Durable Goods: An Explanation for Their Slow Adjustment

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At the microeconomic level, durable purchases are often discontinuous and relatively large. This feature has the potential to explain why aggregate expenditure on durables responds only slowly (relative to the frictionless permanent income model) to wealth and other aggregate innovations. In this paper I develop new results on the problem of dynamic aggregation of stochastically heterogeneous units, which help to characterize the connection between microeconomic behavior and aggregate dynamics in the presence of nonconvex adjustment costs. Using these results and splitting postwar U.S. aggregate durable purchases into different subcategories and time periods, I provide further support for the view that lumpy microeconomic purchases play an important role in explaining the time-series behavior of aggregate expenditure on durable goods.

Following Hall's (1978) "random walk" hypothesis for nondurables consumption, Mankiw (1982) pointed out that a similar pattern should be observed for the services yielded by durable goods. Furthermore, if these services are linear in the stocks, changes in the stock of durables should be white noise; if depreciation is exponential, changes in real durable goods expenditures, ΔE, should follow an

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MA(1) process, with the moving average coefficient equal to negative one minus the depreciation rate:

$$\Delta E_t = e_t - (1 - \delta)e_{t-1}, \tag{1}$$

where e_t is a white-noise innovation equal to the net change in the stock of durables and δ is the depreciation rate of the good. Contrary to this simple theory, the moving average coefficient is not present (at least in the magnitude required) in quarterly postwar U.S. data. The estimated coefficient for the period 1959:1–1990:1 is −0.08 with a standard deviation of 0.09.[1]

Caballero (1990*b*) finds, however, that the negative serial correlation implied by consumer optimization does appear after a longer lag—and in the magnitude required—in the covariogram of changes in expenditure on durable goods. This suggests that the basic permanent income hypothesis (PIH) is the right way to think about the medium-long-run response of durables to aggregate shocks but that further work is required to understand its prolonged dynamic behavior.

Several sensible justifications for the slow adjustment of durable purchases have been advanced in the literature. They range from those based on the presence of strong intertemporal complementarities in consumption (e.g., Heaton 1991) to those based on more traditional mechanisms such as the partial adjustment model and the closely related convex adjustment costs model (e.g., Bernanke 1985). Although able to partially account for the sluggish feature of aggregate data, most of these explanations disregard the apparent lumpy and discontinuous nature of durable purchases at the microeconomic level.[2]

One of the main obstacles faced when considering such realistic microeconomic features is that since lumpiness and discontinuous actions are not dominant features of aggregate expenditure on durable goods, it becomes apparent that it is not possible to apply microeconomic policy functions directly to aggregate data. This aggregation issue is not exclusive to durable purchases, of course, since it arises naturally in situations in which individual units face fixed adjustment costs. Different aspects of it have been studied primarily in

[1] The expression for ΔE corresponds to changes in the level of expenditures. Throughout the paper, however, I use the logs instead of the levels since the time-series properties of both series are similar and detrending and heteroskedasticity corrections are simpler when the log form is used. The data are per capita. Also notice that under the null, time aggregation has little incidence since E is (almost) white noise.

[2] Lumpiness can be an optimal outcome since it arises naturally from increasing returns in the adjustment technology. This is the sense in which the word "lumpiness" is used in this paper, although many aspects of the results and discussion also apply to physical lumpiness.

the context of (S, s) inventory models and menu cost pricing (see, e.g., Blinder 1981; Caplin 1985; Caplin and Spulber 1987; Caballero and Engel 1991, 1992; Caplin and Leahy 1991). Bar-Ilan and Blinder (1992) apply some of these results to the problem of durables; they (as well as Lam [1989], who uses numerical simulation methods) conclude that microeconomic lumpiness can indeed slow down the response of aggregate durable expenditures to aggregate wealth shocks.

After surveying the microeconomic and aggregation literature on (S, s) models, Bertola and Caballero (1990*b*) develop a simple stochastic Markov chain apparatus, where the dynamic problem of tracking down the endogenous evolution of the cross-sectional distribution of durable stocks (in deviation from the corresponding optimal stocks) is made operational.[3] Their empirical findings suggest that models of infrequent microeconomic actions can help explain the time-series departure between actual and frictionless PIH durable purchases.

This paper has a narrower scope than that of Bertola and Caballero (1990*b*) and focuses entirely on the aggregation and empirical implementation of a dynamic (S, s) model to durable purchases. The paper contains new results on the problem of dynamic aggregation of stochastically heterogeneous units, which help to characterize the connection between microeconomic discontinuous actions and aggregate dynamics. The main propositions of the paper describe the dynamic behavior of aggregate durable purchases in terms of the flows of units upgrading and downgrading their stocks, and the connection between these flows and the shape of the cross-sectional distribution of durable stocks (their deviation from PIH) at the microeconomic (purchase and sale) trigger boundaries. In this context, several forces present in the relation between aggregate dynamics and the underlying structural parameters are identified and clarified. Among other things, it is argued that, in general, larger wealth drifts, depreciation rates, and uncertainty (given aggregate uncertainty) reduce aggregate inertia by increasing the relative importance of the deterministic component of the fundamental partial differential equation describing the evolution of the cross-sectional distribution of durable stocks. The paper also provides an empirically useful approximate solution to this complex stochastic partial differential equation.

Implementing the model on postwar U.S. data, I find that, to a large extent, the results support the claim that the prolonged and unstable dynamics of durable goods, and the main differences in the time-series behavior of different subcategories of durable goods, are consistent with what is implied by models in which heterogeneous

[3] See Caballero and Engel (1991, 1992) for a description of the aggregation problem in terms of such a distribution.

microeconomic units face nonconvexities in their adjustment technology. For example, goods likely to exhibit larger secondary market imperfections also exhibit larger departures from the frictionless PIH model, and periods in which aggregate uncertainty rises and total uncertainty and average growth in desired aggregate stocks fall also bring about larger departures from PIH consumption.

This paper is divided into two parts. The first one, the paper itself, contains a description of the main insights and findings. The second part, the Appendices, includes the formulas and technical results behind the propositions and arguments in the main text. Section I develops the basic model and describes the mechanisms through which lumpiness at the microeconomic level and heterogeneity influence aggregate dynamics. Section II presents comparisons of the dynamic behavior of actual expenditures on different types of durables and over different periods, and provides estimates of a structural model of U.S. purchases of new cars and furniture. Section III further describes the time-series properties of durables. Section IV summarizes and discusses several unresolved issues. The technical part of the paper follows in Appendices A and B.

I. General Framework

One way to simplify the description of the dynamic behavior of an economic variable, k, is to decompose it into a "target" or "desired" component, k^*, and a "departure" variable, z. Typically, the target variable can be represented in terms of some simple theory that disregards (to first order) short-run dynamic elements. In this paper k^* will be determined by a simple frictionless PIH model. The departure variable is the difference between the actual stock of durables k (unless otherwise indicated, all variables are in logs) and its corresponding target, k^*. From this definition, one can represent the actual stock of durables at time t as

$$k_t = k_t^* + z_t. \tag{2}$$

The dynamic behavior of the U.S. postwar stock of durables can be described in terms of equation (2). When there is a positive wealth shock, k_t^* rises by more than k_t does; therefore, z_t and k_t^* have a negative contemporaneous correlation. This yields "excess smoothness" of durable goods with respect to wealth and other aggregate innovations. Over time, z_t increases, generating positive serial correlation in the process for Δk_t. Since the changes in k_t are the residuals e_t in the equation for changes in expenditure on durables (no longer white noise), equation (1) now reads

$$\Delta E_t = [1 - (1 - \delta)B]\Delta k_t^* + [1 - (1 - \delta)B]\Delta z_t, \tag{1'}$$

where B is the lag operator, and the white-noise implication of Mankiw's derivation applies only to the terms involving Δk^*. When the dynamic behavior of z_t described above is considered, the MA(1) coefficient of ΔE_t becomes very close to zero.

Therefore, an important part of the success of any explanation for the behavior of durable goods—under the maintained assumption that the simple PIH is a good description of what would happen in the absence of frictions—depends on its ability to generate negative contemporaneous correlation between k_t^* and z_t, as well as serial correlation in z_t. As argued in the Introduction and more extensively below, models of infrequent and lumpy durable purchases at the unit level have these features. This is an interesting explanation to pursue not only because of its microeconomic realism—which is certainly an advantage over alternatives like the simpler convex adjustment costs model—but also because it enriches the description of the dynamic behavior of aggregate durable purchases.

Lumpy behavior arises naturally when purchasing a durable good entails a fixed transaction cost (see, e.g., Bather 1966; Harrison, Sellke, and Taylor 1983; Dixit 1989; Bertola and Caballero 1990*b*; Grossman and Laroque 1990). I refer the reader to any of these papers—especially Grossman and Laroque's—for a complete description of the microeconomic problem; here I outline a simple model of this nature, stressing only the elements that will play an important role at different stages of the aggregation and empirical implementation of the model. Appendix A contains the basic equations underlying the motivation of the microeconomic behavior.

Let the economy be inhabited by a large number of individuals, approximated by a continuum of them and indexed by the subscript $i \in [0, 1]$. Each individual's desired stock of durables is described by[4]

$$dk_{it}^* = dA_t + \sigma_I dW_{it} \tag{3a}$$

and

$$dA_t = \theta dt + \sigma_A dW_{At}, \tag{3b}$$

where W_{it} and W_{At} are independent standard Brownian motions (also independent of the Brownian motions of all other units), and A_t denotes the common stochastic component across units (the "aggregate"). The latter is the only link across units I consider in this paper. In particular, I do not model general equilibrium constraints explicitly, which are a high-priority but very complex problem left for future research; I return to this issue briefly in Section IV.

