

DYNAMIC (S, s) ECONOMIES

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In this paper we provide a framework to study the aggregate dynamic behavior of an economy where individual units follow (S, s) policies. We characterize structural and stochastic heterogeneities that ensure convergence of the economy's aggregate to that of its frictionless counterpart, determine the speed at which convergence takes place, and describe the transitional dynamics of this economy.

KEYWORDS: (S, s) policy, idiosyncratic shocks, heterogeneity, aggregation, synchronization, convergence, speed of convergence.

1. INTRODUCTION

IN RECENT YEARS there has been a surge in the application of formal microeconomic models of discontinuous and lumpy adjustment—originally developed in the early 50's for retail inventories—to a variety of topics in economics, such as cash balances, labor demand, investment, entry and exit, prices, durable goods, and technology upgrade. Yet the possibility of explaining *aggregate* economic phenomena based on these models has remained largely unexplored, primarily because of the technical difficulties involved. Since aggregate data do not look as discontinuous and lumpy as their microeconomic counterparts, in order to apply these models to macroeconomic data aggregation has to be modeled explicitly. This is hard to do when shocks are not purely idiosyncratic but also have a common (or, equivalently, aggregate) component. The few results existent in the literature have provided important insights, but have been limited either to numerical simulations (Blinder (1981)) or to steady state analysis (Caplin (1985), Caplin and Spulber (1987)).² This paper's main contribution is to provide a framework within which the out-of-steady-state aggregate dynamics of an economy with lumpy adjustment at the microeconomic level can be studied analytically.

We simplify the mathematics substantially by only considering a particular, but widely used, adjustment policy: the one sided (S, s) rule. In the last section we argue that many of this paper's insights either carry over directly to more general forms of adjustment rules or provide the natural foundation for their study.

One of the appealing characteristics of (S, s) rules is their simplicity: an individual agent allows his state variable (e.g. inventories) to fall freely until it reaches a certain critical level s ; at this point abrupt action takes place and the state variable is reset to an upper value S from where the cycle starts again.

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²Others have performed comparative statics experiments in models with no aggregate (continuous) shocks (e.g. Akerlof (1979), Tsiddon (1989)).

Examples where the optimality of fixed (S, s) rules has been established go back to the problem of inventories management (Scarf (1959)); a more recent example is price setting in the presence of menu costs (Sheshinski and Weiss (1977, 1983), Caplin and Sheshinski (1987)). Moreover, the fixed (S, s) model has also been extensively used in the Operations Research and Economics literatures as an approximation for more complex optimal rules (e.g. Arrow, Harris, and Marschak (1951), Karlin and Fabens (1959), Blinder (1981), Ehrhardt, Schultz, and Wagner (1981), Blanchard and Fischer (1989, p. 405)).

Whenever microeconomic units adjust discretely and by large amounts, the issue of heterogeneity acquires high priority. The similarity between the economy's aggregate path and the discontinuous and lumpy path of microeconomic units grows with the degree of synchronization of units' actions. In the limit, when all units are identical and act simultaneously (the symmetric equilibria assumption), the aggregate path is indistinguishable from that of an individual unit. On the other hand, if units' actions exhibit little synchronization, the aggregate may depart substantially from the behavior of any single (representative) unit. In this paper we study the aggregate implications of the process of endogenous synchronization and staggering of individual units.

We consider a dynamic economy where agents differ in their initial positions within their bands and face both *stochastic* and *structural* heterogeneity, where the former refers to the presence of (unit specific) idiosyncratic shocks, and the latter to differences in the widths of units' (S, s) bands and their response to aggregate shocks. We study the evolution of the economy's aggregate and the evolution of the difference between this aggregate and that of an economy without microeconomic friction, where the latter pertains to a situation where individual units adjust with no delay to all shocks. We also examine the sensitivity of this difference to common shocks. For example, in the retail inventory problem the aggregate deviation and sensitivity to common shocks correspond to the aggregate inventory level and its sensitivity to aggregate demand shocks, respectively.

In Section 2 we determine conditions under which the microeconomic effect of lumpy adjustment rules has no aggregate impact. Section 3 begins the study of the economy's aggregate (out-of-steady-state) dynamics by discussing the summary variables we use to describe the economy over time. In Section 4 we consider the effect of stochastic heterogeneity on the economy's dynamic aggregate behavior when no structural differences are present. We show that the economy's aggregate converges to that of its counterpart without friction when idiosyncratic shocks spread out without bound over time, and that the speed of convergence increases with the rate at which dispersion occurs; we also show that common shocks play no role in aiding convergence. Structural heterogeneity is incorporated into the analysis in Section 5; we show that it can lead to convergence by itself, that the speed of convergence grows with the degree of structural heterogeneity, and that common shocks aid convergence when structural differences are present. Section 6 shows that, paradoxically, the interaction between both forms of heterogeneity may actually slow down convergence. Section 7 presents final remarks. An extensive appendix follows.

2. BASIC MODEL AND STEADY STATE

We consider an economy composed of a large number of units, and approximate this large number by a continuum, indexed by $i \in [0, 1]$. We let $z_i(t)$ denote the difference between $x_i(t)$, the actual value of unit i 's state variable at time t when an (S, s) policy is followed, and $x_i^*(t)$, the value of the same variable when there is no friction. For example, consider the retail inventory problem, where firms decide on their optimal inventory holding in the presence of uncertain demand and fixed replenishment costs. In this case $x_i^*(t)$ and $x_i(t)$ are accumulated sales and accumulated inventory orders, and $z_i(t)$ is the level of inventories.

We express every frictionless (optimal) variable, $x_i^*(t)$, as the sum of an idiosyncratic component, $v_i(t)$, and the unit's response to an aggregate shock $a(t) \equiv \int x_i^*(t) di$:

$$(1) \quad x_i^*(t) = \theta_i a(t) + v_i(t),$$

where θ_i is unit i 's sensitivity to the common shock.³ For example, in the retail inventory problem $da(t)$ denotes aggregate demand shocks and θ_i the sensitivity of sector i 's demand to these shocks. We normalize the sensitivity parameters so that $\int_0^1 \theta_i di = 1$; this implies that by construction $\int_0^1 v_i(t) di = 0$ for all t .

We assume that, for each unit i , $z_i(t)$ decreases monotonically and continuously until it reaches the unit specific trigger barrier, s_i ; at this point finite control is exerted on x_i to bring z_i back to the unit specific target barrier S_i .⁴

ASSUMPTION 1—Stationarity, symmetry, monotonicity, and continuity:

1. The variable $z_i(t)$ is controlled according to a fixed band, one sided, unit specific (S, s) policy.

2. The (S, s) rules are symmetric: $S_i = -s_i$.⁵

3. The variable $z_i(t)$ decreases monotonically during time periods where no control is exerted.⁶

4. The sample paths of $v_i(t)$ are continuous and those of $a(t)$ are continuous, increasing, and unbounded.

³ Of course, studying the determination of the $x_i^*(t)$'s themselves is a topic in itself.

⁴ We assume that the (S, s) rules followed by units are given exogenously. This has two consequences. First, we do not consider the relation between the economy's aggregate behavior and the determinants of the (S, s) policies' optimal target and trigger points. This can be done easily, yet doing so is beyond the scope of this paper. Second, the results we derive also apply in a broader class of problems, where (S, s) rules are not optimal but can be justified as either simple rules that approximate more complex first best rules or, perhaps equivalently, as arising from near rational behavior.

⁵ The only reason for making this assumption is that it simplifies some of the algebraic expressions. It is easy to work without it, as we did in preliminary versions of this paper. For example, this implies that in the retail inventory problem z_i represents the inventory level in deviation from its long run average.

⁶ This assumption requires that the sum of changes in aggregate and idiosyncratic components always be positive: $\theta_i da(t) + dv_i(t) \geq 0$. We assume that $a(t)$ grows sufficiently fast—compared to the rate at which idiosyncratic shocks disperse—for this assumption to hold. In some cases, however, calculations are simpler if we consider distributions generating idiosyncratic shocks that have infinite tails. Our model is appropriate in this case if the fraction of units violating the monotonicity assumption is small.

