting that $\psi'(a)^2/\mathcal{I}(a)$ is bounded. Hence, by Lemma 9, there exists $\tilde{c} > 0$ such that, for all $t$ and $x \geq 0$, we have

$$\Pr^{\mu}\left(\left|\Omega_t^{\delta}(h)\right| \geq x\right) = \Pr^{\mu}\left(\left|\frac{1-\delta}{\delta}\sum_{t'=1}^{t-1}\frac{\psi'(a_{t'})\,\nu(y_{t'}|a_{t'})}{\mathcal{I}(a_{t'})}\right| \geq x\right) \leq 2\exp\left(-\frac{\tilde{c}x^2}{(1-\delta)^2 t}\right).$$

We can thus apply Lemma 21 to the sequence $\left(\left|\Omega^{\delta}\right|_t\right)_{t\geq 1}$ to conclude that (50) holds for sufficiently large $\delta$.

Thus, for sufficiently large $\delta$, we have

$$\begin{aligned}
& 2\sum_{t=1}^{\infty}\delta^t \int_{h\notin \bar{H}^t}\left(\frac{\tilde{F}(\hat{u}_t(h))}{\bar{F}''(w)} - \Omega_t^{\delta}(h)\,\hat{u}_t(h)\right)d\mu(h) \\
\geq\ & 2\sum_{t=1}^{\infty}\delta^t \int_{h\notin \bar{H}^t}\left(-\frac{\varepsilon_2\hat{u}_t(h)^2}{2\bar{F}''(w)} - \Omega_t^{\delta}(h)\,\hat{u}_t(h)\right)d\mu(h) \\
\geq\ & \frac{\bar{F}''(w)}{\varepsilon_2}\sum_{t=1}^{\infty}\delta^t \int_{h\notin \bar{H}^t}\Omega_t^{\delta}(h)^2\, d\mu(h) \geq \frac{\bar{F}''(w)}{\varepsilon_2}(1-\delta)^{2\varepsilon} \geq -(1-\delta)^{\varepsilon},
\end{aligned}$$

where the first inequality follows by Lemma 20 (as taking a second-order Taylor expansion gives $\tilde{F}(x) \leq -(\varepsilon_2/2)\,x^2$ for all $x \geq -w$), the second follows by minimizing over $x_t(h)$, the third follows by (50), and the fourth follows by $\bar{F}'' < 0$. ■

By (48), (49), and $\Omega_t^\delta(h)^2 > 0$, we see that (46) is no less than

$$-\sum_{t=2}^{\infty} \delta^t \mathbb{E}^\mu \left[\Omega_t^\delta(h)^2\right] + 2\mathbb{E}^\alpha \left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - \max\{\hat{c}, 1\}(1-\delta)^\varepsilon.$$

Since $\left(\psi'(a_t)\nu(y_t|a_t)/\mathcal{I}(a_t)\right)_t$ is a sequence of martingale increments, we have

$$\mathbb{E}_\mu\left[\Omega_t^\delta(h)^2\right] = \left(\frac{1-\delta}{\delta}\right)^2 \sum_{t'=1}^{t-1} \mathbb{E}^\mu\left[\left(\frac{\psi'(a_{t'})\nu(y_{t'}|a_{t'})}{\mathcal{I}(a_{t'})}\right)^2\right] = \left(\frac{1-\delta}{\delta}\right)^2 \sum_{t'=1}^{t-1} \mathbb{E}^{\alpha_{t'}}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right].$$

Therefore, (46) is no less than

$$\begin{aligned} & -\left(\frac{1-\delta}{\delta}\right)^2 \sum_{t=2}^{\infty} \delta^t \sum_{t'=1}^{t-1} \mathbb{E}^{\alpha_{t'}}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] + 2\mathbb{E}^\alpha\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - \max\{\hat{c}, 1\}(1-\delta)^\varepsilon \\ = \ & \mathbb{E}^\alpha\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - \max\{\hat{c}, 1\}(1-\delta)^\varepsilon. \end{aligned}$$

Taking $c = \max\{\hat{c}, 1\}$ completes the proof.