If there were no transaction costs, $dk_{it} = dk_{it}^*$ for all t; however, as

[4] The driving variables are made explicit in the empirical section.

in Grossman and Laroque (1990), I assume that upgrading and downgrading a durable good, by more than the natural depreciation of the stock, require selling the old good and buying the new desired amount. Such a transaction has a "waste" (fixed cost) equal to a fraction λ of the old stock. Obviously, continuous adjustment of the actual stock to the target one cannot be optimal since it would entail infinite transaction costs. Under certain restrictions, the optimal microeconomic policy takes a simple three-barrier form: Most of the time, an individual i does not match changes in its target stock k_{it}^* but lets the actual stock k_{it} be eroded by depreciation: $dk_{it} = -\delta dt$, where δ is the depreciation rate. If, however, the departure z_{it} becomes too large in absolute value, reaching an upper level U (the stock is too large) or lower level L (the stock is too small), abrupt action takes place that brings the departure variable z_{it} to the level C, with $U > C > L$. Figure 1 presents an example of a sample path of z_{it} under the barrier policy described above (also see, e.g., Harrison et al. 1983).

Since the definition of the departure variable is $z_{it} \equiv k_{it} - k_{it}^*$, it is possible to see that when no action is taken, $dz_{it} = -\delta dt - dk_{it}^*$. Thus during the inaction periods, k_{it}^* and z_{it} have a negative contemporaneous correlation, satisfying the smoothness requirement discussed above. Stopping here and disregarding the sporadic large response of z_{it} to changes in k_{it}^* can be highly misleading, however. This is particularly true when consumers are not all in the same position of

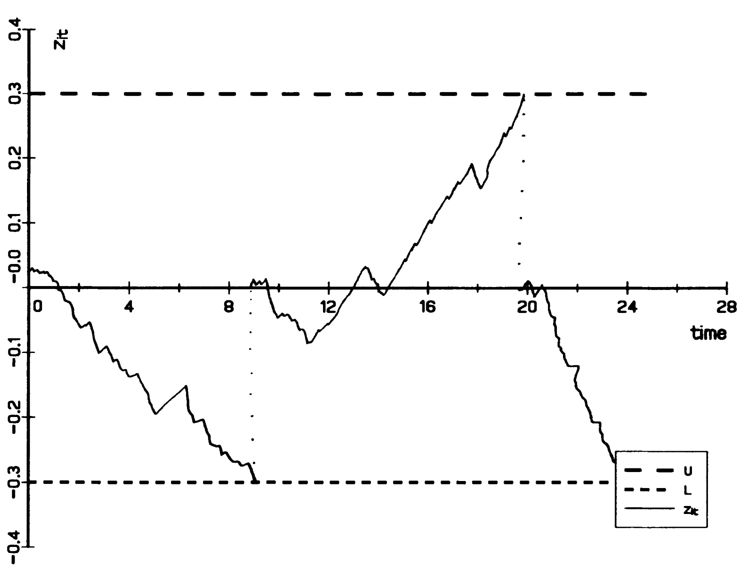

FIG. 1.—Sample path of z_{it}

their state space (i.e., do not have the same z_{it}), since in this case the large positive (i.e., in the same direction of the shock) reaction of some may counteract the negative correlation because of the vast majority.[5] It is at this point that we need to turn to the aggregation problem.

Let $f(z, t)$ be the *cross-sectional density* of the z_i's at time t, and denote its mean—that is, the average of the departures of individual consumers' actual stock of durables from their desired level—by Z_t:

$$Z_t \equiv \int_L^U z f(z, t)\, dz.$$

Taking K_t^* or its generating mechanism as given—as I do here—reduces the problem of describing the behavior of aggregate durable purchases to one of describing the dynamic behavior of Z_t. But since the band policies followed by consumers are fixed, changes in the average of the departures of individual consumers' actual stock of durables from their desired level are entirely due to changes in $f(z, t)$:[6]

$$dZ_t = \int_L^U z\, df(z, t)\, dz; \tag{4}$$

thus the problem has been reduced to describing the path of $f(z, t)$, which I begin doing in proposition 1 below.

PROPOSITION 1. Let individual units follow similar (L, C, U) policies, uncertainty be characterized by equations (3a) and (3b), and $\sigma^2 \equiv \sigma_A^2 + \sigma_I^2$, $\sigma_I > 0$, $\sigma_A \geq 0$. Let $f(z, 0)$ be a given initial cross-sectional density satisfying the boundary conditions below. Then the path of the cross-sectional density of the z_i's satisfying $f(z, t) \geq 0$ and $\int_L^U f(z, t)\, dz = 1$, for all $t \geq 0$, is the solution to the partial differential equation

$$df(z, t) = \left[(\theta + \delta)\frac{\partial f(z, t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z, t)}{\partial z^2}\right] dt + \sigma_A \frac{\partial f(z, t)}{\partial z}\, dW_{At},$$

[5] An extreme example of this is given in Caplin and Spulber (1987), where units are arranged in the state space so that the negative correlation of the many is exactly offset by those that respond by a large amount, eliminating the impact of microeconomic frictions on aggregate dynamics. Caballero (1992) discusses other dynamic fallacies arising from the direct application of microeconomic arguments to aggregate phenomena in these models.

[6] Note that if bands are not constant over time but change smoothly, then eq. (4) still holds since the new terms are $U'(t) f(U, t)$ and $L'(t) f(L, t)$, but $f(U, t) = f(L, t) = 0$ (see proposition 1). Thus any first-order effect of changes in band width on Z_t still comes through their effect on $f(z, t)$.

for $\{z \in (L, C) \cup (C, U)\}$, with boundary conditions

$$f(U, t) = f(L, t) = 0,$$

$$\frac{\partial f(C, t)^+}{\partial z} - \frac{\partial f(C, t)^-}{\partial z} = \frac{\partial f(U, t)^-}{\partial z} - \frac{\partial f(L, t)^+}{\partial z},$$

and

$$f(C, t)^+ = f(C, t)^-.$$

Proof. See Appendix B.

If the stochastic term in the partial differential equation (PDE) were suppressed, proposition 1 would describe the problem whose solution is the path of the conditional (on the initial distribution) density of a Brownian motion controlled according to the impulse control (L, C, U) policy described above (see, e.g., Feller 1954). That is, if shocks across individuals were uncorrelated ($\sigma_A = 0$), one could describe the path of the cross-sectional density of deviations by the path of the probability density describing the location of the individual deviation of a randomly selected consumer (with initial probability density $f(z, 0)$); this is just an application of the Glivenko-Cantelli theorem (see, e.g., Billingsley 1986). By extension, it is straightforward to show that in this case (i.e., when there is no aggregate uncertainty) the cross-sectional density eventually converges to a unique piecewise exponential density, $f(z)$, regardless of initial conditions (see, e.g., App. B). I shall refer to this limit as the "steady-state" density. In this limit, $dZ = 0$, implying that the aggregate dynamic behavior of an economy with microeconomic adjustment costs is indistinguishable from the behavior of an economy without adjustment costs.[7]

Restoring aggregate uncertainty complicates the problem. In order to go from probabilistic statements at the individual level to statements about the cross-sectional density, one must first remove the correlation of shocks across individuals. This can be done by conditioning on the path of aggregate shocks. Proposition 1 does precisely this; that is, it presents the problem whose solution is the path of the probability density describing the location of the individual deviation of a randomly selected consumer (with initial probability density $f(z, 0)$) conditional on the path of aggregate shocks. A (strong) law of large numbers argument similar to the one used above for the case without aggregate uncertainty can now be used to exchange conditional (on aggregate shocks) probability densities for cross-sectional densities.

[7] Of course, in this limit the frictionless case is not very interesting since it is fully deterministic and growing at a constant rate.

As long as there are aggregate shocks, the cross-sectional density never settles down to an invariant distribution. This can be understood by rewriting the stochastic PDE in proposition 1 as

$$df(z,t) = (dA_t + \delta dt)\frac{\partial f(z,t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z,t)}{\partial z^2}dt;$$

thus aggregate shocks prevent the cross-sectional density from ever settling into an invariant state by continuously "changing the drift" (note that the diffusion term dA_t replaces the drift term θdt of the case without aggregate shocks). This informal description constitutes the basis for the empirical implementation of the model later in the paper.

Although the cross-sectional density never settles down, the same forces that yield convergence in the case without aggregate shocks are continuously in play. Aggregate shocks are innovations that continuously perturb the density away from its resting state, $f(z)$; however, the latter is always "pulling" the cross-sectional density toward it.[8] In general, a larger drift and total variance relative to aggregate uncertainty imply a stronger effect of the deterministic part of the PDE and therefore a smaller and less persistent impact of aggregate shocks on the cross-sectional distribution; this in turn implies that Z will move less to offset fluctuations in K^*. Henceforth I shall use the expression "steady-state density" to refer to the limit (as t goes to infinity) of $f(z, t)$ when the stochastic term of the PDE in proposition 1 is suppressed.

The main reason to study the evolution of the cross-sectional density of deviations is not the density itself but its first moment, Z_t, which summarizes the difference between frictionless and actual aggregate paths of the stock of durables. Proposition 2 below shows that the path of this moment can be conveniently expressed in terms of the flow of consumers upgrading and downgrading their stocks.

PROPOSITION 2. Under the assumptions of proposition 1,

$$dZ_t = -\delta dt - dK_t^* + \frac{\sigma^2}{2}[f_z(L,t)^+(C-L) + f_z(U,t)^-(U-C)]dt.$$

Proof. See Appendix B.