This framework can accommodate many well known problems, apart from the retail inventory problem mentioned above. A few of them are:

- The Pricing Problem, where firms pay a menu-cost when they adjust their nominal prices. In this case $x_i^*(t)$ is the frictionless optimal price and $x_i(t)$ the actual price charged.
- The Cash-Balance Problem, where consumers decide on the optimal level of cash holdings when adjusting their cash-balances is costly. In this case, $x_i^*(t)$ and $x_i(t)$ are accumulated expenditures and accumulated withdrawals, and $z_i(t)$ is the current cash balance.
- The Technology Update Problem, where firms decide on whether to scrap their current machines and update them or not. In this case, $x_i^*(t)$ and $x_i(t)$ are desired and actual state of technology, and $z_i(t)$ is the gap between them.
- The Durable Goods Problem, where consumers decide when to buy a durable good and adjusting the stock they have is costly. In this case, $x_i^*(t)$ and $x_i(t)$ are desired and actual levels of the stock of durable goods, and $z_i(t)$ is the gap between them. In this problem, the sensitivity parameters could correspond to the marginal propensities to consume.
- The Capital Stock Adjustment Problem, where firms decide when to adjust their capital stock when there are nonconvex costs of adjustment. In this case, $x_i^*(t)$ and $x_i(t)$ are desired and actual levels of the stock of capital, and $z_i(t)$ is the gap between them.⁷

The main goal of this paper is to examine the behavior of the variable we call “the aggregate,” defined as the integral of the $x_i(t)$ ’s over all i ’s and denoted by $X(t)$. Using the definition of the $z_i(t)$ ’s, and letting $Z(t) \equiv \int_0^1 z_i(t) di$, leads to the following expression for $X(t)$:

$$(2) \quad X(t) = a(t) + Z(t).$$

When there is no microeconomic friction, all the z_i ’s are identically zero; thus $X(t) = X^*(t) = a(t)$. As we are interested in the effects of microeconomic (S, s) policies on the departure of $X(t)$ from $X^*(t)$, we focus on the mean of the cross-section distribution of individual departures, $Z(t)$.⁸ The entire analysis carried out in this paper—in particular the computation of the latter mean—is conditional on the actual path of the aggregate shock $a(t)$. It turns out that the results we derive do not depend on any particular features of this path, as long as $a(t)$ is continuous, increasing, and tends to infinity (see Assumption 1). We therefore do not need to specify the stochastic mechanism underlying common shocks. The fact that we consider the dynamic path of the actual cross-section distribution—and not that of the joint distribution of all units—in spite of the presence of aggregate shocks, is one of the building blocks of the methodology

⁷The monotonicity assumption is appropriate in the inventory problem when returns are dominated by new sales and the holding cost does not vary much; in the pricing problem, when core inflation is sufficiently large; in the cash balance problem, when expenditures dominate the interest rate variability; and in the technology, consumer durables, and investment problems, when the obsolescence and depreciation rates dominate the uncertainty faced by firms and consumers.

⁸To reconstruct $X(t)$ based on $Z(t)$ we need to know the value of $a(t)$. This is usually obtained from a theoretical model for the frictionless economy.

we develop in this paper. Its usefulness is best appreciated when we consider convergence issues in Sections 4 and 5. We therefore postpone discussing its importance until the final section.

Instead of working directly with $z_i(t)$, it is notationally convenient to describe the problem in terms of the fraction unit i has covered of its (S, s) band at time t , $c_i(t)$. We therefore define

$$(3) \quad c_i(t) \equiv \frac{1}{2} - \frac{z_i(t)}{\lambda_i},$$

where $\lambda_i \equiv S_i - s_i$ denotes unit i 's bandwidth. The variable $c_i(t)$ takes values in $[0, 1)$; it starts its cycle when $c_i(t) = 0$ (i.e. when $z_i(t) = S_i$) and ends it when $c_i(t)$ reaches one (i.e. when $z_i(t)$ reaches s_i). Substituting $x_i(t) - x_i^*(t)$ for $z_i(t)$ in (3) yields

$$c_i(t) \equiv \left(\frac{1}{2} - \frac{x_i(t) - x_i^*(t)}{\lambda_i} \right).$$

Substituting $x_i^*(t)$ by (1), adding and subtracting $c_i(0)$, and noting that $(x_i(t) - x_i(0))/\lambda_i$ is always an integer, yields

$$(4) \quad c_i(t) = \left(c_i(0) + \frac{\theta_i a(t) + v_i(t)}{\lambda_i} \right) \pmod{1},$$

where $x \pmod{1}$ denotes the difference between the real number x and its integer part and we set $a(0)$ and $v_i(0)$ equal to zero without loss of generality.

We let c_t , v_t , Θ , and Λ denote random variables with a joint *probability* distribution identical to that of the joint *cross-section* distribution of the $c_i(t)$'s, the $v_i(t)$'s, the θ_i 's, and the λ_i 's.⁹ Thus, we have that

$$(5) \quad c_t = \left(c_0 + \frac{\Theta a(t) + v_t}{\Lambda} \right) \pmod{1}.$$

An expression for the aggregate deviation, $Z(t)$, can be obtained directly in terms of the variables we defined above. All that is needed to determine $Z(t)$ is the value of the current aggregate shock, $a(t)$, and the cross-section distribution of the random vector $(c_0, v_t, \Lambda, \Theta)$.

⁹ Note that Θ and Λ do not have time subindices, indicating that units' sensitivity parameters and bandwidths do not change over time.

PROPOSITION 1: *Suppose Assumption 1 holds. Then $Z(t) = g(a(t), t)$ and $X(t) = a(t) + g(a(t), t)$, where*

$$g(a, t) \equiv \frac{1}{2}E(\Lambda) - E\left[\Lambda\left(\left(c_0 + \frac{\Theta a + v_t}{\Lambda}\right) \pmod{1}\right)\right].$$

PROOF: Follows directly from equations (3), (4), and (5).

Q.E.D.

The intermittent and lumpy microeconomic behavior is irrelevant at the aggregate level when—for any realization of the stochastic mechanism underlying aggregate and idiosyncratic shocks— $Z(t)$ remains constant over time. Without loss of generality we suppose that the constant aggregate deviation is equal to zero in what follows.

DEFINITION 1: The aggregate deviation of an economy satisfying Assumption 1 is at its steady state at time $t = 0$ if $g(a, t) = 0$ for all $a \geq a(0) \equiv 0$ and all $t \geq 0$, with $g(a, t)$ defined in Proposition 1.

Whether the economy's aggregate deviation is at its steady state or not depends on the stochastic mechanism underlying the model. There are various sets of conditions under which the aggregate deviation remains equal to zero as time passes. In this paper we consider conditions that can be expressed only in terms of *the cross-section distributions* defined above. In the following proposition—which is an extension of Proposition 1 in Caballero and Engel (1989b)—we show that when units' initial positions within their cycle are distributed uniformly on $[0, 1)$ and independent from the remaining sources of heterogeneity, the economy's aggregate deviation is at its steady state.

PROPOSITION 2 (Caballero and Engel (1989b)): *Given Assumption 1, the economy's aggregate deviation is at its steady state at time $t = 0$ if c_0 is uniform on $[0, 1)$ and independent from Λ , Θ and v_t for all $t > 0$. Furthermore, c_t is uniform on $[0, 1)$ for all $t > 0$.*

This result shows that when units' positions within their cycle are independent from the sources of structural and stochastic heterogeneity, there exists a cross-section (or empirical) distribution of the c_i 's that is invariant under continuous, monotone, aggregate shocks. This distribution is uniform. It follows from equation (3) that there also exists a cross-section distribution of the z_i 's that is invariant under the same class of shocks. This distribution is determined by the probability distribution of Λ ; it is uniform only when bandwidths do not vary across units.

Proposition 2 presents an economy with strong forms of microeconomic rigidity that has an aggregate behavior indistinguishable from that of an economy without friction. This is a generalization of the insightful result in Caplin and Spulber (1987). They consider the case where all units have the same

bandwidth, no idiosyncratic shocks are present, and common shocks have the same impact on all units' $x_i^*(t)$'s. None of these conditions are required for Proposition 2 to hold. In addition, the scenario described in Proposition 2 has a realistic feature that is absent in an economy without structural or stochastic heterogeneity: the relative positions of units within their cycle changes over time. The order in which units adjust their state variable does not repeat itself from one cycle to another.

Proposition 2 assumes that the initial cross-section distribution of the c_i 's is independent from the joint distribution of idiosyncratic shocks, bandwidths, and sensitivity parameters. If this is not the case, the cross-section distribution of the c_i 's generally does not remain uniform on $[0, 1)$ and the aggregate deviation, $Z(t)$, does not remain constant. This happens, for example, when units with smaller bandwidths—or larger sensitivity parameters—are initially concentrated at the beginning of their cycle, as is further illustrated in Section 5.

3. DESCRIPTION OF NON-STEADY-STATE DYNAMICS

There are many reasons why Assumption 1 may be momentarily violated and the (S, s) economy's aggregate deviation be forced away from the steady state described in Proposition 2. For example, in the case of the pricing problem, a finite (discrete) change in $a(t)$, like an oil shock or a large monetary shock, bunches a fraction of units at the beginning of their cycle. Alternatively, a widening of units' bands—due, for example, to an increase in the rate of core inflation in an economy where bands are set optimally—leaves a fraction of the new state space initially with no units. In the time period following any one of these “structural changes,” the aggregate deviation typically does not remain constant and the economy therefore is not at its steady state anymore.

In the following three sections we study the dynamic behavior of the economy outside of its steady state. We consider idiosyncratic shocks and structural heterogeneity as possible sources of convergence; the latter meaning differences in bandwidths and sensitivity parameters. For expository simplicity, we study the effects of these factors separately before considering their interaction.

Proposition 1 characterizes the dynamic path of the difference of the aggregates from economies with and without frictions. For example, it can be used to determine the evolution of the average level of inventories after an oil shock. Yet we may not only be interested in the level of aggregate inventories at a given point in time, but also in the potential impact of a small aggregate demand shock on this aggregate. This impact on $Z(t)$ is equal to the (partial) derivative of the function g —defined in Proposition 1—with respect to a , evaluated at $(a(t), t)$. We denote this derivative by $J(t) \equiv \partial g / \partial a$.