## G.2 Proof of Lemma 11

If $\alpha$ assigns probability 1 to $a = \bar{a}(w)$ then

$$\begin{aligned} & \min\left\{\frac{1-\delta}{\delta}\mathbb{E}^\alpha\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]\frac{\bar{F}''(w)}{2} + c(1-\delta)^{1+\varepsilon}, -\varepsilon_1 \mathbb{E}^\alpha\left[(a-\bar{a}(w))^2\right]\right\} \\ \geq \ & \frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2}. \end{aligned}$$

Hence, the optimal $\alpha$ satisfies

$$\mathbb{E}^\alpha\left[(a-\bar{a}(w))^2\right] \leq \frac{1}{\varepsilon_1}\frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{-\bar{F}''(w)}{2}. \tag{51}$$

Since $\psi'(a)^2/\mathcal{I}(a)$ is Lipschitz continuous, there exists $\kappa > 0$ such that

$$\begin{aligned}
&\frac{1-\delta}{\delta}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]\frac{\bar{F}''(w)}{2}+c(1-\delta)^{1+\varepsilon} \\
\leq\; &\frac{1-\delta}{\delta}\left(\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}+(\kappa\mathbb{E}^{\alpha}[|a-\bar{a}(w)|])\right)\frac{\bar{F}''(w)}{2}+c(1-\delta)^{1+\varepsilon}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1-\delta}{\delta}\frac{-\bar{F}''(w)}{2}\kappa\mathbb{E}^{\alpha}[|a-\bar{a}(w)|] &\leq \frac{1-\delta}{\delta}\frac{-\bar{F}''(w)}{2}\kappa\sqrt{\mathbb{E}^{\alpha}\left[(a-\bar{a}(w))^2\right]} \\
&\leq \left(\frac{1-\delta}{\delta}\frac{-\bar{F}''(w)}{2}\right)^{\frac{3}{2}}\kappa\sqrt{\frac{\psi'(\bar{a}(w))^2}{\varepsilon_1\mathcal{I}(\bar{a}(w))}},
\end{aligned}$$

where the first inequality follows by Cauchy-Schwarz, and the second follows by (51). Since this expression is of order $(1-\delta)^{3/2}$, there exists $\tilde{c} > 0$ such that, for sufficiently large $\delta$, we have

$$\frac{1-\delta}{\delta}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right]\frac{\bar{F}''(w)}{2}+c(1-\delta)^{1+\varepsilon} \leq \frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2}+\tilde{c}(1-\delta)^{1+\varepsilon},$$

completing the proof.

## G.3 Construction of $x_a(y)$

**Lemma 25** *There exists $\bar{I} < \infty$ such that, for any sufficiently large $\delta < 1$ and any $\bar{a} \in A$, there exists $x_{\bar{a}} : Y \to [-(1-\delta)^{-1/4}, (1-\delta)^{-1/4}]$ satisfying*

$$\bar{a} \in \operatorname*{argmax}_{a\in A}\int_y x_{\bar{a}}(y)\,p(y|a)\,dy - \psi(a), \tag{52}$$

$$\int_y x_{\bar{a}}(y)\,p(y|\bar{a})\,dy = 0, \tag{53}$$

$$\int_y x_{\bar{a}}(y)^2\,p(y|\bar{a})\,dy \leq \frac{1}{\delta}\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}, \qquad \textit{and} \tag{54}$$

$$\int_y \exp(\theta x(y|\bar{a}))\,p(y|\bar{a})\,dy \leq \exp\left(\theta^2\bar{I}\right). \tag{55}$$

**Proof.** Define, in turn,

$$\begin{aligned}
\varphi_{\bar{a}} &= \frac{\mathcal{I}(\bar{a})}{\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right]}, && (56)\\
\varepsilon_{\bar{a}} &= -\varphi_{\bar{a}}\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right], \qquad \text{and} && (57)\\
x_{\bar{a}}(y) &= \varphi_{\bar{a}}\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\mathbf{1}\left\{|\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a}) + \varepsilon_{\bar{a}}.
\end{aligned}$$