Proposition 2 shows that the dynamic behavior of this complex economy can be characterized in terms of the slopes of the cross-sectional density at each point in time, evaluated at the boundaries. To understand this, suppose, for a moment, that no consumer up-

[8] This is the same as in any stationary process that tends to return to its unconditional mean but is continuously prevented from doing so by innovations. In a previous version of this paper I referred to this effect as the "attractor" effect.

grades or downgrades (by more than the depreciation) his stock of durables in the time interval dt; then $dZ_t = -\delta dt - dK_t^*$ and the actual shock is driven only by depreciation, $dK_t = -\delta dt$. Next, consider the case in which some units upgrade their stocks; then dK_t declines by less than the depreciation, and this is reflected one for one in dZ_t. By how much less depends on the size of the increase in the stock of durables of those that decide to upgrade $(C - L)$ times the fraction of units that upgrade their stocks. The latter depends on the density in the "neighborhood" of L, captured by its (right) derivative at the boundary, $f_z(L, t)^+$, and by the total uncertainty faced by each unit, σ.[9] The parameter σ matters since, as it rises, L can be reached by units farther away within the time interval dt; hence the "neighborhood" is increasing with respect to σ. The arguments run in the same way, although with the opposite sign, when one considers the units that decide to downgrade their stocks.

Proposition 2 also shows that an innovation in K^* is fully offset, contemporaneously, by a change in Z_t. Over time, this is reversed by the purchases and sales flows, summarized by the behavior of the slopes of the cross-sectional density at the boundaries that respond to the (local) change in drift due to the innovation in K^*. When this continuous-time result is mapped into finite time intervals, the correlation between innovations in K^* and Z still remains negative but less (in absolute value) than one, as changes in the slopes at the boundaries become random variables (conditional on information available at the beginning of the time interval).[10] This smoothing, combined with the serial correlation in ΔZ_t due to the behavior of the slopes at the boundaries, yields the properties described in the Introduction for actual durable goods purchases. An example of this mechanism is presented a few paragraphs below, after obtaining an expression for the rate of growth of the aggregate stock of durables.

Proposition 3 below adds the path of the frictionless stock of durables to that of the first moment of the cross-sectional density to obtain an expression for the path of the aggregate stock of durables.

PROPOSITION 3. Under the assumptions of proposition 1 and letting $f(z)$ represent the steady-state density and $\dot{x} \equiv dx/dt$, we get

$$\dot{K}_t = \theta + \frac{\sigma^2}{2}\{[f_z(L, t)^+ - f_z(L)^+](C - L) + [f_z(U, t)^- - f_z(U)^-](U - C)\}$$

[9] Remember that the density is zero at L and U (trigger points); thus the first derivatives are the leading nonnegligible terms.

[10] In continuous time the slopes are locally predictable.

or

$$\dot{K}_t = -\delta + \frac{\sigma^2}{2}[f_z(L,t)^+(C - L) + f_z(U,t)^-(U - C)].$$

Proof. See Appendix B.

Proposition 3 shows that even though all sources of uncertainty in the economy are infinite variation Wiener processes, the path of the aggregate stock of durables has *finite variation.* In contrast to Brownian motion, the sample path of K_t is differentiable; the fluctuation in the cross-sectional distribution not only attenuates the volatility of the aggregate stock variable but also changes the fundamental stochastic nature of this stock.[11] The latter is not an empirically important consideration since actual sample paths are not observed continuously; however, it reflects the smoothing property of both, adjustment costs and aggregation.

Recalling that the cross-sectional density is equal to the conditional (on the aggregate path) density of an individual, we see that the nontrivial dynamic behavior of the stock of durables is described by the difference between the slopes at the boundaries of the conditional and the unconditional densities, weighted by the size of the upward and downward jumps. Figure 2 provides an example. The solid curve represents the steady-state density and the solid lines its slopes at the boundaries. The skewness of this density is due to the negative drift in the process for individual deviations, $-(\theta + \delta)$; in terms of the flows, the drift is reflected in a steeper slope at L than at U. Suppose now that the economy starts from a depressed situation, where $\dot{K}_t$ is less than its steady-state value, θ: this corresponds to $f_z(L, t)^+ < f_z(L)^+$ and $f_z(U, t)^- < f_z(U)^-$ in the figure. If good or normal times follow, initially few units replenish their durables; thus the rate of growth of durables is still below its long-run average. However, the slopes start increasing as more units approach the neighborhood of L and fewer remain in the neighborhood of U. Eventually, if the upward pressure continues, the rate of growth of the stock of durables exceeds its long-run average; at this point the large net upgrading flow puts pressure for the density to return to its steady-state value.

Since the paper is mostly aimed at studying the features arising from aggregation of microeconomic discontinuous adjustment, the discussion up to now has taken as given the band policy followed by individual consumers. However, with the empirical section in mind, it is worth discussing briefly the aggregate consequences of a wider

[11] This does not mean that the path of the actual stock of durables is deterministic since the slopes at the boundaries are random variables; it just says that the stock of durables is locally predictable.

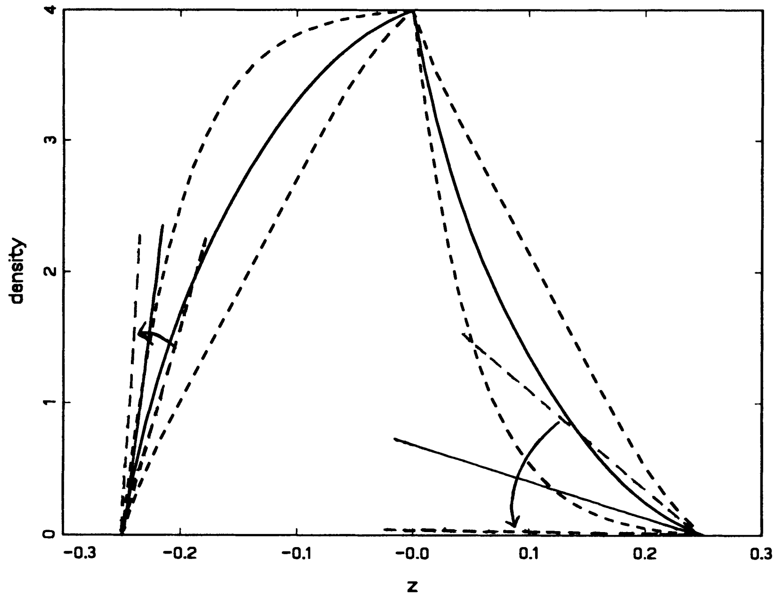

FIG. 2.—Cross-sectional densities

inaction range for given uncertainty and drift parameters.[12] As the inaction range widens, the fraction of units near the barriers at each point in time decreases. Since these are the units responsible for offsetting the negative correlation between K^* and Z, the enlargement of the inaction range raises the excess smoothness of the actual stock of durables for a given path of K^*. The counterpart of this is an increase in the (positive) serial correlation of the stock of durables and a further reduction (in absolute value) of the MA(1) coefficient in the expenditure (its changes) series.[13]

Figure 3*a* illustrates this: Using the model described in Appendices A and B and in the next section, I generate a sample path for the aggregate shocks A_t and plot the path of ΔK_t under three different values of the width on the inaction range, indexed by the adjustment cost parameter λ (.00, .05, and .25). Other parameters are $\sigma_A = .05$, $\sigma = .15$, and $\delta = .05$. The figure confirms the previous conclusion: as the inaction band widens (larger λ's), the path of the actual stock of durables becomes more sluggish, reflecting the delayed dynamic

[12] For example, this may be due to larger transaction costs or the concavity of the flow cost function. See App. A.

[13] For these effects to occur, it is important that the reduction in number of units near the bands be larger than the implicit increase in the size of the jumps. This is generally the case, although an important exception is the one-sided (S, s) model.

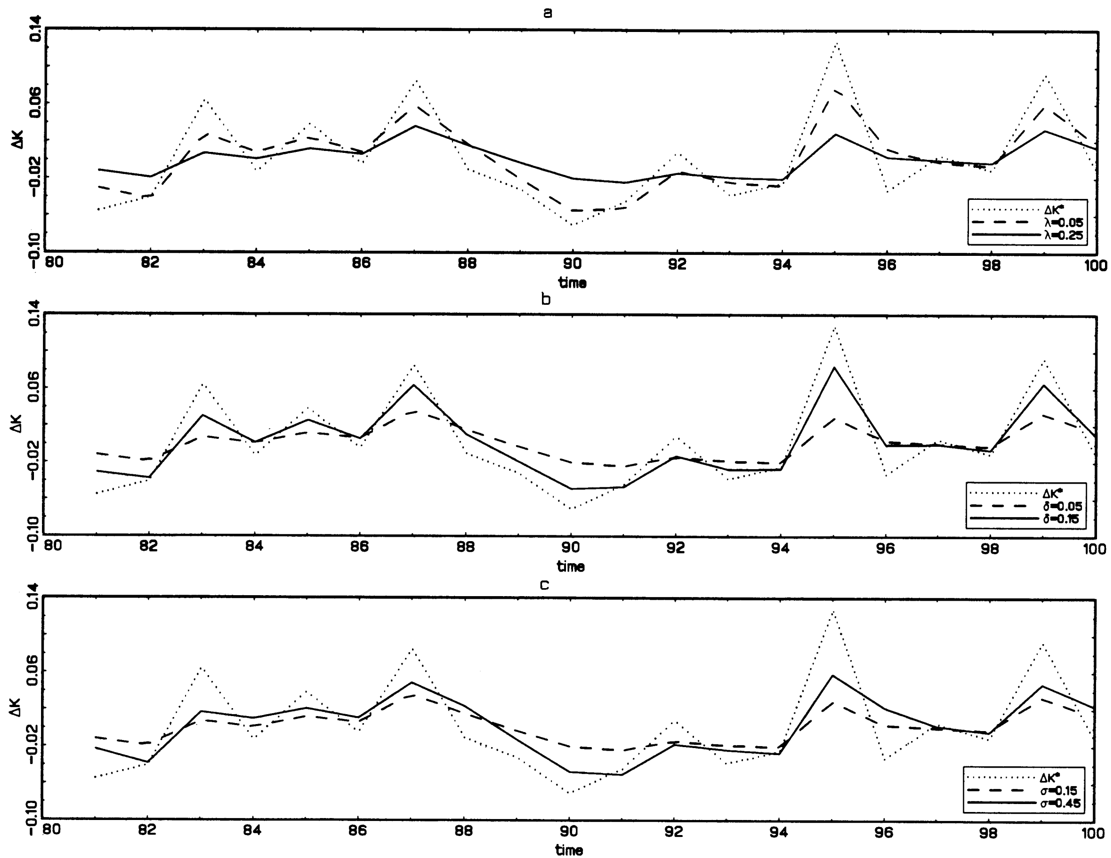

FIG. 3.—Simulated sample paths

TABLE 1

EXAMPLE'S STATISTICS

	λ = .00	λ = .05	λ = .25
$\sigma_{\Delta K}$	.023	.022	.013
$\rho_{\Delta K}$	.037	.091	.305

NOTE.—The corresponding values for $U - L$ are .00, .54, and .98, respectively.

response due to the endogenous microeconomic lumpiness. Table 1 reports several statistics arising from this example; they reveal the smoothing and serial correlation implications of an increase in the adjustment cost parameter.