The relation between $J(t)$ and the cross-section distribution of firms' positions within their cycle is best understood if we look at the effect of a small common shock, Δa , on the aggregate deviation, $Z(t)$, when sensitivity parameters do not vary across units ($\Theta \equiv 1$). We begin with the units that are forced to start a new cycle. The common shock forces a unit with bandwidth λ to adjust

only if it has covered a fraction larger than or equal to $1 - (\Delta a/\lambda)$ of its cycle before the shock. The fraction of units with bandwidth λ that reach their trigger point is proportional to $\Delta a \cdot f_{(c_t|\Lambda=\lambda)}(1^-)/\lambda + O((\Delta a)^2)$,¹⁰ where $f_X(\lambda)$ denotes the density of the random variable X . Other things equal, this fraction is smaller the larger the common bandwidth. This effect is exactly offset by the fact that $Z(t)$ grows more when a unit with a larger bandwidth restarts its cycle. Thus, the contribution to the aggregate deviation of those units that adjust and have bandwidth equal to λ is proportional to $\Delta a \cdot f_{(c_t|\Lambda=\lambda)}(1^-)f_\Lambda(\lambda)$; the total increase in $Z(t)$ due to units reaching their trigger point is then equal to $\Delta a \cdot \int f_{(c_t|\Lambda=\lambda)}(1^-)f_\Lambda(\lambda) d\lambda = \Delta a \cdot f_{c_t}(1^-)$. Next we consider those units that do not start a new cycle after the aggregate shock. Every unit that does not adjust decreases its contribution to $Z(t)$ by Δa ; their total contribution is equal to Δa (minus a term of order $(\Delta a)^2$ that accounts for the fact that not all units belong to the group that does not start a new cycle). We have therefore shown that $\Delta g/\Delta a$ is equal to $f_{c_t}(1^-) - 1 + O(\Delta a)$. Letting Δa approach zero we conclude that $J(t) = f_{c_t}(1^-) - 1$.

It is apparent from the previous paragraph and Proposition 1 that both $J(t)$ and $Z(t)$ may be equal to zero even when the economy's aggregate deviation is "far away" from its steady state. On the one hand, $J(t)$ is equal to zero every time $f_{c_t}(1^-)$ is equal to one;¹¹ on the other hand, Proposition 1 implies that $Z(t) = 0$ every time $\int \lambda E(c_t|\Lambda = \lambda)f_\Lambda(\lambda) d\lambda = 0$. Therefore $Z(t) = 0$ whenever the weighted average of the "sectoral" aggregates is equal to zero, where the latter are defined as the aggregates conditional on a common bandwidth. Since these sectoral aggregates may evolve in rather arbitrary ways, there is no reason why their average should remain equal to zero in the future.

It is tempting to argue, based on Proposition 2, that the economy's aggregate deviation is at its steady state every time the cross-section distribution of units within their cycle, c_t , is uniform on $[0, 1)$. This intuition is supported by the fact that $Z(t) = E\{\Lambda(\frac{1}{2} - c_t)\}$ is equal to zero when c_t is uniform on $[0, 1)$ and independent from Λ . Yet this argument is not correct, since c_0 is generally not independent from Λ . For example, consider the case where a fraction of units is bunched at the beginning of their cycle after the economy is perturbed away from its steady state. Other things equal, units with larger bandwidths move a smaller fraction of their cycle in a given period of time, so that the correlation between Λ and c_t is negative in the time period following the perturbation. We conclude that although $Z(t)$, $J(t)$, and the shape of the cross-section distribution of units' positions within their cycle are interesting summary variables of

¹⁰ The 1^- is used in place of 1 to remind us that there are no units with $c_t(t) = 1$, since this is a trigger point. Strictly speaking, this notation is unnecessary since the density of an absolutely continuous random variable is determined up to a set of Lebesgue measure zero. What we have in mind is a continuous version of this density.

¹¹ This assumes that all sensitivity parameters are the same across units. The expression for $J(t)$ is extended to the general case as follows. We apply the argument given in the text with the density of c_t conditional on the value of Θ instead of f_{c_t} , and take expectation with respect to Θ , concluding that $J(t) = E_\Theta[\theta f_{(c_t|\Theta=\theta)}(1^-)] - 1$. The assertion that $J(t_0) = 0$ does not imply that $J(t)$ remains equal to zero is still valid.

the economy's aggregate deviation at any particular instant in time, neither of them has the property of capturing how much the economy's aggregate behavior differs from that of its counterpart without frictions. Next we consider two indices that do have this property:

$$Z^*(t) \equiv \sup_{\{a \geq a(t), s \geq t\}} |g(a, s)|,$$

and

$$J^*(t) \equiv \sup_{\{a \geq a(t), s \geq t\}} \left| \frac{\partial g(a, s)}{\partial a} \right|.$$

The definition of these indices is now illustrated by describing how one of them, $Z^*(t)$, is evaluated at a given instant in time, t_0 . Suppose that accumulated common shocks at time t_0 are equal to a_0 . Consider all possible future paths of the aggregate shock, $\{a(s), s \geq t\}$, that satisfy Assumption 1 and have $a(t_0) = a_0$, and calculate the maximum (absolute) aggregate deviation for every one of them. The index $Z^*(t)$ then is equal to the largest among these maxima. The aggregate deviation and its sensitivity to small aggregate shocks have (absolute) values that are bounded from above by $Z^*(t)$ and $J^*(t)$ for any future trajectory of the common shock that satisfies Assumption 1.¹² Accordingly, we define “convergence of the economy's aggregate to that of its counterpart with no friction” as follows.

DEFINITION 2: The aggregate of an economy that satisfies Assumption 1 converges to that of its frictionless counterpart if $Z^*(t)$ and $J^*(t)$ tend to zero as t tends to infinity.

In sum, we describe the dynamic behavior of an (S, s) economy using four summary variables. We look at the economy's aggregate deviation from the frictionless counterpart, at the sensitivity of this index to common shocks, and at the suprema of these indices over all possible realizations of the underlying stochastic mechanism.

4. CONVERGENCE AND IDIOSYNCRATIC SHOCKS

4.1. Convergence

In this section we isolate stochastic heterogeneity as the only source of convergence by assuming that all units have the same bandwidth ($\lambda_i \equiv \lambda$) and the same sensitivity parameters ($\theta_i \equiv 1$).

There are many ways in which the economy's aggregate may converge to that of its frictionless counterpart. For example, convergence takes place if idiosyncratic shocks are correlated with c_0 in such a way that they exactly fill in the gaps between the density of c_i and a density uniform on $[0, 1)$ in finite time and,

¹² Considering suprema in the definitions above is just one possible choice. We could work with a weighted average—over all possible values of $a \geq a(t)$ and $s \geq t$ —where the weights reflect the likelihood of different sample paths of the common shock and the time discount rate.

after this happens, become independent of units' positions within their cycle so that c_t remains uniform on $[0, 1)$ (see Proposition 2). This way of achieving convergence is rather far-fetched; there exist other scenarios where convergence takes place that are even more arbitrary. In this section we consider conditions that ensure convergence when there is no systematic relation between c_0 and the realizations of the idiosyncratic shocks.

ASSUMPTION 2—Independence: *The random variables c_0 and v_t are independent for all $t > 0$, or, equivalently, dv_t is independent from c_s , for all $s \leq t$.*

Under the independence assumption, convergence is not achieved by filling in the gaps in finite time, but by making initial conditions irrelevant as time passes. This happens when the cross-section distribution of idiosyncratic shocks, v_t , folded back into the unit interval, converges to a distribution uniform on $[0, 1)$ and thereby “washes away” the initial cross-section distribution of units' positions within their cycle. An example is useful at this point. Suppose that the process generating any unit's idiosyncratic shocks, $(v_t(t), t \geq 0)$, is Gaussian with variance $\sigma^2(t)$ growing as time passes. Since these processes are independent across units, the cross-section distribution of idiosyncratic shocks, v_t , also is normal and has the same variance. This follows from the Glivenko-Cantelli Theorem; see e.g. Billingsley (1986). Figure 1a illustrates how the cross-section density of idiosyncratic shocks flattens out; the corresponding evolution of the density of c_t is illustrated in Figure 1b—where we have abstracted from the value of the common shock $a(t)$ —for the case where c_0 is a spike at 0.5. Since the expected value of c_t approaches one half, $Z(t)$ tends to zero, and since the cross-section density of c_t is approaching one, $J(t) = f_{c_t}(1^-) - 1$ tends to zero. Furthermore, since bandwidths and sensitivity parameters are the same across units, aggregate shocks do not act as a unit separating mechanism; all they do is move units around their cycle. Figure 1c illustrates this by showing how the density of c_t varies for different values of the common shock at a fixed instant in time ($t = 1.0$). It follows that $Z^*(t)$ and $J^*(t)$ both tend to zero. Thus Figure 1 suggests that all summary variables converge to zero when the cross-section density of idiosyncratic shocks flattens out as time passes. This assumes that densities are unimodal, or at least that they do not oscillate too much. The following assumption makes these intuitive conditions on the density of the v_t 's precise.

ASSUMPTION 3—Flattening out of densities: *The total variation of the density of v_t tends to zero as t tends to infinity.*¹³

¹³ The total variation of a function $f(x)$ is equal to $\sup \sum_k |f(x_{k+1}) - f(x_k)|$, where the supremum is taken over all finite increasing sequences $x_1 < x_2 < x_3 < \dots$. It follows directly from this definition (see, e.g., Proposition 3.8 in Engel (1991)) that the total variation of a unimodal function is equal to twice the maximum value it attains. More generally, if $f(x)$ is piecewise continuously differentiable, with jumps of absolute magnitude $\delta_1, \delta_2, \dots$, then its total variation is equal to $\sum \delta_k + \int |f'(x)| dx$.