Note that $\varphi_{\bar{a}} \geq 1$, since $\mathcal{I}(\bar{a}) = \mathbb{E}\left[\nu(y|\bar{a})^2\right]$. We will prove that $x_{\bar{a}}(y) \in [-(1-\delta)^{-1/4}, (1-\delta)^{-1/4}]$ for all $y$, and that (52)–(55) hold. We first establish that, for any sufficiently large $\delta < 1$ and any $\bar{a} \in \left[0, \bar{A}\right]$, we have

$$\begin{aligned}
|\varphi_{\bar{a}} - 1| &\leq \exp\left(-(1-\delta)^{\frac{1}{4}}\right) \qquad \text{and} && (58)\\
|\varepsilon_{\bar{a}}| &\leq \frac{\psi'(\bar{a})}{\sqrt{\mathcal{I}(\bar{a})}}\exp\left(-(1-\delta)^{-\frac{1}{5}}\right). && (59)
\end{aligned}$$

Note that (58) and (59) immediately imply that $x_{\bar{a}}(y) \in [-(1-\delta)^{-1/4}, (1-\delta)^{-1/4}]$.

For (58), note that

$$\begin{aligned}
0 \leq \varphi_{\bar{a}} &= \frac{\mathcal{I}(\bar{a})}{\mathcal{I}(\bar{a}) - \mathbb{E}\left[1\left\{|\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right]}\\
&\leq \frac{\mathcal{I}(\bar{a})}{\mathcal{I}(\bar{a}) - \sqrt{\Pr\left(|\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}}\right)\mathbb{E}\left[\nu(y|\bar{a})^4\right]}}.
\end{aligned}$$

where the second line follows by Cauchy-Schwarz. By Assumption 2(iii), we have

$$\int_y \exp(\theta\nu(y|\bar{a}))\, p(y|\bar{a})\, dy \leq \exp\left(\theta^2\mathcal{I}(\bar{a})/2\right) \qquad \text{for all } \bar{a} \in A \text{ and } \theta \in \mathbb{R}. \tag{60}$$

Since $\mathcal{I}(\bar{a})$ is uniformly bounded in $\bar{a}$ given Assumption 2(ii), there exists $c > 0$ such that, for all $\bar{a} \in A$, $\varphi_{\bar{a}}$ is bounded by

$$\frac{\mathcal{I}(\bar{a})}{\mathcal{I}(\bar{a}) - \exp\left(-\frac{(1-\delta)^{-\frac{2}{5}}}{c}\right)16\Gamma(2)\,\mathcal{I}(\bar{a})} = \frac{1}{1 - \exp\left(-\frac{(1-\delta)^{-\frac{2}{5}}}{c}\right)16\Gamma(2)},$$

which implies (58).

For (59), note that

$$
\begin{aligned}
|\varepsilon_{\bar{a}}| &= \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\left|\mathbb{E}\left[\varphi_{\bar{a}}\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right]\right| \\
&\le \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\left(\begin{array}{c}\left|\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right]\right| \\ +(\varphi_{\bar{a}}-1)\left|\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right]\right|\end{array}\right)
\end{aligned}
$$

where the inequality is by $\mathbb{E}[\nu(y|\bar{a})] = 0$ and the triangle inequality. As above, applying Cauchy-Schwarz and Assumption 2(ii)&(iii), we have

$$
\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\left|\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right]\right| \le \exp\left(-\frac{(1-\delta)^{-\frac{2}{5}}}{c}\right)\frac{\psi'(\bar{a})}{\sqrt{\mathcal{I}(\bar{a})}}.
$$

Again applying Cauchy-Schwarz, together with (58), we have

$$
\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}(\varphi_{\bar{a}}-1)\left|\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right]\right| \le \frac{\psi'(\bar{a})}{\sqrt{\mathcal{I}(\bar{a})}}\exp\left(-(1-\delta)^{\frac{1}{4}}\right).
$$

Taking $\delta < 1$ sufficiently large so that $\exp\left(-(1-\delta)^{-2/5}/c\right)+\exp\left(-(1-\delta)^{1/4}\right) \le \exp\left(-(1-\delta)^{1/5}\right)$, we have (59).