Figures 3*b* and 3*c* show the impact of the same sample path of aggregate shocks for different drifts (indexed by different depreciation rates) and idiosyncratic uncertainty, respectively, confirming the previous discussion: an increase of either of these reduces aggregate sluggishness.[14] Other parameters are $\sigma_A = .05$, $\sigma = .15$, and $\lambda = .15$ (fig. 3*b*) and $\sigma_A = .05$, $\delta = .05$, and $\lambda = .15$ (fig. 3*c*).

II. Empirical Evidence

Figure 3 above, the evidence in Bertola and Caballero (1990*b*), and the simulations in Lam (1989) show that the type of models discussed here has the potential to account for the sluggishness of the aggregate stock of durables. Besides implementing a continuous-time version of aggregate (S, s) models—which offers empirical advantages discussed later in the paper—in this section I study whether these models can explain subtler characteristics of the evolution of aggregate durables purchases and stocks. Using the model and effects described above, I attempt to explain the differences in the dynamic behavior of different goods within consumer durables (cars and furniture) and across time (within cars).[15] The results provide further support for the view that principal aspects of aggregate stock dynamics are generated by the lumpy state-dependent nature of microeconomic purchases.

[14] The effect of idiosyncratic uncertainty on sluggishness is nonmonotonic. There are two effects that counteract the one stressed in the text: First, consumers widen their bands to reduce the impact of uncertainty on the frequency with which bands are reached. Second, when total unit level uncertainty is low relative to the drift, there is a "beating the drift" effect. That is, given aggregate uncertainty, an increase in idiosyncratic uncertainty may introduce more dynamics rather than less by letting the movements in k_i^* be nonmonotonic, moving the problem away from the Caplin and Spulber paradox.

[15] Cars correspond to new cars, and furniture to furniture and furnishings as described in Citibase for the period 1959:1–1990:1.

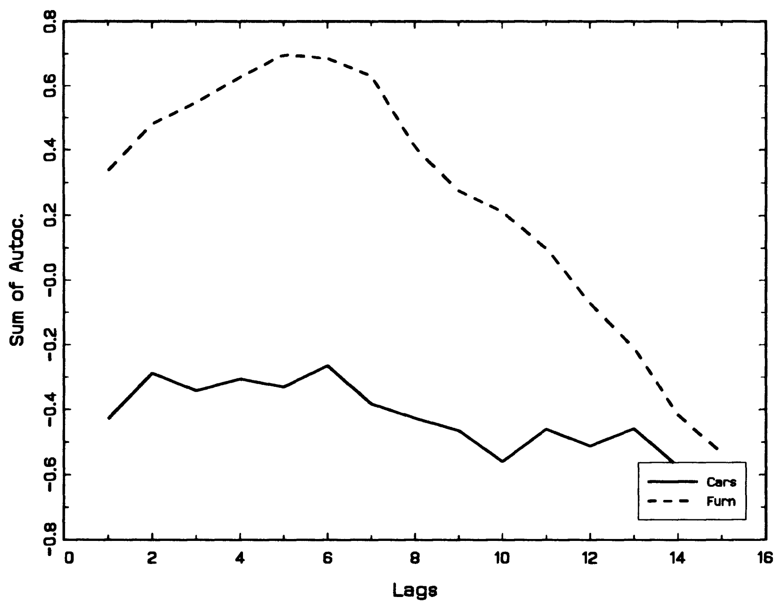

FIG. 4.—Sum of autocorrelations (1959:1–1990:1)

A. Diagnostic

I start the diagnostic of the data by estimating MA(1) models for both goods. Under this metric, cars are clearly closer to the PIH than furniture. For the period 1959:1–1990:1, cars have a moving average coefficient of −.39 and furniture has a positive moving average coefficient equal to .28, both of them significant.

A problem of the simple MA(1) comparison is that all cases fail to match the PIH (where the moving average coefficient should be approximately between −.95 and −1.00 under the simplest frictionless PIH model); thus it is possible that even though the initial response of cars is faster than that of furniture, it may be overturned later along the adjustment process. To see whether this is the case, I look at the plot of the sum of the autocorrelations of both series, having in mind that under the simple PIH model they should be flat and at about −.5. Figure 4 shows that the sum of the autocorrelations for cars is consistently smaller, and closer to −.5, than that of furniture.[16] It is interesting to note, however, that both sums of autocorrelations eventually settle at about −.5, providing further support for

[16] Similarly, the spectra show that the lower frequencies are more important in furniture than in cars.

TABLE 2

Sample Instability: Cars

	1959:1–1972:4	1973:1–1979:4	1980:1–1990:1
MA(1)	−.447 (.124)	−.147 (.195)	−.571 (.133)

Note.—Standard errors are in parentheses.

the idea that the PIH is the right way to think about the medium long run (Caballero 1990*b*).

As discussed above, the dynamic behavior of the aggregate depends in intricate ways on parameters like the amount and composition (idiosyncratic vs. common) of uncertainty. Whatever the sign of these effects, one would not expect them to be constant during the 1970s, especially for cars for which the price of gasoline is an important determinant of its user cost. Table 2 presents MA(1) coefficients for cars in three sample periods, 1959:1–1972:4, 1973:1–1979:4, and 1980:1–1990:1. The χ^2 tests for the difference in the moving average coefficients between the first and second periods, second and third periods, and second and all periods are 2.7, 4.7, and 5.3, respectively. They are all significant at the 10 percent level.[17] Therefore, in spite of the low power due to the few observations in each period (especially the middle one), there is suggestive evidence pointing in the direction of a larger departure from the PIH during the 1970s.

In sum, furniture departs more from the PIH than cars, and within cars the departure is more important during the 1970s. In the next subsections I provide a structural interpretation of these findings.

B. Estimation Strategy

The empirical implementation of the model developed in Section I and the Appendices requires—at least conceptually—two distinct steps: First, the frictionless model and the realizations of the aggregate shocks are estimated. Second, the dynamic component of the model and its parameters are identified.

I initially estimate the target stock of each good by exploiting the cointegrating properties between the observed stock and the desired one.[18] For this, I postulate that the desired stock of each good follows

[17] All the tests have free constants.

[18] A similar procedure is used in Bertola and Caballero (1990*a*, 1990*b*). Caballero (1990*a*) discusses this approach in a more general context.

a simple PIH type relation, enriched by (long-run) price effects:

$$K_t^* = \beta_0 + \beta_1 H_t + \beta_2 P_t + \beta_3 R_t,$$

where H_t is the log of wealth,[19] P_t is the log of the ratio of the price of the durable good in question and the price of nondurables, and R_t is a user cost index equal to the log of the price of gasoline over the price of nondurables in cars, and equal to zero in furniture.[20] The coefficients β_1, β_2, and β_3 are determined from the cointegrating relationship $K_t = K_t^* + Z_t$.[21]

There are two important caveats worth mentioning at this point. First, models with important dynamic elements due to adjustment costs are prone to a strong bias of their coefficients toward zero when estimated with static ordinary least squares procedures (Caballero 1990*a*); a useful procedure to reduce this problem is the Stock and Watson (1989) dynamic ordinary least squares approach. In implementing their method, I include in the right-hand side four leads and 20 lags of the first difference of the variables driving K^*. Second, the specification of K_t^* includes long-run relative price variables but does not include the real interest rate in terms of durables. The technical reason for this is that since this variable is stationary, its impact on K_t^* cannot be assessed using a cointegrating regression. If in spite of this the durables' real interest rate is included in the regressions, it comes out insignificant in both cars and furniture (the *t*-statistics are -0.15 and 1.3, respectively). Another, also imperfect, alternative is to estimate the interest rate coefficient in the second stage. I report these results later in the paper. Anticipating the results of this experiment, I proceed in this section as though these transitory variables played a secondary role.

Table 3 reports the results together with the statistics of the variables of main concern. I have imposed the constraint $\beta_1 = 1$ so that the (frictionless) share of each stock remains constant if relative prices

[19] This is constructed as the expected present value of per capita disposable income, under the (nonrejected) assumption that its logarithm follows a random walk. The main results of the paper are unchanged by using nondurables consumption instead of H_t as the main driving variable. The motivation to include nondurables comes from the first-order conditions of the consumer problem. Following an earlier version of this paper, Beaulieu (1991) has pursued this insight further and shown that it can be derived from a model à la Grossman and Laroque (1990) expanded to consider nondurable goods. I have chosen the more ad hoc specification of the paper mainly because it is simpler and more robust to frictions affecting the nondurables consumption series itself.

[20] Throughout the paper I am adopting the assumption that the system is decoupled, i.e., that it can be solved separately across goods.