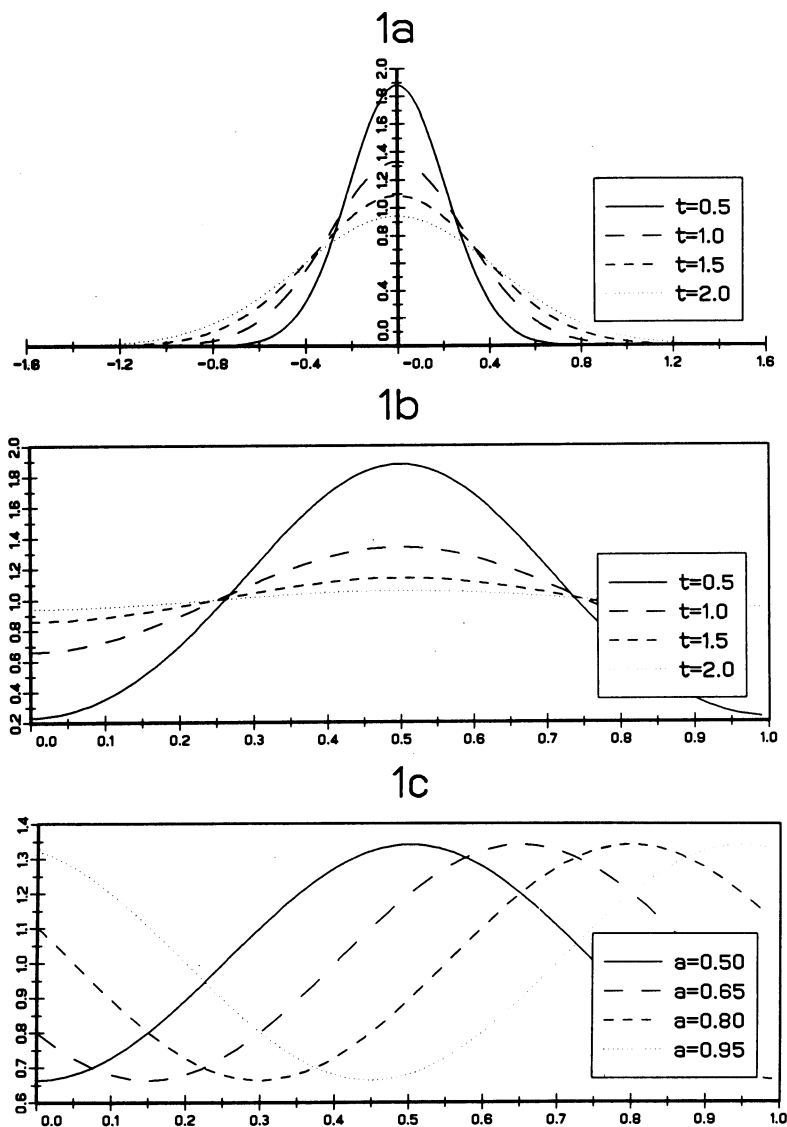


FIGURE 1.

Assumption 3 holds when the density of v_t is unimodal and its largest value tends to zero as t tends to infinity. Two situations where this happens are when the v_t 's are normal and their variance tends to infinity, and when the v_t 's are absolutely continuous and are an integrated process. The proposition that follows provides general conditions under which convergence occurs.

PROPOSITION 3: *Suppose idiosyncratic shocks and differences in units' initial positions within their cycle are the only sources of heterogeneity and Assumptions*

1–3 hold. Then the economy's aggregate converges to that of its counterpart without friction and c_t converges to a distribution uniform on $[0, 1)$.

PROOF: See the Appendix.

Q.E.D.

The assumptions of Proposition 3 are on the cross-section distribution of idiosyncratic shocks, not on the processes generating individual units' shocks. Since we have a continuum of units, the Glivenko-Cantelli Theorem (see Billingsley (1986)) provides a link between assumptions on the $v_i(t)$'s and assumptions on v_t . For example, if idiosyncratic shocks are i.i.d. across units, then the cross-section distribution of idiosyncratic shocks is equal to the probability distribution generating individual shocks. Another example is when the $v_i(t)$'s are of the form $\gamma_i w_i(t)$, with the $w_i(t)$'s i.i.d. across units and γ_i a fixed, unit specific parameter (that could depend on θ_i and λ_i). In this case v_t has the same probability distribution as the product of the independent random variables Γ and w_t , where Γ corresponds to the cross-section distribution γ_i 's, and w_t to the common distribution of $w_i(t)$'s.

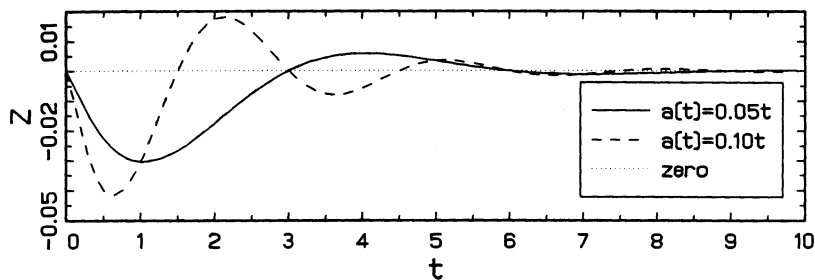
4.2. Speed of Convergence

Figure 1 suggests that convergence is faster when the variance of idiosyncratic shocks, relative to the common bandwidth, is larger. It also shows that the speed at which the economy's aggregate behavior approaches that of an economy with no friction—as measured by $Z^*(t)$ and $J^*(t)$ —does not depend on the sample path of the common shock $a(t)$. We illustrate these issues with an example.

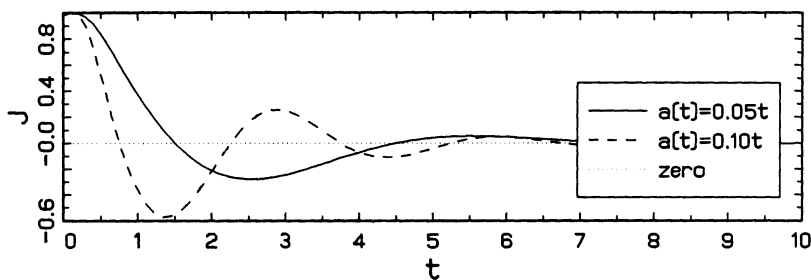
Suppose the economy's aggregate deviation is at its steady state, when an increase in the variance of shocks leads all units to increase their bandwidths by 50%, and that the new idiosyncratic shocks follow a Brownian motion with instantaneous standard deviation equal to 5%. From the symmetry assumption it follows that c_0 is uniform on $[1/4, 3/4)$. Figure 2a shows the resulting paths of the aggregate deviation $Z(t)$ for two economies which only differ in the realizations of the common shock, $a(t)$. The explicit dependence of $g(a(t), t)$ on t (via v_t) is reflected in the dampening of the oscillations of the sample paths of $Z(t)$. The dependence of g on $a(t)$ determines the speed at which the actual sample paths oscillate; the number of oscillations grows with the speed at which common shocks accumulate. Figure 2b illustrates the corresponding paths of $J(t)$.

The convergence mechanism we consider in this section ensures that the cross-section distribution of units' positions within their cycle converges to a distribution U uniform on $[0, 1)$. It is therefore not surprising that the summary variables $Z^*(t)$ and $J^*(t)$ are closely related to particular notions of distance between c_t and U . Since the corresponding relation for $J^*(t)$ can be derived intuitively, we only consider this case. From our discussion in Section 3, we have that $J(t)$ is equal to $f_{c_t}(1^-) - 1$. The index $J^*(t)$ is obtained by maximizing $\partial g / \partial a$ over all values of $a \geq a(t)$ and all values of $s \geq t$. Modifying the value of

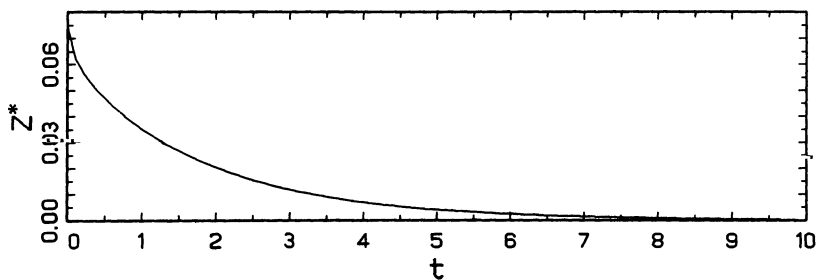
2(a)



2(b)



2(c)



2(d)

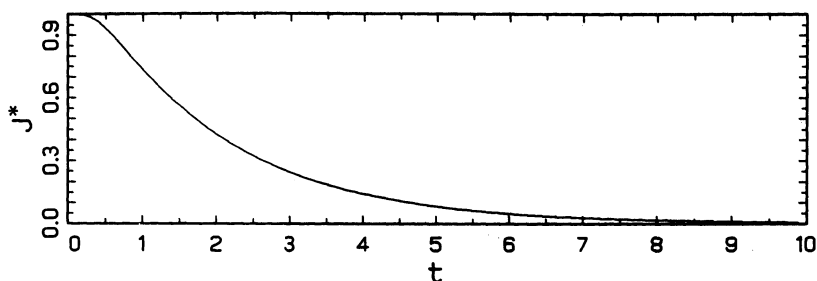


FIGURE 2.