We now establish (52)–(55). Note that (53) follows directly from (57). For (52), for any $a \neq \bar{a}$, we have

$$
\begin{aligned}
&\mathbb{E}^{\bar{a}}[x_{\bar{a}}(y)] - \psi(\bar{a}) - (\mathbb{E}^{a}[x_{\bar{a}}(y)] - \psi(a)) \\
=\ &\psi(a) - \psi(\bar{a}) - \varphi_{\bar{a}}\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\int_y \mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\left(\begin{array}{c} p(y|a) \\ -p(y|\bar{a})\end{array}\right)dy.
\end{aligned} \tag{61}
$$

We bound the second line as follows. For any $\gamma \in (0, \inf_{a\in\bar{A}}\psi''(a))$ and any $\bar{a} \in A$, we have

$$
\psi(a) - \psi(\bar{a}) \ge \psi'(a)(a-\bar{a}) + \frac{\gamma}{2}(a-\bar{a})^2;
$$

Taking a second-order Taylor expansion of $p(y|a)$ around $a = \bar{a}$, there exists $\hat{a} \in A$ such that

$$\begin{aligned}
& \varphi_{\bar{a}} \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})} \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a}) \left( p(y|a) - p(y|\bar{a}) \right) dy \\
= \; & (a-\bar{a}) \varphi_{\bar{a}} \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})} \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a})^2 p(y|a) \, dy \\
& + \frac{(a-\bar{a})^2}{2} \varphi_{\bar{a}} \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})} \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a}) p_{aa}(y|\hat{a}) \, dy.
\end{aligned}$$

Substituting (56), (61) is no less than $(a-\bar{a})^2/2$ multiplied by

$$\gamma - \varphi_{\bar{a}} \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})} \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a}) p_{aa}(y|\hat{a}) \, dy.$$

It remains to show that, for any sufficiently large $\delta < 1$ and any $\bar{a}, \hat{a}$, this expression is non-negative. Since $\psi'(\bar{a})/\mathcal{I}(\bar{a})$ is bounded, by (12) it suffices to show that

$$\lim_{\delta \to 1} \sup_{\bar{a},\hat{a}} \int_y \left( \varphi_{\bar{a}} \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} - 1 \right) \nu(y|\bar{a}) p_{aa}(y|\hat{a}) \, dy = 0.$$

In turn, it suffices to show that both

$$\begin{aligned}
& \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a}) p_{aa}(y|\hat{a}) \, dy \qquad \text{and} \\
& (\varphi_{\bar{a}} - 1) \int_y \mathbf{1}\left\{ |\nu(y|\bar{a})| \leq (1-\delta)^{-\frac{1}{5}} \right\} \nu(y|\bar{a}) p_{aa}(y|\hat{a}) \, dy
\end{aligned}$$

converge to 0 as $\delta \to 1$, uniformly in $(\bar{a}, \hat{a})$.

By Cauchy-Schwarz, the first line is bounded by

$$\Pr\left( |\nu(y|\bar{a})| > (1-\delta)^{-\frac{1}{5}} \right)^{\frac{1}{4}} \left( \int_y \nu(y|\bar{a})^4 p(y|\bar{a}) \, dy \right)^{\frac{1}{4}} \left( \int_y \left( \frac{p_{aa}(y|\hat{a})}{p(y|\bar{a})} \right)^2 p(y|\bar{a}) \, dy \right)^{\frac{1}{2}}.$$

By (60), the first term of this product converges to 0 uniformly in $\bar{a}$ as $\delta \to 1$, and the second term is bounded uniformly in $\bar{a}$ given Assumptions 2(i) and (ii). Moreover, (13) ensures the last term is bounded uniformly in $(\bar{a}, \hat{a})$. So the entire product converges to 0 uniformly in $(\bar{a}, \hat{a})$.