[21] Note that if adjustment costs are important, Z_t has a strong serial correlation. Therefore, there is no power to test whether K_t is cointegrated with the right-hand-side variables.

TABLE 3

Frictionless Model and Basic Statistics (1963:1–1989:1)

	Cars	Furniture
θ	.028	.027
$\sigma_{\Delta K}$	.012	.005
$\sigma_{\Delta K^*}$	.022	.020
$\sigma_{\Delta K}/\sigma_{\Delta K^*}$	.546	.259
$\rho_{\Delta K}$	.695	.894
	(.071)	(.039)
$\rho_{\Delta K^*}$	.175	.083
	(.098)	(.100)
β_1	1.000	1.000
β_2	−.283	−.346
	(.018)	(.020)
β_3	−.103	...
	(.041)	

Note.—All equations include a constant. Standard errors are in parentheses. The standard deviations and θ's are annualized using the relations $\sigma\sqrt{dt}$ and θdt (the same applies to the rest of the tables). The sample, 1963:1–1989:1, corresponds to the observations left after using four leads and 20 lags in the Stock and Watson procedure.

do not change. The rest of the coefficients are precisely estimated. Relative price effects are important for both goods, and the price of gasoline plays a significant role in explaining the path of the target stock of cars. The statistics are also quite interesting. Consistent with the arguments presented up to now, they indicate that excess smoothness is more important in furniture than in cars: a ratio of the standard deviation of the actual stock growth series versus the target series of .26 for furniture versus .55 for cars. Furthermore, the positive serial correlation of the rate of growth of the stock of furniture is stronger than the one for cars (.89 vs. .70), which is the serial correlation that counteracts the negative MA(1) coefficient in the series of changes in expenditures. It is also interesting to notice that in spite of the significant price effects, changes in the frictionless stocks do not display significant serial correlation.[22]

These estimates, together with an exogenous depreciation rate of 15 percent per year for both goods, yield estimates of the discrete time analogue of the terms $(\theta + \delta)dt$ and $\{\sigma_A dW_{At}\}$ in proposition 1's Kolmogorov equation (see App. B).

The next step in estimating the path of Z_t is to find an initial cross-

[22] This provides some support for using a Brownian motion as an approximation of the true driving process.

sectional density, which I arbitrarily choose to be the steady-state density, $f(z)$,[23] the variance of idiosyncratic shocks, and the parameters L, U, and C. Three caveats concerning what can be inferred from aggregate data alone are in order, however: First, given the free constant in the equation for the desired durables' stock (β_0), the location of the cross-sectional density is not identified; thus I set $C = 0$. Second, the asymmetry in the jumps at the microeconomic level is not likely to have strong implications for aggregate time series, at least when looked at with a model that is driven by smooth processes (Caballero 1992). Given that imposing symmetry simplifies the numerical routine for tracking down the path of the cross-sectional density, I impose it.[24] Third, for technical reasons, identifying the idiosyncratic uncertainty faced by consumers is difficult in the range of parameters obtained. As a result, I first estimate the model with both U and σ_I free but then pick the combination of parameters within the flat region of the objective function (see below) with the lowest value of σ_I.[25] I return to this issue later when presenting the results.

Feeding the estimated aggregate shocks into the PDE of proposition 1 (see App. B for the connection between discrete data and the continuous nature of the setup), computing the derivatives of the estimated density at the boundaries, and using proposition 3, I compute an estimate (hats denote estimates) of the path of the rate of growth of the actual stock of durables for given U and σ_I:

$$\widehat{\Delta K}_t(U, \sigma_I) \equiv \int_{t-\Delta t}^{t} \hat{\dot{K}}_s(U, \sigma_I)\, ds.$$

I then search over U and σ_I to minimize the sample variance of the residuals:

$$\min_{U,\sigma_I} \frac{1}{N} \sum_t (e_t - \bar{e})^2,$$

where N is the number of observations,

$$e_t \equiv \Delta K_t - \widehat{\Delta K}_t(U, \sigma_I),$$

and $\bar{e}$ is the sample mean of the residuals.

[23] The first 10 observations are excluded from the sum of squared residuals to reduce the impact of this arbitrary selection.

[24] The cost of this is that now U represents both the size of the jumps and one-half the inaction range.

[25] "Flat" is defined as no change (up to two digits) in the R^2 of the model after changing idiosyncratic uncertainty by 5 percent. The reason why the objective function becomes flat is that once U is large (as is the case here), the model becomes almost linear with respect to aggregate shocks. In the limit, when the model is linear, only U/σ_I is identified.

TABLE 4

STRUCTURAL ESTIMATES

	CARS			
	1963:1–1989:1	1973:1–1979:4	1963:1–1972:4, 1980:1–1989:1	FURNITURE: 1963:1–1989:1
U	.83	.87	.91	2.53
	[.72]	[.77]	[.80]	[2.15]
	{.92}	{.92}	{1.01}	{2.92}
θ	.03	.01	.03	.03
σ_A	.02	.03	.02	.02
σ_I	.30	.30	.35	.50
	[.25]	[.25]	[.30]	[.45]
	{.35}	{.35}	{.40}	{.55}
σ_A/σ	.07	.08	.06	.04
	[.09]	[.10]	[.07]	[.04]
	{.06}	{.07}	{.05}	{.04}
$E[T]$	4.32	4.86	4.35	13.5
	[3.91]	[4.56]	[4.04]	[14.6]
	{4.50}	{4.71}	{4.54}	{15.4}
$R^2_{\Delta Z}$	.80	.90	.75	.94
	[.79]	[.89]	[.74]	[.93]
	{.80}	{.90}	{.75}	{.94}

NOTE.—All equations include a constant. Results in brackets and braces represent the values for the cases σ_I − .05 and σ_I + .05, respectively, with respect to the base case. All estimates use the full sample to generate the path of the cross-sectional density. Five Fourier coefficients were used (see App. B).

C. *Results: Cars*

Column 1 in table 4 provides the full sample results for cars. The main results correspond to the case in which good-specific idiosyncratic uncertainty at the individual level is 30 percent per year.[26] Once idiosyncratic uncertainty is fixed, the estimates of the inaction range are very precise; consequently, instead of reporting standard errors, I have chosen to report the impact of changing σ_I by 5 percent. Hence, the numbers in brackets and braces in column 1 correspond to the results in which σ_I equals 25 percent and 35 percent, respectively.

The estimate of U (or L)—if interpreted as the size of the control jumps—implies that the (service) value of the car when upgraded is about 2.3 times the value of the old car and that, on average, individuals change cars about 4.3 years after the previous purchase. The

[26] It is important to notice that this is the uncertainty as seen by the econometrician. Individuals could, and are likely to, be able to foresee many things that appear as idiosyncratic shocks for an outside observer. Moreover, structural differences, as generated by heterogeneous wealth and price elasticities of the frictionless stocks across individuals, are also likely to have implications very similar to those of idiosyncratic shocks (Caballero and Engel 1991). Finally, taste shocks also form part of idiosyncratic shocks, and they may be quite large for individual consumption goods.

expected time to hit any of the barriers starting from the target point is given by (see, e.g., Harrison 1985, p. 52, problem 12)

$$E[T] = -\frac{U}{\theta + \delta}\left[1 - 2\frac{1 - e^{-2(\theta+\delta)U/\sigma^2}}{1 - e^{-4(\theta+\delta)U/\sigma^2}}\right],$$

where T is a hitting time.

The estimate of σ_A/σ indicates that aggregate uncertainty accounts for about 7.5 percent of total uncertainty.

Overall, the fit of the model is good: it accounts for 90 percent of the departure in the rate of growth of actual and frictionless stocks. The reduction in the R^2 when σ_I is lowered and the negligible change on it when σ is raised by 5 percent reflect the two-stage estimation strategy discussed above (looking for the minimum σ_I within the range of [almost] flat likelihood).

Columns 2 and 3 split the estimates between the period 1973:1–1979:4 and the periods 1963:1–1972:4 and 1980:1–1989:1. In both cases the entire sample was used to generate the path of the cross-sectional density and its slopes at the boundaries; the only difference between the two cases lies in the residuals that were included in the objective function. The following findings emerge from comparing these two columns: First, total uncertainty is larger during the non-1970s. Second, aggregate uncertainty and its relative importance in total uncertainty are larger during the 1970s. Third, the drift is substantially smaller during the 1970s.[27] Fourth, the bands are slightly narrower during the 1970s.

As discussed in the previous section, all these features but the last one point in the direction of making aggregate durable purchases more sluggish during the 1970s,[28] which is consistent with the evidence reported in table 2.

Figure 5 decomposes the rate of change of the stock into the units that upgrade their stocks and those that downgrade theirs plus depreciation. It is apparent that given the strong drift, the model implies that units seldom exercise control to reduce their stocks, which also seems realistic.

Thus far I have disregarded the effect of transitory components on K^*. Suppose that instead of an estimate of Z_t, the residual from the first stage is an estimate of $Z_t + x_t\alpha$. In particular, let x_t be the

[27] Moreover, a sequence of negative shocks starting from a cross-sectional density that is very bowed toward the upgrading barrier exacerbates the sluggishness of durable purchases. This effect goes beyond the pure "long-run" drift mechanism discussed in the previous section.

[28] From the row that shows that the expected time to hit any of the barriers starting from the target point rises during the 1970s, we know that the former effects dominate the band-widening effect.