a for a fixed instant in time s rotates the density of c_s , without affecting its shape; for this see Figure 1c and imagine joining both ends of the x -axis to form a circular diagram, as in Caplin and Spulber (1987). It follows that $\sup_a |\partial g(a, s) / \partial a|$ is equal to $\sup_a |f_{c_s}(a) - 1|$. The latter expression is the sup-distance between the densities of c_s and U , which we denote by $R(c_s, U)$. It is equal to the largest relative error made when approximating the distribution of c_s by a distribution uniform on $[0, 1]$.¹⁴ We therefore have that $J^*(t) = \sup_{s \geq t} R(c_s, U)$. Figure 1b indicates that it is quite likely that $R(c_s, U)$ decreases monotonically over time. This is indeed true when the v_i 's have independent increments, as is shown in Proposition A4 in the Appendix. It then follows that $J^*(t) = R(c_t, U)$, that is, that the largest (percentage) error made when approximating the probability of an event under c_t by the corresponding probability under U is equal to the largest sensitivity of the aggregate deviation to small common shocks over all possible future sample paths of $a(t)$.

Figures 2c and 2d show the trajectories of $Z^*(t)$ and $J^*(t)$ that correspond to Figures 2a and 2b. These do not depend on the particular paths of $a(t)$. It follows from the formulas we derive for the summary statistics in the Appendix that the speed of convergence increases with the relative importance of idiosyncratic shocks compared with the common bandwidth. For example, if in the experiment of Figure 2 $x_i(t)$ and $x_i^*(t)$ are the logarithms of economically meaningful variables and time is measured in years, then it takes about 18 years before $J^*(t)$ is below 5 percent when σ/λ is equal to 0.1; if σ/λ is equal to 0.5 it takes only about 9 months.¹⁵

In Proposition A1 in the Appendix, we provide general expressions for the indices used to construct these figures. They are all expressed in terms of the Fourier coefficients of v_i , and show that, loosely speaking, the smaller the Fourier coefficients, the faster all indices converge to zero. This can be understood in terms of the example given in Figure 1 above, since Fourier coefficients measure how fast v_i spreads out.¹⁶ Moreover, in the particular case where idiosyncratic shocks have independent increments, all the indices converge to zero at the same rate as $|k|^t$, where k denotes the first nontrivial Fourier

¹⁴ Formally:

$$R(c_s, U) = \sup_{\mathcal{A}} \left| \frac{\Pr\{c_s \in \mathcal{A}\}}{\Pr\{U \in \mathcal{A}\}} - 1 \right|,$$

where the supremum is taken over all Borel sets with positive Lebesgue measure. The proof may be found in Caballero and Engel (1989a).

¹⁵ We have limited our attention to cases where the economy converges to the steady state, but the same approach can be used when this does not happen. In Caballero and Engel (1989b) we show that when the $v_i(t)$'s are stationary, the synchronizing features of large aggregate shocks can only be partially undone by stationary idiosyncratic shocks.

¹⁶ Given a random variable X , the real and imaginary parts of its first Fourier coefficient are equal to the expected value of $\cos(2\pi X)$ and $\sin(2\pi X)$. Since the sine and cosine functions are periodic, these expectations are equal to those of $\cos(2\pi X(\bmod 1))$ and $\sin(2\pi X(\bmod 1))$ and therefore measure how near to a uniform distribution the random variable X is after being folded back onto the unit interval.

coefficient of v_1/λ that differs from zero.¹⁷ Hence the speed of convergence is faster, the smaller the first nontrivial Fourier coefficient of v_1/λ . For example, when idiosyncratic shocks follow a Brownian motion with instantaneous variance σ^2 , we have that $|k| = \exp(-2\pi^2\sigma^2/\lambda^2)$. Since the variance of the random variable that is folded back into the unit interval (see equation (4)) is $(\sigma/\lambda)^2$, it is not surprising that convergence is faster when this ratio is larger.

5. CONVERGENCE AND STRUCTURAL HETEROGENEITY

Structural heterogeneity—namely, differences in bandwidths and sensitivity parameters—is a second source of convergence. It ensures convergence by itself, even if no idiosyncratic uncertainty is present. It also adds various new features to the analysis of convergence and speed of convergence. Most prominently, aggregate shocks stop being irrelevant—as was the case in Section 4—and become the driving force behind convergence.

In this section we isolate structural heterogeneity as a source of convergence, by assuming that there are no idiosyncratic shocks. We consider both sources of convergence simultaneously in Section 6. We find it convenient to study separately the cases where differences in bandwidths and differences in sensitivity parameters are the only sources of convergence. We begin with the former case.

5.1. *Heterogeneous Bandwidths*

When structural heterogeneity due to different bandwidths is present, equation (3) may be used to show that

$$(6) \quad Z(t) = \int \lambda f_A(\lambda) \left\{ \frac{1}{2} - C(t|\lambda) \right\} d\lambda,$$

where $f_A(\lambda)$ denotes the probability density of bandwidths and $C(t|\lambda)$ the average position within their cycle of the “sector” of the economy formed by units with bandwidths equal to λ .

Equation (6) shows that, as mentioned in Section 3, units with a larger bandwidth have a larger weight when determining the deviation of the aggregate from its frictionless counterpart. The weight is proportional to both the size of the bandwidth and the size of the sector. This equation also shows that the aggregate path of the economy may converge to that of its frictionless counterpart in one of two ways. First, convergence takes place if units within each sector approach a distribution uniform on their common bandwidth. Each sector then behaves as in a frictionless economy, and adding over all sectors shows that the economy’s aggregate mimics that of its frictionless counterpart. Convergence occurs in this way when the density of idiosyncratic shocks spreads out without limit as time passes (see Section 4). Yet convergence may take place

¹⁷ When we say that $g(t)$ converges to zero at the same rate as a positive decreasing function $h(t)$, we mean that

$$0 < \lim_{t \rightarrow +\infty} \left(\limsup_{u \geq t} |g(u)|/h(t) \right) < +\infty.$$

even when the aggregate deviation of units with the same bandwidth does not converge at all, but synchronization among the aggregate deviations of different bandwidths breaks down over time. This is the case with sufficient differences in bandwidths.

5.1.1. Convergence

We start our discussion of convergence by presenting an example where differences in bandwidths are the only source of convergence. All θ_i 's are the same, there are no idiosyncratic shocks, and all units start off at the beginning of their cycle. We consider a cross-section distribution of the inverse-bandwidths—the $1/\lambda_i$'s—that is uniform on $[10, 20]$ and assume that the x_i 's and x_i^* 's are the logarithms of economically meaningful variables. Bandwidths therefore vary between 5 and 10 percentage points. Since there are no idiosyncratic shocks, and all units start off at the beginning of their cycle, we may imagine that there is only one unit in each sector. The deviation of any given sector does not approach zero; it exhibits cycles that do not dampen out over time.

As common shocks begin to accumulate, units with different bands move in a fully synchronized manner within their bandwidths until they start completing their first cycle (when $a(t) = 0.05$). The times at which units complete their cycles vary because bandwidths differ across units; this is the source of convergence in this example.

From equation (5) we have that $c_t = (a(t)/\Lambda) \pmod{1}$, therefore the distribution of c_t is uniform on $[0, 1)$ every time accumulated common shocks are equal to a multiple of 0.1. It departs from this distribution after every visit, yet every time by less. A visit to the uniform distribution is characterized by the fact that the correlation between units' positions within their cycle and their bandwidths decreases when compared to the previous visit. Figures 3a and 3b show the paths of $Z(t)$ and $J(t)$. The discontinuities in $J(t)$ (the vertical lines in Figure 3b) are due to the fact that the density of $1/\Lambda$ is not continuous at its endpoints. Modifying this density slightly at these points would lead to the same qualitative behavior without jumps. The figure shows that the aggregate deviation, $Z(t)$, and its sensitivity to small common shocks, $J(t)$, oscillate on their way to zero.

When there is no stochastic heterogeneity, both $Z(t)$ and $J(t)$ only depend on time through the current value of $a(t)$; $g(a, t) \equiv g(a)$ remains constant as t varies (see Proposition 1). Hence the path of $Z^*(t) = \sup_{a \geq a(t)} |g(a)|$ and $J^*(t) = \sup_{a \geq a(t)} |g'(a)|$ are both equal to the envelopes of the sample path of $Z(t)$ and $J(t)$. Figures 3c and 3d show how $Z^*(t)$ and $J^*(t)$ evolve over time. This example also serves to show that convergence may take place even if units' initial positions within their cycles are highly correlated with their bandwidths. Structural heterogeneity achieves convergence by breaking down the correlation between the aggregates of different sectors.