Similarly, again by Cauchy-Schwarz, the second line is bounded by

$$(\varphi_{\bar{a}} - 1) \left( \int_y \nu(y|\bar{a})^2 p(y|\bar{a}) \, dy \right)^{\frac{1}{2}} \left( \int_y \left( \frac{p_{aa}(y|\hat{a})}{p(y|\bar{a})} \right)^2 p(y|\bar{a}) \, dy \right)^{\frac{1}{2}}.$$

By (58), the first term of this product converges to 0 uniformly in $\bar{a}$ as $\delta \to 1$; and, as above, the other terms are bounded uniformly in $(\bar{a}, \hat{a})$. The product thus converges to 0 uniformly in $(\bar{a}, \hat{a})$. This establishes (52).

We next establish (54). By construction, we have

$$\begin{aligned} \mathbb{E}\left[x_{\bar{a}}(y)^2\right] - \frac{1}{\delta}\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})} &= \varphi_{\bar{a}}^2 \left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\right)^2 \mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right] \\ &+2\varepsilon_{\bar{a}}\varphi_{\bar{a}}\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right] + \varepsilon_{\bar{a}}^2 - \frac{1}{\delta}\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}. \end{aligned}$$

By (56),

$$\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\right)^2 \varphi_{\bar{a}}\mathbb{E}\left[\mathbf{1}\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right] = \frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}.$$

Thus, the above expression equals

$$\begin{aligned} &\varphi_{\bar{a}}(\varphi_{\bar{a}}-1)\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\right)^2 \mathbb{E}\left[1\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right] \\ &+2\varepsilon_{\bar{a}}\varphi_{\bar{a}}\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\mathbb{E}\left[1\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right] + \varepsilon_{\bar{a}}^2 - \frac{1-\delta}{\delta}\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}. \end{aligned}$$

Together with

$$\begin{aligned} \left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\right)^2 \mathbb{E}\left[1\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})^2\right] &\le \frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}, \\ \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\mathbb{E}\left[1\left\{|\nu(y|\bar{a})| \le (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a})\right] &\le \frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\sqrt{\mathbb{E}\left[\nu(y|\bar{a})^2\right]} = \frac{\psi'(\bar{a})}{\sqrt{\mathcal{I}(\bar{a})}}, \end{aligned}$$

(58), and (59), this in turn is bounded by $\psi'(\bar{a})^2/\mathcal{I}(\bar{a})$ multiplied by

$$\varphi_{\bar{a}}\exp\left(-(1-\delta)^{\frac{1}{4}}\right) + 2\varphi_{\bar{a}}\exp\left(-(1-\delta)^{-\frac{1}{5}}\right) + \exp\left(-2(1-\delta)^{-\frac{1}{5}}\right) - \frac{1-\delta}{\delta},$$

which is non-positive uniformly in $\bar{a}$ for sufficiently large $\delta < 1$.

We finally establish (55). It suffices to show that, for any $I \ge \sup_{a\in A}\psi'(\bar{a})^2/\mathcal{I}(\bar{a})$, we have

$$\Pr(|x_{\bar{a}}(y)| \ge \lambda) \le 2\exp\left(-\lambda^2/(4I)\right).$$

This is immediate if $\lambda \le \sqrt{(\log 2)(4I)}$. Next, for any sufficiently large $\delta$, $\bar{a} \in A$, and

$\lambda \geq \sqrt{(\log 2)(4I)}$, we have $\lambda - \varepsilon_{\bar{a}} > 0$ and $-\lambda - \varepsilon_{\bar{a}} < 0$ by (59). Hence, we have

$$\begin{aligned}
\Pr\left(\left|x_{\bar{a}}(y)\right| \geq \lambda\right) &= \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\varphi_{\bar{a}}\mathbf{1}\left\{\left|\nu(y|\bar{a})\right| \leq (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a}) \leq -\lambda - \varepsilon_{\bar{a}}\right) \\
&\quad + \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\varphi_{\bar{a}}\mathbf{1}\left\{\left|\nu(y|\bar{a})\right| \leq (1-\delta)^{-\frac{1}{5}}\right\}\nu(y|\bar{a}) \geq \lambda - \varepsilon_{\bar{a}}\right) \\
&\leq \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \leq \frac{-\lambda + \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right) + \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \geq \frac{\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right).
\end{aligned}$$