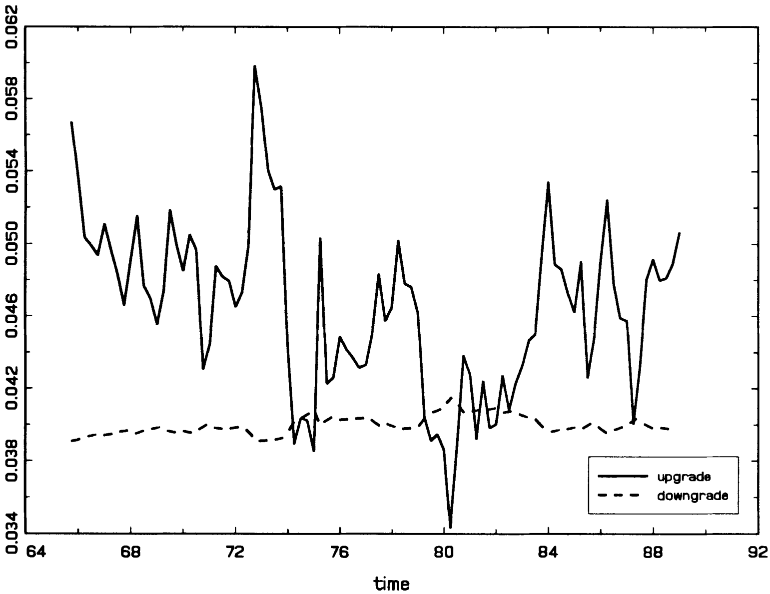

FIG. 5.—Stock upgrades and downgrades (cars)

ex ante real interest rate in terms of durables.[29] I rerun the second stage under these new assumptions: the estimate of α is equal to $-.005$ and insignificant. More important, the estimate of the inaction range remains at .83 for the whole sample.[30]

D. *Results: Furniture*

Column 4 of table 4 reports the results for furniture. The model explains 97 percent of the departure of the path of the actual growth series from the path implied by the PIH model with no dynamics. The estimates of U and of total uncertainty are much larger than those for cars; both combined give a substantially larger expected hitting time. This suggests that transaction costs, perhaps reflecting the less developed secondary markets, are larger in furniture than in cars.[31]

[29] This corresponds to the 3-month Treasury bill rate minus expected inflation (conditional on inflation lags) of the corresponding durable.

[30] This procedure only cleans x_t from the residual of the first stage; it does not correct the estimates of K_t^*. Doing so would induce a strong bias due to a problem similar to the small sample problem of the (uncorrected) first stage.

[31] Aggregation across goods brings about issues similar to those of aggregation across individuals; thus the path of the composite furniture good for each individual is likely to have smoother behavior than that of each individual good.

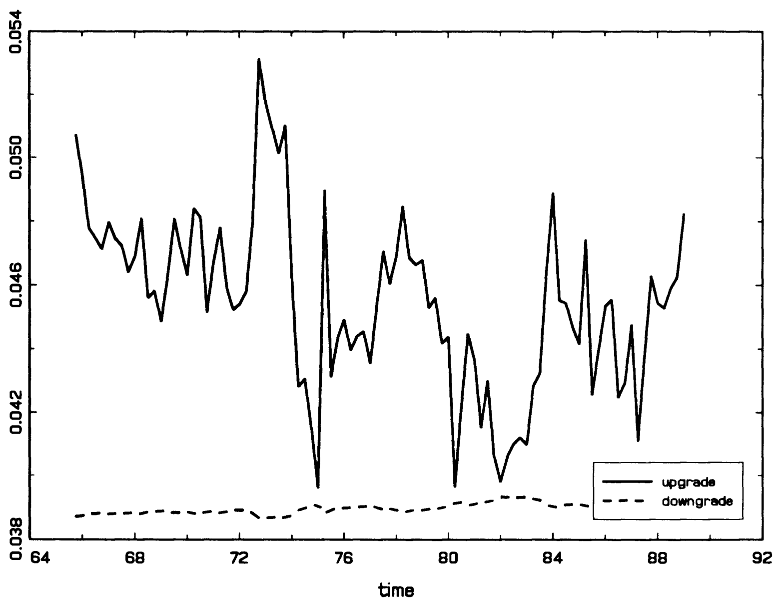

FIG. 6.—Stock upgrades and downgrades (furniture)

Figure 6 illustrates the decomposition of the rate of change of the stock into the units that upgrade their stocks and those that downgrade theirs plus depreciation. That the source of fluctuations comes from consumers' stock upgrades rather than downgrades is even more apparent than in cars, and this is consistent with the idea of a less developed secondary market for furniture than for cars.

Finally, I rerun the second stage including the real interest rate measure with results similar to those obtained for cars. The estimate of α is equal to $-.058$ and insignificant. More important, the estimate of the inaction range remains unchanged.

III. ARMA Representation and Impulse Responses

The model presented in this paper is nonlinear. The response of durable purchases to aggregate shocks depends on past history in intricate ways; the previous section provides this "historical filter." In this section, on the other hand, I approximate the generating mechanism of durable purchases by low-order autoregressive moving average (ARMA) models. This eases the comparison with previous results.

Since the rate of growth of the actual stocks is equal to the rate of

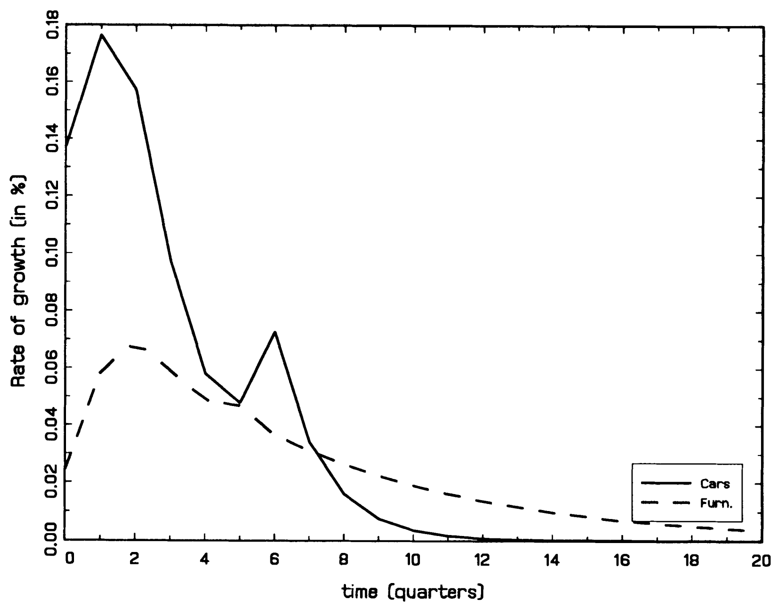

FIG. 7.—Impulse responses (1 percent long-run change)

growth of the frictionless stocks plus the change in the mean of the cross-sectional density and the latter is a function of the current and past rates of growth of the frictionless stocks, one may construct an approximate (average) impulse response function in which the innovations to the frictionless stock are the impulses. For this I run nonparsimonious regressions of ΔK on its own lag and the current and six lags of ΔK^*. Figure 7 portrays the response of the actual and frictionless rate of growth of cars and furniture to an impulse yielding a 1 percent long-run increase in the stock (approximately the area under the curves). The shapes are broadly consistent with the description given in the paper.

IV. Concluding Remarks

In this paper I have shown how lumpiness at the microeconomic level can aid in explaining different features of the time-series behavior of durable goods. For the sake of clarity, factors (other than prices) introducing serial correlation in the frictionless stock of capital growth series have been excluded. Allowing for other realistic features like habit formation (e.g., Constantinides 1990; Heaton 1991) and nonseparabilities across goods and time (e.g., Eichenbaum and Hansen 1987; Heaton 1991) should enrich the characterization of the target stock, K^*, and reduce the need for large inaction range estimates (especially in furniture) to account for the large departure of durables from the simplest PIH.

An important limitation of the approach followed here is the absence of an explicit description of how and whether markets clear. In the paper, at each point in time there is a large number of consumers wishing to sell to upgrade and a smaller number of consumers wishing to downgrade. The net of these two is the demand for new goods. Prices of new goods are taken as exogenous. Meanwhile, secondary markets must clear a large number of buy and sell orders for goods of different age (or size); modeling these transactions explicitly is an important issue left unexplored. Despite this, I believe that the continuous-time representation of the problem in propositions 1, 2, and 3 will facilitate this task. In particular, the simple expressions for gross flows in terms of slopes of the cross-sectional density should be a starting point for price determination in models that combine heterogeneous agents and fixed costs.[32] These expressions should also facilitate work on other areas in which gross flows play a central role; labor markets are a natural example.

The model presented here exhibits history dependence and nonlinearities, and it yields a framework to interpret the ARMA representations of durables and their sample instability. I have found sharp differences across goods and times, which I have tentatively attributed to the degree of development of secondary markets and changes in the underlying stochastic processes. Further research on the nature and extent of transaction costs is much needed. Microeconomic evidence and case studies should provide a natural complement to this work. The papers by Eberly (1990) and Beaulieu (1991) take important steps along these lines; their results provide strong microeconomic support for the type of models discussed in this paper.

Appendix A

The Individual Consumer Problem

Let each individual i have a stock (level) of durables D_{it} and a strictly positive desired (or target) stock D^*_{it}.[33] In terms of dollars, the flow utility cost of departing from the target stock level is summarized by the expression

[32] For example, a model with the condition that the net flow is zero must search for prices such that the slope at the upgrading boundary is equal to (minus) the slope at the downgrading boundary.

[33] Grossman and Laroque (1990) develop a model of portfolio choice and durable purchases in the presence of transaction costs. Since the only objective of the microeconomic model in this paper is to motivate a general type of inaction policies used, here I develop a less realistic but simpler version of their model.