Consider any cross-section distribution of $1/\Lambda$ that has a sufficiently smooth density. Partition the set of possible bandwidths into a finite number of

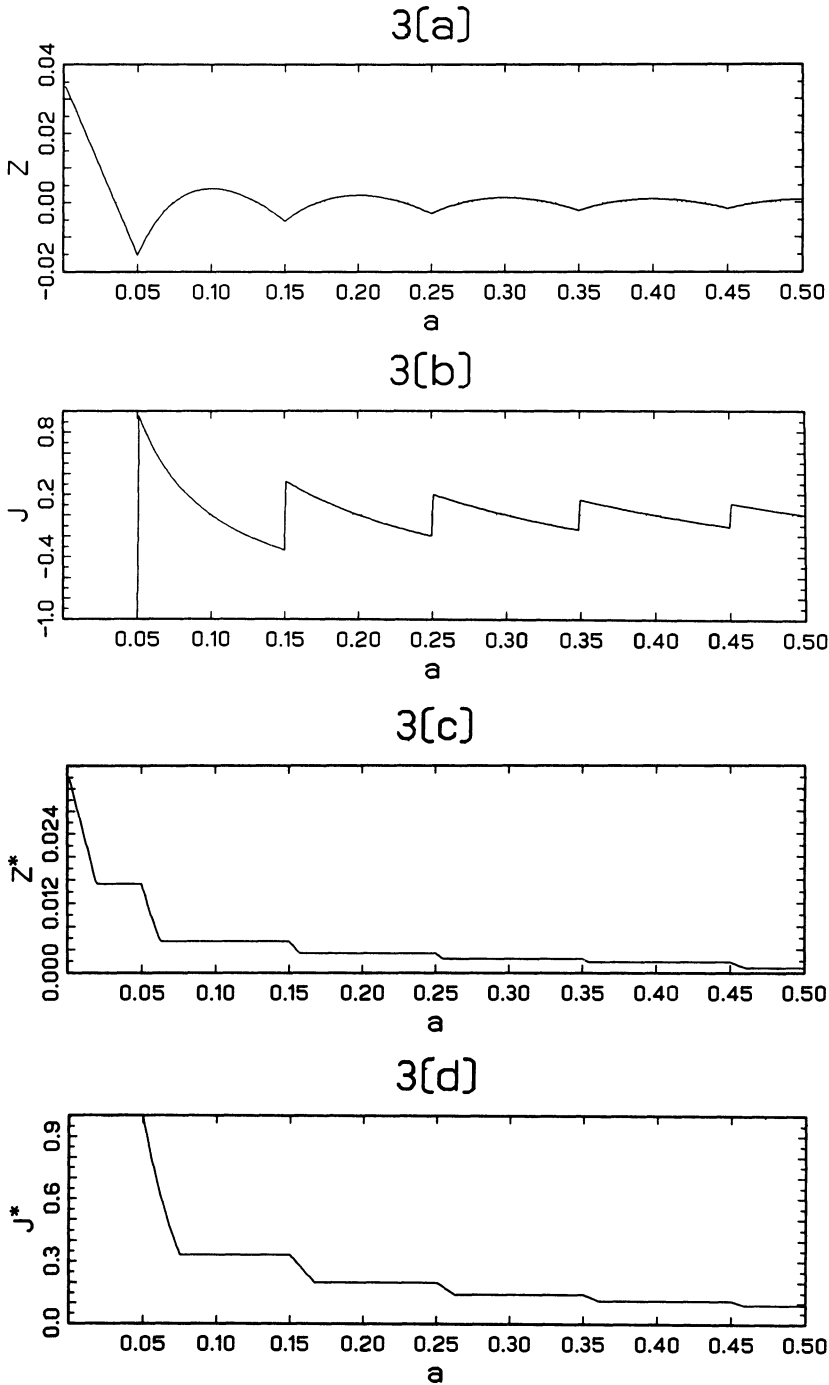


FIGURE 3.

intervals, and approximate the cross-section distribution of bandwidths within each interval by a uniform distribution. The argument given above applies to the sector composed of units with bandwidths in any one interval; these units' aggregate deviation therefore converges. It follows that the behavior of the entire economy's aggregate converges to that of its counterpart without friction. This argument explains why, when differences in bandwidths is the only source of heterogeneity, convergence takes place when the inverse of units' bandwidths have a smooth density. Since our model has a continuum of units, this is a relatively weak assumption.

The previous argument is based on assuming that c_0 has all its mass concentrated at a point. It can be generalized to the case where c_0 and Λ are not "perfectly" correlated by requiring that the density of $1/\Lambda$, conditional on any value of c_0 , be sufficiently smooth.

ASSUMPTION 4—Smoothness (1): *The random variable Λ has finite expectation and the density of $1/\Lambda$, conditional on any value of c_0 , has bounded variation $V(\Lambda^{-1}|c_0)$ such that $E_{c_0}V(\Lambda^{-1}|c_0)$ is finite.*

Below we provide a proposition generalizing and formalizing the insights of this example.

PROPOSITION 4: *Suppose that differences in bandwidths and units' initial positions within their cycle are the only source of heterogeneity, and that Assumptions 1 and 4 hold. Then the economy's aggregate behavior converges to that of its counterpart with no friction and c_t converges to a distribution uniform on the interval $[0, 1)$.*

PROOF: See the Appendix.

Q.E.D.

5.1.2. Speed of Convergence

The example above shows that the rate at which the common shock $a(t)$ grows—which is irrelevant in the case of only stochastic heterogeneity—is crucial when heterogeneity in bandwidths is the only source of convergence. The mechanism that leads to convergence in this case is not based upon spreading units out, but on having them move around their cycles at different speeds. This mixing effect grows with $a(t)$.

The example above also shows that the distance between the cross-section distribution of units' positions within their cycle and a distribution uniform on $[0, 1)$ does not decrease monotonically over time. Even though the distribution of units within their cycle approaches a distribution uniform on $[0, 1)$, there are periods when units "catch up" with each other and the distance between c_t and its limiting distribution increases. This differs from what we saw in Section 4, since the distance between c_t and a distribution uniform on $[0, 1)$ decreases monotonically over time when stochastic heterogeneity is the only source of convergence and idiosyncratic shocks have independent increments.

When there is no stochastic heterogeneity, $Z^*(t)$ and $J^*(t)$ depend on t only through the value of $a(t)$; it follows that the speed of convergence grows with the rate at which aggregate shocks accumulate. It is shown in the Appendix that, under the assumptions of Proposition 4, $J^*(t)$ is bounded from above by $k/a(t)$ for some constant k that depends on how smooth the corresponding densities are. This bound cannot be improved upon; it is sharp when the cross-section distribution of units' bandwidths within their cycle is uniform.

5.2. Heterogeneous Sensitivity Parameters

When different sensitivity parameters are the only source of convergence, equation (3) may be used to show that

$$(7) \quad Z(t) = \lambda \int \left\{ \frac{1}{2} - C(t|\theta) \right\} f_{\theta}(\theta) d\theta,$$

where $f_{\theta}(\theta)$ denotes the probability density of sensitivity parameters and $C(t|\theta)$ the average position within their cycle of the "sector" of the economy formed by units with sensitivity parameter equal to θ . As in the case with different bandwidths, when differences in sensitivity parameters are the only source of heterogeneity, aggregate shocks achieve convergence by gradually eliminating the synchronization between sectoral aggregates instead of by having every sectoral aggregate deviation converge.

When all bandwidths are the same (without loss of generality $\lambda_i \equiv 1$) and there are no idiosyncratic shocks, equation (5) implies that $c_t = (c_0 + a(t)\Theta) \pmod{1}$; hence c_t converges to a distribution uniform on $[0, 1)$ because $a(t)\Theta$ flattens out without bound as aggregate shocks accumulate. The correlation between the position within their cycle of units with different sensitivity parameters decreases over time, since common shocks affect them differently and these differences accumulate.¹⁸ As long as Θ has a sufficiently smooth density, conditional on any value of c_0 , the economy's aggregate deviation converges to that of its frictionless counterpart.

ASSUMPTION 5—Smoothness (2): *The random variable Θ has a density $f_{\Theta}(\theta)$ such that $f_{\Theta}(\theta)$ and $\theta f_{\Theta}(\theta)$, conditional on any value of c_0 , have bounded variation $V(f_{\Theta}(\theta)|c_0)$ and $V(\theta f_{\Theta}(\theta)|c_0)$; and $E_{c_0} V(f_{\Theta}(\theta)|c_0)$ and $E_{c_0} V(\theta f_{\Theta}(\theta)|c_0)$ are both finite.*

PROPOSITION 5: *Suppose that differences in sensitivity parameters and units' initial positions within their cycle are the only source of heterogeneity, and that Assumptions 1 and 5 hold. Then the economy's aggregate behavior converges to*

¹⁸ Looking at a particular example—say, $c_0 \equiv 0$, $\Lambda \equiv 1$, and Θ uniform on $[1/2, 3/2]$ —helps build the intuition behind how convergence takes place in this case. Since such an analysis is entirely analogous to the one we made in Section 5.1, we omit it.

that of its counterpart with no friction and c_i converges to a distribution uniform on the interval $[0, 1)$.

PROOF: See the Appendix.

Q.E.D.