Since $\nu(y|\bar{a})$ is sub-Gaussian with variance-proxy $\mathcal{I}(\bar{a})$ by Assumption 2(iii), we have

$$\Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \leq \frac{-\lambda + \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right) + \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \geq \frac{\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right) \leq 2\exp\left(-\frac{\left(\frac{\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right)^2}{2\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}}\right).$$

Finally, note that

$$\left(\frac{\left(\frac{\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right)^2}{2\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}}\right) \Big/ \left(\frac{\lambda^2}{4I}\right) \geq 2\left(1 - \frac{2\varepsilon_{\bar{a}}}{\lambda}\right)\frac{1}{\varphi_{\bar{a}}^2}\frac{1}{\frac{\psi'(\bar{a})^2}{\mathcal{I}(\bar{a})}/I} \geq 2\left(1 - \frac{2\varepsilon_{\bar{a}}}{\sqrt{(\log 2)(4I)}}\right)\frac{1}{\varphi_{\bar{a}}^2},$$

which is greater than one uniformly in $\bar{a}$ for sufficiently large $\delta$, by (58) and (59). We thus have

$$\Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \leq \frac{-\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right) + \Pr\left(\frac{\psi'(\bar{a})}{\mathcal{I}(\bar{a})}\nu(y|\bar{a}) \geq \frac{\lambda - \varepsilon_{\bar{a}}}{\varphi_{\bar{a}}}\right) \leq 2\exp\left(-\frac{\lambda^2}{4I}\right),$$

as desired. ■

## G.4 Equilibrium Verification

We verify that the contract defined in the main appendix, with $x_a(y)$ defined as in Lemma 25, satisfies incentive compatibility and promise keeping, and hence is an equilibrium.

**Lemma 26** *For each $h^t$, we have*

$$a_t\left(h^t\right) \in \underset{a}{\operatorname{argmax}}\,(1-\delta)\left(u\left(c_t\left(h^t\right)\right) - \psi(a)\right) + \delta\mathbb{E}\left[w_{t+1}\left(h^{t+1}\right)|h^t, a\right], \tag{62}$$

$$w_t\left(h^t\right) = (1-\delta)\left(u\left(c_t\left(h^t\right)\right) - \psi\left(a_t\left(h^t\right)\right)\right) + \delta\mathbb{E}\left[w_{t+1}\left(h^{t+1}\right)|h^t, a_t\left(h^t\right)\right]. \tag{63}$$

**Proof.** The conclusion is immediate if $h^t$ is irregular. If $h^t$ is regular, then (62) follows from (52) since

$$\operatorname*{argmax}_a (1-\delta)\left(u\left(c_t\left(h^t\right)\right)-\psi\left(a\right)\right)+\delta\mathbb{E}\left[w_{t+1}\left(h^{t+1}\right)|h^t,a\right] = \operatorname*{argmax}_a \mathbb{E}\left[x_{\bar{a}(w_t(h^t))}\left(y_t\right)|h^t,a\right]-\psi\left(a\right).$$

Moreover, (63) holds as $u\left(\bar{c}\left(w_t\left(h^t\right)\right)\right)-\psi\left(\bar{a}\left(w_t\left(h^t\right)\right)\right)=w_t\left(h^t\right)$ by definition of $\bar{c}$ and $\bar{a}$, and $\mathbb{E}\left[w_{t+1}\left(h^{t+1}\right)|h^t,\bar{a}\left(w_t\left(h^t\right)\right)\right]=w_t\left(h^t\right)$ by (53). ∎

## G.5 Proof of Lemma 12

Let $\hat{H}^t=\left\{h^t:\left|w_t\left(h^t\right)-w\right|\le(1-\delta)^{1/2-\varepsilon}\right\}$ be the set of regular period $t$ histories. Let

$$x_{a_t}^{h^t}\left(y_t\right)=\begin{cases} x_{a_t}\left(y_t\right) & \text{if } h^t\in\hat{H}^t, \\ 0 & \text{if } h^t\notin\hat{H}^t, \end{cases} \qquad \text{and} \qquad X\left(h^t\right)=\frac{1-\delta}{\delta}\sum_{t'=1}^{t-1}x_{a_{t'}}^{h^{t'}}\left(y_{t'}\right),$$

where in the latter definition $h^{t'}$ is the period $t'$ truncation of $h^t$. Note that $w_t\left(h^t\right)=w+X\left(h^t\right)$ for all $t$ and $h^t$, and that $h^t\in\hat{H}^t$ iff $\left|X\left(h^t\right)\right|\le(1-\delta)^{1/2-\varepsilon}$.