$aD_{it}[\ln(D_{it}/D^*_{it})]^2$, $a > 0$; that is, the cost is quadratic in the percentage departure and is scaled by the level of the durable held by the individual. The transaction cost incurred when changing the durable at any given time τ is proportional to the old durable sold: $\lambda D_{i\tau^-}$. At any normalized time 0, the consumer's problem is to minimize the present value cost:

$$V(D_{i0}, D^*_{i0}) = \inf_{\tau, D_{i\tau}} E_0\left[a\int_0^\tau e^{-\rho t}D_{it}\left[\ln\left(\frac{D_{it}}{D^*_{it}}\right)\right]^2 dt \right.$$
$$\left. + \lambda D_{i\tau^-}e^{-\rho\tau} + e^{-\rho\tau}V(D_{i\tau}, D^*_{i\tau})\right],$$

where ρ is the discount rate and τ is the first stopping time.[34]

Letting $z_{it} \equiv \ln(D_{it}/D^*_{it})$ $(\equiv k_{it} - k^*_{it})$, one can write $V(D_{it}, D^*_{it})$ as $D_{it}J(z_{it})$. If one assumes that the durable depreciates exponentially at the rate δ and defines $M \equiv \inf_z e^z J(z)$, the problem can be written in terms of a single state variable:

$$J(z_{i0}) = \inf_\tau E_0\left[a\int_0^\tau e^{-\eta t}z^2_{it}dt + \lambda e^{-\eta\tau} + e^{-\eta\tau}e^{-z_{i\tau^-}}M\right],$$

where $\eta \equiv \rho + \delta$.

If the target stock level, D^*_{it}, is assumed to follow a geometric Brownian motion with $d\ln D^*_{it} = \theta dt + \sigma dW_{it}$, so $dz_{it} = \mu dt - \sigma dW_{it}$, with $\mu \equiv -(\delta + \theta)$, it is easy to verify that the optimal policy consists of lower and upper trigger points, denoted by L and U, respectively, and a common target point denoted by C (see, e.g., Harrison et al. 1983; Grossman and Laroque 1990; Bertola and Caballero 1990*b*), with $L < C < U$.[35] It is well known that the inaction range is generally increasing with respect to λ, σ, and $1/a$.

Appendix B

Aggregation

The final objective of the paper is to characterize the path of the rate of growth of the aggregate stock of durables (or subcategory of durables). In order to reconcile this with the variable denoted dK_t in the text—defined as the average rate of growth of individual stocks—one must assume that the rate of growth of individual purchases is independent of the share of individuals on the initial aggregate stock. The same must be true of the frictionless stocks.[36] Once this has been done, the aggregation problem consists in characterizing equation (4), which depends on the path of $f(z, t)$.

[34] Note that $D_{it} > 0$ since $\lim_{D_{it}\to 0} D_{it}[\ln(D_{it}/D^*_{it})]^2 = \infty$.

[35] See earlier drafts of this paper for the corresponding differential equation, value matching, and smooth pasting conditions.

[36] Alternatively, one can describe the aggregate rate of growth in terms of a (size) weighted average of the rate of growth of consumers' stocks indexed by their initial deviation, z. In this case one needs to assume that either the distribution of initial stocks is independent of the initial deviations or the rates of growth of the stocks indexed by the initial deviations are independent of the initial shares.

Proof of Proposition 1

Formal derivations of the stochastic PDE in the proposition and its properties (without the boundary conditions) can be found in Krylov and Rozovski (1977, 1978). Here I provide a less rigorous but simple derivation starting from the discrete time/space analogue of the driving processes used in this paper. Thus

$$\Delta A_{t+dt} = \begin{cases} \sigma_A \sqrt{\Delta t} & \text{with prob. } ½\left(1 + \dfrac{\theta \sqrt{\Delta t}}{\sigma_A}\right) \\ -\sigma_A \sqrt{\Delta t} & \text{with prob. } ½\left(1 - \dfrac{\theta \sqrt{\Delta t}}{\sigma_A}\right). \end{cases}$$

If $\Delta A_{t+dt} = \sigma_A \sqrt{\Delta t}$,

$$\Delta k^*_{i,t+dt} = \begin{cases} \sigma \sqrt{\Delta t} & \text{with prob. } ½(1 + \gamma) \\ -\sigma \sqrt{\Delta t} & \text{with prob. } ½(1 - \gamma), \end{cases}$$

and if $\Delta A_{t+dt} = \sigma_A \sqrt{\Delta t}$,

$$\Delta k^*_{i,t+dt} = \begin{cases} \sigma \sqrt{\Delta t} & \text{with prob. } ½(1 - \gamma) \\ -\sigma \sqrt{\Delta t} & \text{with prob. } ½(1 + \gamma), \end{cases}$$

where $\gamma \equiv -\sigma_A/\sigma$.

Except for the boundary and center points, the probabilities are communicated across time and space by the Kolmogorov equation:

$$\begin{aligned} f(z, t) = & \\ & 1[\Delta A_t > 0][f(z - \Delta z, t - \Delta t)½(1 - \gamma) + f(z + \Delta z, t - \Delta t)½(1 + \gamma)] \\ & + 1[\Delta A_t < 0][f(z - \Delta z, t - \Delta t)½(1 + \gamma) + f(z + \Delta z, t - \Delta t)½(1 - \gamma)], \end{aligned}$$

where $1[x]$ is an indicator function that takes the value one when x is satisfied.

Rearranging this balance equation, using the relation $(\Delta z)^2 = \sigma^2 \Delta t$, and taking the limit of Δt as this converges to the infinitesimal quantity dt yield the Kolmogorov equation in the text:

$$df(z, t) = \left[-\mu \frac{\partial f(z, t)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 f(z, t)}{\partial z^2} \right] dt + \sigma_A \frac{\partial f(z, t)}{\partial z} dW_{At}.$$

In the discrete state space case, the first boundary conditions are obtained directly from the fact that L and U are trigger points, so no unit ever spends any time at them. In the continuous state space case, this is not sufficient. The proof, however, is obtained from evaluating the discrete Kolmogorov equation at the boundaries, noticing that the density is identically zero beyond these boundaries, and taking the limit as Δz goes to zero.

The next two boundary conditions can be obtained directly from the stochastic PDE. Integrating both sides of this, using the fact that the density integrates to one, and exchanging differentials with derivatives yield

$$0 = -\int_L^U \left[-\mu \frac{\partial f(z, t)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 f(z, t)}{\partial z^2} \right] dz dt + \sigma_A \int_L^U \frac{\partial f(z, t)}{\partial z} dz dW_{At},$$

which is satisfied for all t if

$$\int_L^U \frac{\partial f(z,t)}{\partial z} dz = 0 \tag{B1}$$

and

$$\int_L^U \frac{\partial^2 f(z,t)}{\partial z^2} dz = 0. \tag{B2}$$

Integration by parts transforms condition (B1) into the condition

$$f(C,t)^- - f(L,t) + f(U,t) - f(C,t)^+ = 0,$$

which, when combined with boundary conditions $f(L, t) = f(U, t) = 0$, yields the boundary condition $f(C, t)^+ = f(C, t)^-$. Finally, integrating equation (B2) by parts yields the (conservation) boundary condition involving the derivatives of the cross-sectional density. Q.E.D.

Proof of Proposition 2

Noticing that dZ can be written as

$$dZ = \int_L^{C^-} zdf(z,t)\,dz + \int_{C^+}^U zdf(z,t)\,dz$$

and replacing the stochastic PDE directly in this expression yield

$$\begin{aligned} dZ = {} & -\mu\left[\int_L^{C^-} zf_z(z,t)\,dz + \int_{C^+}^U zf_z(z,t)\,dz\right]dt \\ & + \frac{\sigma^2}{2}\left[\int_L^{C^-} zf_{zz}(z,t)\,dz + \int_{C^+}^U zf_{zz}(z,t)\,dz\right]dt \\ & + \sigma_A\left[\int_L^{C^-} zf_z(z,t)\,dz + \int_{C^+}^U zf_z(z,t)\,dz\right]dW_{At}. \end{aligned}$$

Solving the integrals and replacing the boundary conditions in the solution prove the proposition. Q.E.D.

Proof of Proposition 3

Direct substitution of the relation $dK_t = dK_t^* + dZ_t$ in proposition 2 yields

$$\dot{K}_t = -\delta + \frac{\sigma^2}{2}[f_z(L,t)^+(C-L) + f_z(U,t)^-(U-C)].$$

If the ergodic density replaces the cross-section density on the right-hand side, $\dot{K}_t = \theta$. Noticing this proves the proposition. Q.E.D.

An interesting extension is obtained for the case in which the return (or target) points from L and U are different: C_L and C_U, respectively.

PROPOSITION A1. Let individual units satisfy the assumptions in proposition 1 but follow (L, C_L, C_U, U) policies. Then propositions 1, 2, and 3 hold, with the following modifications: The boundary conditions of the PDE are

$$f(U,t) = f(L,t) = 0,$$

$$\frac{\partial f(C_L,t)^+}{\partial z} - \frac{\partial f(C_L,t)^-}{\partial z} + \frac{\partial f(L,t)^+}{\partial z} = 0,$$

$$\frac{\partial f(C_U,t)^+}{\partial z} - \frac{\partial f(C_U,t)^-}{\partial z} - \frac{\partial f(U,t)^-}{\partial z} = 0,$$

$$f(C_L,t)^+ = f(C_L,t)^-,$$

and

$$f(C_U,t)^+ = f(C_U,t)^-.$$

The path of the mean of the cross-sectional distribution is

$$dZ_t = -\delta dt - dK_t^* + \frac{\sigma^2}{2}[f_z(L,t)^+(C_L - L) + f_z(U,t)^-(U - C_U)]dt.$$

The path of K_t is given by

$$\dot{K}_t = \theta + \frac{\sigma^2}{2}\{[f_z(L,t)^+ - f_z(L)^+](C_L - L) + [f_z(U,t)^- - f_z(U)^-](U - C_U)\}.$$

Proof. It follows trivially from applying the same steps used in the proofs of propositions 1, 2, and 3. Q.E.D.