The speed at which the economy's aggregate converges to that of its frictionless counterpart increases with the rate at which $a(t)$ grows; it is shown in the Appendix that $Z^*(t)$ and $J^*(t)$ are both bounded from above by $k/a(t)$, where k depends on how smooth the corresponding densities are. This bound is sharp when sensitivity parameters have a uniform distribution.¹⁹

The mechanism that leads to convergence in this case combines those present when either idiosyncratic shocks or differences in bandwidths are the sole source of heterogeneity. On the one hand, aggregate shocks are the main determinant of convergence; on the other, these shocks achieve convergence by spreading out indefinitely the $x_i^*(t)$'s, as idiosyncratic shocks did in Section 4.

6. INTERACTIONS

We have found conditions under which stochastic and structural heterogeneity yield convergence separately. It follows that convergence is more likely to occur when both sources of heterogeneity are present. We formalize this intuition at the end of Section A2 in the Appendix.

The results on the speed of convergence are, however, far less transparent. There is a broad set of parameters for which the intuitive assertion that when a second mechanism is added, convergence speeds up, is valid; surprisingly, however, this is not universally true.

Figure 4 presents an example of this paradox. It shows that adding structural heterogeneity to stochastic heterogeneity may slow down the speed of convergence. In this example, idiosyncratic shocks follow a Brownian motion (with instantaneous variance equal to 0.4 and $1/\Lambda$ is normal with mean 0.4 and variance η^2). All units have the same sensitivity parameters and their initial distribution within their cycle is uniform on $[0, 0.2]$. Figure 4 shows the path of $J^*(t)$ for three values of the parameter η . It is apparent that—beyond a certain time threshold—convergence is faster when stochastic heterogeneity is the only source of convergence ($\eta = 0$) than when structural heterogeneity is also present ($\eta > 0$).²⁰

Figure 4 is best understood by comparing the aggregate deviation without structural heterogeneity, $Z(t)$, with the aggregate deviation of a sector composed of units with a common bandwidth larger than average after structural

¹⁹ It is interesting to note that if instead of Assumptions 4 and 5 we only assume that Λ and Θ have a density, then Propositions 4 and 5 continue holding but the corresponding rates of convergence may be arbitrarily slow.

²⁰ A similar phenomenon takes place for $Z^*(t)$.

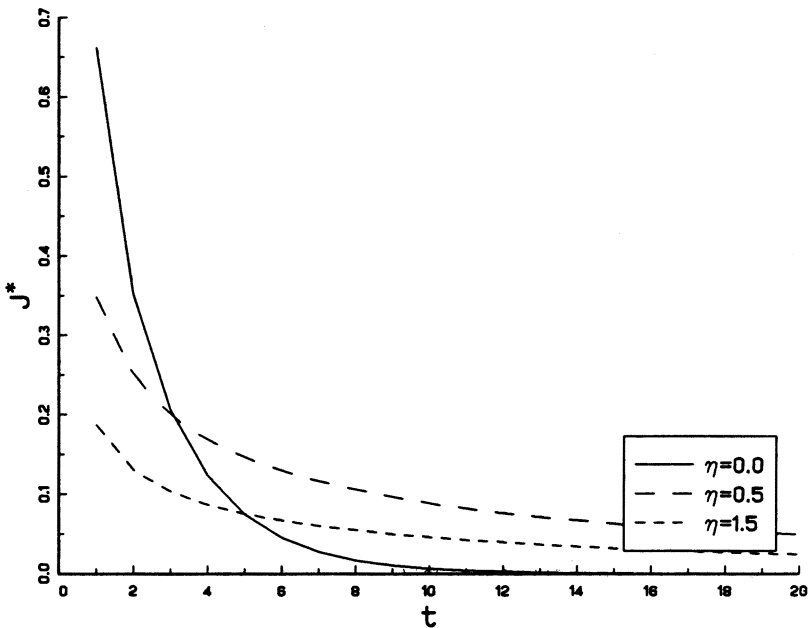


FIGURE 4.

heterogeneity is added. When structural heterogeneity is added to idiosyncratic uncertainty, the sectoral aggregates corresponding to larger bandwidths converge slower than $Z(t)$, since structural heterogeneity reduces the variance of their idiosyncratic shocks relative to their bandwidths and it is this ratio that determines the speed of convergence (see Section 4). For the same reason the sectoral aggregates corresponding to smaller bandwidths converge faster than $Z(t)$. Figure 4 shows an example where the slowdown of units with bandwidths larger than average dominates over the combined effect of the acceleration of units with bandwidths smaller than average and the decrease in synchronization between sectoral aggregates (see Section 5).

Perverse interactions may also be present when we add stochastic heterogeneity to an economy where structural heterogeneity—in the form of differences in bandwidths—leads to convergence by itself. This is best understood when we consider the case where there are no differences in sensitivity parameters and we group units into sectors according to the value of their idiosyncratic shock at time t , $v_i(t)$. Since the effect of $v_i(t)$ on units *within a sector* is the same as the effect of having a common shock equal to $a(t) + v_i(t)$ instead of $a(t)$, the discussion in Section 5 shows that sectors with positive $v_i(t)$'s are typically nearer to their steady state than they would be if there were no idiosyncratic shocks, while sectors with negative realizations are farther away. When adding sectoral aggregate deviations, structural heterogeneity decreases the degree of synchronization; yet it may happen that units with negative shocks determine

the overall speed of convergence. We have constructed examples where this is the case.²¹

Finally, we consider the case where idiosyncratic shocks interact with differences in sensitivity parameters. For simplicity we suppose that bandwidths are the same across units. If the $v_i(t)$'s are i.i.d. across units and independent from Θ , then adding idiosyncratic shocks speeds up convergence (this follows from Proposition A4 in the Appendix). Yet when $v_i(t)$ depends on θ_i , there are cases where adding structural heterogeneity—in the form of differences in sensitivity parameters—slows down the speed at which an economy with stochastic heterogeneity converges.

7. FINAL REMARKS

In this paper we study the dynamic behavior of an (S, s) economy where units face idiosyncratic shocks and differ in both their bandwidths and their responses to aggregate shocks. We develop a framework that provides a meaningful characterization of the out-of-steady state dynamics of an (S, s) economy, and study its convergence properties and the speed at which this occurs.

The major building block in our approach is to work with the *cross-section distribution* of units' positions within their cycle, *conditional on the sample path of the aggregate shock*. This distribution, combined with that of structural differences and sensitivity parameters, describes the actual state of the economy at a given instant in time and—if the number of units is sufficiently large—does not depend on the value taken by every particular agent's idiosyncratic shock but only on the common distribution function originating them. This insight follows from the Glivenko-Cantelli Theorem (see Billingsley (1986)); it allows us to apply results from probability theory when studying convergence and speed of convergence. Although we work with (S, s) rules, the summary variables $Z(t)$, $J(t)$, $Z^*(t)$, and $J^*(t)$ should be applicable to a much broader set of circumstances where microeconomic frictions influence aggregate dynamics.

We have implicitly assumed in our analysis that the redefinition of initial conditions—i.e. whatever moves the economy away from its steady state—occurs infrequently enough so that the economy has time to converge back to its steady state; the insights developed here, however, apply even when this is not the case (see Caballero and Engel (1989a), and the working paper version of this paper, Caballero and Engel (1990)). In general, there is a permanent tension between the natural tendency for the economy's aggregate deviation to converge back to the steady state and the impact of repeated large (finite) aggregate shocks. Given any process generating the latter, the *average* distance of the

²¹ The argument given above assumes that units' idiosyncratic shocks are independent from their bandwidths. If v_i and Λ are correlated, the perverse effect described above may still happen. One exception, though, is when the $v_i(t)$'s are identically distributed except for a scale parameter that is proportional to λ_i ; in this case adding idiosyncratic shocks to differences in bandwidths always speeds up convergence. This follows from Proposition A4 and the Glivenko-Cantelli Theorem.

economy from the steady state decreases with an increase in the importance of stochastic and structural heterogeneity (with the caveats of Section 6).

The techniques developed here have already found applications beyond the framework of this paper. For example, Caplin and Leahy (1989) use them to prove convergence (up to a location parameter) in the context of a fully symmetric two sided (S, s) economy where heterogeneity is negligible; and Caballero and Engel (1989b) have used the concept of synchronization developed here to show that when strategic interactions are present, multiple equilibria can be ruled out once the cross-section distribution is sufficiently close to its steady state.

To conclude, we stress that the principle of conditioning on the aggregate in order to keep track of the evolution of the cross-section distribution is far more general than the framework of this paper. This may be one of the building blocks of future work on aggregation of heterogeneous units in the presence of nonvanishing correlation across units.

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APPENDIX A1. EXACT FORMULAS FOR THE SUMMARY VARIABLES

DEFINITION A1: Given nonnegative real numbers a and t , we define the random variables $C(a, t)$ and $Y(a, t)$ as follows:

$$(8) \quad C(a, t) = \left(c_0 + \frac{a\Theta + v_t}{\Lambda} \right),$$

$$(9) \quad Y(a, t) = [C(a, t)](\text{mod } 1);$$

with c_0 , v_t , Θ , and Λ as in Section 2. We then have that the function $g(a, t)$ defined in Section 2 is equal to $\frac{1}{2}E\Lambda - E\{\Lambda Y(a, t)\}$.

The following three propositions provide expressions to calculate $g(a, t)$ and $\partial g/\partial a$ when only one of the three sources of heterogeneity considered in this paper—idiosyncratic shocks, differences in bandwidths, and differences in sensitivity parameters—is present, and all units have the same initial position within their cycle. Following these results we show how a simple conditioning argument extends them to the case where more than one source of convergence is present and units differ in their initial positions within their cycle.