We first bound the weight on irregular histories under the equilibrium outcome $\mu$.

**Lemma 27** *For any sufficiently large $\delta<1$, we have $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pr^{\mu}\left(h^t\notin\hat{H}^t\right)\le(1-\delta)^2$.*

**Proof.** Note that $x_{a_t}^{h^t}\left(y_t\right)$ is a sequence of martingale increments. Moreover, by (55),

$$\mathbb{E}^{\mu}\left[\exp\left(\theta\frac{1-\delta}{\delta}x_{a_t}^{h^t}\left(y_t\right)\right)|h^t,a_t\right]\le\exp\left(\theta^2\left(\frac{1-\delta}{\delta}\right)^2\bar{I}\right) \qquad \text{for all } \theta,t,a_t.$$

Therefore, by Lemma 9,

$$\Pr^{\mu}\left(X\left(h^t\right)>x\right)\le 2\exp\left(-\frac{x^2}{2\left(\frac{1-\delta}{\delta}\right)^2\bar{I}t}\right) \qquad \text{for all } x>0.$$

We can now apply Lemma 21 with $\vartheta=0$ to conclude that, for any sufficiently large $\delta$,

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pr^{\mu}\left(h^t\notin\hat{H}^t\right)=(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pr^{\mu}\left(\left|X\left(h^t\right)\right|>(1-\delta)^{1/2-\varepsilon}\right)\le(1-\delta)^2.$$

∎

Recall that $\varepsilon < 1/8$. By Taylor expansion, since $3(1/2 - \varepsilon) > 1 + \varepsilon$ and $|w_t(h^t) - w| \leq (1-\delta)^{1/2-\varepsilon}$ for all $h^t \in \hat{H}^t$, for any sufficiently large $\delta < 1$ and any $h^t \in \hat{H}^t$, we have

$$\begin{aligned} a_t\left(h^t\right) - c_t\left(h^t\right) &= \bar{a}\left(w_t\left(h^t\right)\right) - \bar{c}\left(w_t\left(h^t\right)\right) \\ &= \bar{F}\left(w_t\left(h^t\right)\right) \\ &\geq \bar{F}(w) + \bar{F}'(w)\left(w_t\left(h^t\right) - w\right) + \frac{\bar{F}''(w)}{2}\left(w_t\left(h^t\right) - w\right)^2 - (1-\delta)^{1+\varepsilon} \\ &= \bar{F}(w) + \bar{F}'(w)\left(w_t\left(h^t\right) - w\right) + \frac{\bar{F}''(w)}{2} X\left(h^t\right)^2 - (1-\delta)^{1+\varepsilon}. \end{aligned}$$

At the same time, since $w_t(h^t) \in [0, 2w]$, $a_t(h^t) - c_t(h^t)$ and $\bar{F}(w) + \bar{F}'(w)(w_t(h^t) - w)$ are bounded, and $\bar{F}''(w) \leq 0$, there exists $c_1 > 0$ such that, for any $\delta$ and $h^t \notin \hat{H}^t$, we have

$$a_t\left(h^t\right) - c_t\left(h^t\right) \geq \bar{F}(w) + \bar{F}'(w)\left(w_t\left(h^t\right) - w\right) + \frac{\bar{F}''(w)}{2} X\left(h^t\right)^2 - c_1.$$