Implementation

In practice, data are observed only at discrete time intervals, but the discrete change in the aggregate stock of durables can be obtained by integrating dK_s over the time interval $(t - \Delta t, t]$:

$$\Delta K_t = \theta\Delta t + \frac{\sigma^2}{2}\int_{t-\Delta t}^{t}\{[f_z(L,s)^+ - f_z(L)^+](C - L) + [f_z(U,s)^- - f_z(U)^-](U - C)\}ds.$$

A more difficult problem is that the shape of the cross-sectional density at time t depends on the realization of the entire path of the aggregate shocks; by having information only at discrete intervals, we cannot know the exact position of the distribution. One possibility is to solve a filtering problem. Although appropriate, this is rather intractable. Instead, I take a shortcut that simplifies the problem substantially: I assume that the realization of the aggregate is homogeneously distributed within the observation periods. This not only simplifies matters by avoiding the filtering problem, but also makes the stochastic PDE deterministic within each observation period.[37]

In this context, the drift for period $h = (t - \Delta t, t]$ is defined as ϕ_h and is

[37] A similar shortcut is taken in Bertola and Caballero (1990*a*) for the case of irreversible investment. Some of the problems of this approximation in terms of the quadratic variation of the aggregate process are discussed there.

computed by adding to the depreciation the observed change in the frictionless stock divided by the time interval's length:

$$\phi_h = \delta + \frac{K_t^* - K_{t-\Delta t}^*}{\Delta t},$$

and the PDE for period h simplifies to

$$\frac{\partial f(z, t)}{\partial t} = \phi_h \frac{\partial f(z, t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z, t)}{\partial z^2}.$$

Even after we reduce the complexity of the PDE by letting it be deterministic in the small time interval $h = (t - \Delta t, t]$, the problem is not simple since it involves nonhomogeneous (because of the connection of the PDEs across time) and nonlocal (because of the finite jumps implied by the optimal microeconomic policies) boundary conditions. The fact that the first and second derivatives of the empirical density are only piecewise continuous (because of the kink at the return point) also makes the algebra more tedious.

The deterministic PDE can be solved with the method of separation of variables. For this, let me postulate that the particular solutions take the form $f(z, t) = H(z)M(t)$. Plugging this back into the PDE, we can decompose the problem into two ordinary differential equations linked by a real-valued parameter ψ, $M'(t) - \psi M(t) = 0$, and the Sturm-Liouville problem

$$H''(z) + \xi_h H'(z) - \frac{\xi_h}{\phi_h}\psi H(z) = 0,$$

with $\xi_h \equiv 2\phi_h/\sigma^2$ and boundary conditions

$$H(L) = H(U) = 0,$$

$$H'(C)^+ - H'(C)^- = H'(U)^- - H'(L)^+,$$

$$H(C)^+ = H(C)^-,$$

and the initial condition $f(z, 0) = g(z)$, where $g(z)$ is the empirical density at the end of period $h - 1$.

The roots β_1 and β_2 of the Sturm-Liouville problem are given by

$$\beta_1 = -\frac{\xi_h}{2} + \frac{1}{2}\sqrt{\xi_h^2 + \frac{4}{\phi_h}\psi}$$

and

$$\beta_2 = -\frac{\xi_h}{2} - \frac{1}{2}\sqrt{\xi_h^2 + \frac{4}{\phi_h}\psi}.$$

When $\psi \geq -\phi_h \xi_h/4$, the two roots are real and the solutions have the form

$$H(z) = \begin{cases} A_1 e^{\beta_1 z} + A_2 e^{\beta_2 z} & \text{for } L \leq z \leq C \\ A_3 e^{\beta_1 z} + A_4 e^{\beta_2 z} & \text{for } C \leq z \leq U, \end{cases}$$

where the constants A_1, A_2, A_3, and A_4 are to be determined from the boundary conditions. It is easy to verify that in the case of real roots the only nontrivial solution for these constants occurs when $\psi = 0$. Moreover, this solution gives the unconditional or long-run distribution under the drift ϕ_h.

Simple algebra shows that $H(z; \psi = 0)$ takes the form

$$H(z; \psi = 0) = \begin{cases} A_0(e^{-\xi_h z} - e^{-\xi_h L}) & \text{for } L \le z \le C \\ A_0\left(\dfrac{e^{-\xi_h L} - e^{-\xi_h C}}{e^{-\xi_h U} - e^{-\xi_h C}}\right)(e^{-\xi_h z} - e^{-\xi_h U}) & \text{for } C \le z \le U, \end{cases}$$

where A_0 is to be determined later. On the other hand, when $\psi < \phi_h \xi_h/4$, the roots are imaginary:

$$\beta_1 = -\frac{\xi_h}{2} + i\gamma(\psi)$$

and

$$\beta_2 = -\frac{\xi_h}{2} - i\gamma(\psi),$$

where i stands for $\sqrt{-1}$ and

$$\gamma \equiv \gamma(\psi) = \frac{1}{2}\sqrt{-\left(\xi_h^2 + \frac{4}{\phi_h}\psi\right)}.$$

In this case the solutions of the Sturm-Liouville problem have the form

$$H(z) = \begin{cases} e^{-(\xi_h/2)z}[A_1 \cos(\gamma z) + A_2 \sin(\gamma z)] & \text{for } L \le z \le C \\ e^{-(\xi_h/2)z}[A_3 \cos(\gamma z) + A_4 \sin(\gamma z)] & \text{for } C \le z \le U, \end{cases}$$

and the boundary conditions reduce to

$$A_1 \cos(\gamma L) + A_2 \sin(\gamma L) = 0,$$

$$A_3 \cos(\gamma U) + A_4 \sin(\gamma U) = 0,$$

$$(A_1 - A_3)\cos(\gamma C) + (A_2 - A_4)\sin(\gamma C) = 0,$$

and

$$(A_1 - A_3)\sin(\gamma C) - (A_2 - A_4)\cos(\gamma C) = [A_1 \sin(\gamma L) - A_2 \cos(\gamma L)]e^{(\xi_h/2)(C-L)} - [A_3 \sin(\gamma U) - A_4 \cos(\gamma U)]e^{(\xi_h/2)(C-U)}.$$

Earlier drafts of the paper contain the solution for the general case (available on request); for briefness, here I describe the solution for the case used in estimation only. In the empirical implementation of the model I normalize the problem so $C = 0$ and make the simplifying assumption $L = -U$. In this case the boundary conditions system has nontrivial solutions only when the condition $\sin(\gamma U) = 0$ is met.

The parameters ψ_n for which this condition is satisfied are called the eigenvalues of the Sturm-Liouville problem, and there are a countable infinity of them: if $\psi(\gamma_1)$ denotes the solution associated with the smallest positive γ that solves the condition above, then it is straightforward to verify that all the eigenvalues $\psi(n\gamma_1)$, for positive integers n, satisfy the same condition. It is also easy to see that in this case $\gamma_1 = 2\pi/U$.

Associated with each eigenvalue there is a solution of the Sturm-Liouville problem:

$$H(z; \psi_n) = \begin{cases} A_n e^{-(\xi_h/2)z} \sin(\gamma_n z) & \text{for } U \le z \le 0 \\ A_n(-1)^{n+1} e^{(\xi_h/2)U} e^{-(\xi_h/2)z} \sin(\gamma_n z) & \text{for } 0 \le z \le U, \end{cases}$$

where $A_n \equiv A(\psi_n)$ for $n = 1, \ldots, \infty$, and

$$\psi_n = -\frac{\sigma^2}{2}\left(\frac{\pi^2 n^2}{U^2} + \frac{\phi_h^2}{\sigma^4}\right) < 0.$$

Consistently, the ordinary differential equation for the time component of the PDE has a solution $M(t) = e^{\psi_n t}$, and the general solution of the PDE has the form

$$f(z, t) = \sum_{n=0}^{\infty} H(z; \psi_n) e^{\psi_n t},$$

with $\psi_0 = 0$ and

$$H(z; \psi = 0) = \begin{cases} A_0(e^{-\xi_h z} - e^{\xi_h U}) & \text{for } -U \leq z \leq 0 \\ A_0\left(\dfrac{e^{\xi_h U} - 1}{e^{-\xi_h U} - 1}\right)(e^{-\xi_h z} - e^{-\xi_h U}) & \text{for } 0 \leq z \leq U, \end{cases}$$

where A_0 is determined by the adding-up constraint of the probability distribution

$$A_0 = \frac{1}{U(1 - e^{\xi_h U})},$$

and the rest of the constants are determined from the nonhomogeneous boundary condition

$$\sum_{n=0}^{\infty} H(z; \psi_n) = g(z)$$

or, equivalently,

$$\sum_{n=1}^{\infty} H(z; \psi_n) = p(z),$$

where $p(z) = g(z) - H(z; \psi = 0)$.

Multiplying each side by $e^{\xi_h z} H(z; \psi_k)/A_k$ and noticing that the eigenfunctions of the Sturm-Liouville problem are orthogonal under $e^{\xi_h z}$ yield

$$A_n = \frac{\int_L^U p(z)[H(z; \psi_n)/A_n] e^{\xi_h z} dz}{\int_L^U [H(z; \psi_n)/A_n]^2 e^{\xi_h z} dz}.$$

The denominator has a closed form that I omit since it is not informative. The path of the cross-sectional density together with its derivatives at the boundaries can now be computed numerically.

Although analytically complex, the continuous time implementation described here has several advantages over discrete time/space Markov chain alternatives. First, by working with a continuous state space, one avoids the discontinuities in the objective function arising from the integer nature of the endogenously determined number of states of the discrete case (see, e.g., Bertola and Caballero 1990*b*). Second, although aggregate shocks are assumed to be homogeneous within a time period, idiosyncratic shocks occur continuously. Third, by varying the number of Fourier coefficients used, one can efficiently exploit the trade-off between precision and computational time. For example, in this paper I did all the preliminary exploration and study of global properties of the objective function using only two Fourier terms. After identifying the region with the global maximum, I improved precision by using five Fourier coefficients.[38]

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