Combining these bounds, we have

$$\begin{aligned} & (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\mathbb{E}^{\mu}\left[a_t\left(h^t\right) - c_t\left(h^t\right)\right] \\ \geq\ & \bar{F}(w) + \frac{\bar{F}''(w)}{2}(1-\delta)\sum_{t=2}^{\infty}\delta^{t-1}\int_h X\left(h^t\right)^2 d\mu(h) \\ & -(1-\delta)^{1+\varepsilon} - c_1(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pr{}^{\mu}\left(h_t \notin \hat{H}^t\right). \end{aligned} \tag{64}$$

Moreover, since $(x_{a_t}(y_t))_t$ is a sequence of martingale increments with variance bounded by (54), we have

$$\begin{aligned} & (1-\delta)\sum_{t=2}^{\infty}\delta^{t-1}\sum_h \mu(h)\left(\frac{1-\delta}{\delta}\sum_{t'=1}^{t-1} x_{a_{t'}}(y_{t'})\right)^2 \\ =\ & \frac{1-\delta}{\delta}\left((1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\mathbb{E}^{a\sim\alpha_t}\left[\mathbb{E}^{y\sim p(y|a)}\left[x_a(y)^2\right]\right]\right) \leq \frac{1-\delta}{\delta^2}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right], \end{aligned}$$

Together with Lemma 27, (64) now implies that the principal's payoff is no less than

$$\bar{F}(w) + \frac{\bar{F}''(w)}{2}\frac{1-\delta}{\delta^2}\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - (1-\delta)^{1+\varepsilon} - (1-\delta)^2.$$

It remains to bound $\mathbb{E}^{\alpha}\left[\psi'(a)^2/\mathcal{I}(a)\right]$. Since $\psi'(a)^2/\mathcal{I}(a)$ is Lipschitz continuous, there exists $\kappa > 0$ such that

$$\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - \frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))} \geq -\kappa \mathbb{E}^{\alpha}\left[|a - \bar{a}(w)|\right].$$

Since $\bar{a}(w)$ is continuously differentiable and $w \in [0, 2w]$, there exists $c_2$ such that, for any $t$ and $h^t \in \hat{H}^t$,

$$\left|a_t\left(h^t\right) - \bar{a}(w)\right| = \left|\bar{a}\left(w_t\left(h^t\right)\right) - \bar{a}(w)\right| \leq c_2 \left|w_t\left(h^t\right) - w\right| \leq c_2 (1-\delta)^{1/2-\varepsilon}.$$

Since $|a_t(h^t) - \bar{a}(w)| \leq \bar{A}$ for all $t$ and $h^t$, we have

$$\begin{aligned}
\mathbb{E}^{\alpha}\left[|a - \bar{a}(w)|\right] &\leq (1-\delta)\sum_t \delta^{t-1}\left(\int_{h^t \in \hat{H}^t} c_2 \left|w_t\left(h^t\right) - w\right| d\mu\left(h^t\right) + \Pr^{\mu}\left(h_t \notin \hat{H}^t\right)\bar{A}\right) \\
&\leq c_2 (1-\delta)^{1/2-\varepsilon} + 2\bar{A}(1-\delta)\sum_t \delta^{t-1}\Pr^{\mu}\left(h_t \notin \hat{H}^t\right) \\
&\leq c_2 (1-\delta)^{1/2-\varepsilon} + 2\bar{A}(1-\delta)^2,
\end{aligned}$$

where the third inequality follows from Lemma 27. We thus have

$$\mathbb{E}^{\alpha}\left[\frac{\psi'(a)^2}{\mathcal{I}(a)}\right] - \frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))} \geq -\kappa c_2 (1-\delta)^{\frac{1}{2}-\varepsilon} - 2\kappa\bar{A}(1-\delta)^2.$$

Therefore, by (64) (and using $(1-\delta)/\delta^2 = (1-\delta)/\delta + O\left((1-\delta)^2\right)$), there exists $c_3$ such that, for any sufficiently large $\delta < 1$, the principal's payoff is no less than

$$\bar{F}(w) + \frac{1-\delta}{\delta}\frac{\psi'(\bar{a}(w))^2}{\mathcal{I}(\bar{a}(w))}\frac{\bar{F}''(w)}{2} - c_3(1-\delta)^{1+\varepsilon},$$

completing the proof.