LEMMA A1: *Let X be a random variable whose density $f(x)$ has bounded variation. Then $X(\text{mod } 1)$ also has a density, $f_1(x)$, and $f_1(x) = \sum_k f(x+k)$.*

PROOF: This is a well known result in probability theory; for a proof under the assumptions made above, see Proposition 3.4 in Engel (1991). Q.E.D.

LEMMA A2: Let X denote a random variable whose characteristic function $\hat{f}(z)$ satisfies $\sum_{k \geq 1} |\hat{f}(2\pi k)| < +\infty$. Then

$$(10) \quad E[X(\text{mod } 1)] = \frac{1}{2} - \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \Im[\hat{f}(2\pi k)],$$

where $\Im[z]$ denotes the imaginary part of the complex number z and $x(\text{mod } 1)$ the difference between x and the largest integer less than or equal to x .

PROOF: The Fourier coefficients of X and $X(\text{mod } 1)$ are the same (see, e.g., Lemma 3.1 in Engel (1991)); hence the Fourier coefficients of $X(\text{mod } 1)$ are summable and $X(\text{mod } 1)$ has a continuous density, $f_1(x)$, with bounded variation. Applying Poisson's Summation Formula (see Butzer and Nessel (1971, p. 202) for the version being used here) we then have that $f_1(x) = \sum_k \hat{f}(2\pi k) e^{-i2\pi kx}$. Substituting this expression for $f_1(x)$ in $E[X(\text{mod } 1)] = \int x f_1(x) dx$, interchanging the order of integration and summation,²² and integrating the resulting terms, leads to equation (10). *Q.E.D.*

PROPOSITION A1: Suppose that $c_0 \equiv c$, $\Lambda \equiv 1$,²³ and $\Theta \equiv 1$ in equation (10), and assume that the density of v_t has a characteristic function, $\hat{f}_t(z)$, that satisfies $\sum_{k \geq 1} |\hat{f}_t(2\pi k)| < +\infty$. Then

$$(11) \quad g(a, t) = \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \Im[\hat{f}_t(2\pi k) e^{i2\pi k(c+a)}],$$

$$(12) \quad \frac{\partial g}{\partial a}(a, t) = 2 \sum_{k \geq 1} \Re[\hat{f}_t(2\pi k) e^{i2\pi k(c+a)}],$$

where $\Re[z]$ and $\Im[z]$ denote the real and imaginary parts of the complex number z .

PROOF: The expression for $g(a, t)$ follows directly from equation (10) in Lemma A1, letting $c + a + v_t$ play the role of X .

The expression for $\partial g/\partial a$ can be derived formally by differentiating the sum in (11) term by term. The change in the order of summation and differentiation is made rigorous by applying Lebesgue's Dominated Convergence Theorem (see, e.g., Billingsley (1986)) and using the assumption that the Fourier coefficients of v_t are summable. *Q.E.D.*

PROPOSITION A2: Suppose that $c_0 \equiv c$ with $c \in [0, 1)$, $v_t \equiv 0$, and $\Theta \equiv 1$, that $E\Lambda$ is finite, and that $\lambda_{f_\Lambda}(\lambda)$ has finite total variation, where $f_\Lambda(\lambda)$ denotes the density of Λ . Then

$$(13) \quad g(a, t) = \left(\frac{1}{2} - c\right) E\Lambda - a + \sum_{k \geq 1} \int_0^{a/(k-c)} \lambda f_\Lambda(\lambda) d\lambda,$$

$$(14) \quad \frac{\partial g}{\partial a}(a, t) = -1 + \sum_{k \geq 1} \frac{a}{(k-c)^2} f_\Lambda\left(\frac{a}{k-c}\right).$$

PROOF: The expression for $g(a, t)$ follows from

$$\begin{aligned} E[AY(a, t)] &= \int \lambda \left[\left(c + \frac{a}{\lambda}\right) (\text{mod } 1) \right] f_\Lambda(\lambda) d\lambda \\ &= \sum_{k \geq 1} \int_{a/(k+1-c)}^{a/(k-c)} \lambda \left(c + \frac{a}{\lambda} - k\right) f_\Lambda(\lambda) d\lambda + \int_{a/(1-c)}^{+\infty} \lambda \left(c + \frac{a}{\lambda}\right) f_\Lambda(\lambda) d\lambda \\ &= cE\Lambda + a - \sum_{k \geq 1} k \int_{a/(k+1-c)}^{a/(k-c)} \lambda f_\Lambda(\lambda) d\lambda \\ &= cE\Lambda + a - \sum_{k \geq 1} \int_0^{a/(k-c)} \lambda f_\Lambda(\lambda) d\lambda. \end{aligned}$$

²² This step is based on Fubini's Theorem. It is here where we use the assumption that the Fourier coefficients of X are summable.

²³ If $\Lambda \equiv \lambda$ we let v_t/λ play the role of v_t .

The expression for $\partial g/\partial a$ is obtained by differentiating the latter expression. The assumption that $\lambda f_\Lambda(\lambda)$ has bounded variation is used when interchanging the order of differentiation and summation. *Q.E.D.*

PROPOSITION A3: Suppose that $c_0 \equiv c$ with c in $[0, 1)$, $v_i \equiv 0$, and $\Lambda \equiv 1$, and that $\theta f_\Theta(\theta)$ has bounded variation, where $f_\Theta(\theta)$ denotes the density of Θ . Then

$$g(a, t) = \frac{1}{2} - a + \sum_k (k - c) \left\{ F_\Theta \left(\frac{k + 1 - c}{a} \right) - F_\Theta \left(\frac{k - c}{a} \right) \right\},$$

$$\frac{\partial g}{\partial a}(a, t) = -1 + \frac{1}{a^2} \sum_k (k - c) f_\Theta \left(\frac{k - c}{a} \right),$$

where $F_\Theta(\theta)$ denotes the cumulative distribution function of Θ .

PROOF: The expression for $g(a, t)$ is obtained using Lemma A1 as follows:

$$\begin{aligned} E[(c + a\Theta)(\text{mod } 1)] &= \frac{1}{a} \sum_k \int_0^1 x f_\Theta \left(\frac{x + k - c}{a} \right) dx \\ &= \sum_k \int_{(k-c)/a}^{(k+1-c)/a} (ua - k + c) f_\Theta(u) du \\ &= aE\Theta - \sum_k (k - c) \int_{(k-c)/a}^{(k+1-c)/a} f_\Theta(u) du \\ &= a - \sum_k (k - c) \left\{ F_\Theta \left(\frac{k + 1 - c}{a} \right) - F_\Theta \left(\frac{k - c}{a} \right) \right\}. \end{aligned}$$

The expression for $\partial g/\partial a$ is obtained by differentiating the latter expression; the assumption that $\theta f_\Theta(\theta)$ has bounded variation is used when interchanging the order of summation and differentiation. *Q.E.D.*

Generalizations

When more than one source of convergence is present, we obtain expressions for $g(a, t)$ and $\partial g/\partial a$ by calculating $E[\Lambda Y(a, t) | X = x]$ —and the corresponding derivative—for an appropriately chosen random vector X using one of the above propositions, and then taking expected value with respect to X . This argument is based on the fact that $E[f(X, Y)] = E_X[f(X, Y) | X = x]$. It requires that the corresponding proposition's regularity conditions hold conditional on $X = x$ for any x , and, when calculating $\partial g/\partial a$, that they hold uniformly in x . Next we show explicitly how to apply this argument for every one of the propositions derived above.

1. If we want to apply Proposition A1 we let $((v_i/\lambda) | c_0 = \bar{c}, \Lambda = \lambda, \Theta = \theta)$ play the role of v_i and $\bar{c} + (a\theta/\lambda)$ the role of c . This leads to the following expressions:

$$g(a, t) = \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \Im \left[E_\Lambda \left\{ \Lambda e^{i2\pi k C(a, t)} | \Lambda = \lambda \right\} \right],$$

$$\frac{\partial g}{\partial a}(a, t) = 2 \sum_{k \geq 1} \Re \left[E_\Theta \left\{ \Theta e^{i2\pi k C(a, t)} | \Theta = \theta \right\} \right],$$

with $C(a, t)$ defined in (8).

2. If we want to apply Proposition A2 we let $(\Lambda | c_0 = \bar{c}, \Theta = \theta, v_i = v)$ play the role of Λ , $a\theta + v$ that of a , and \bar{c} that of c .

3. If we want to apply Proposition A3 we let $(\Theta | c_0 = \bar{c}, \Lambda = \lambda, v_i = v)$ play the role of Θ , $\bar{c} + (v/\lambda)$ that of c , and (a/λ) that of a .

APPENDIX A2. CONVERGENCE

LEMMA A3: Let X denote a random variable that has a density, $f(x)$, with finite total variation equal to $V(f)$. Denote the density of $X(\text{mod } 1)$ by $f_1(x)$, and the sup-distance between $X(\text{mod } 1)$ and a distribution uniform on $[0, 1]$ by $R(X(\text{mod } 1), U)$. Then

$$(15) \quad |E[X(\text{mod } 1)] - \frac{1}{2}| \leq \frac{1}{2}R(X(\text{mod } 1), U),$$

$$(16) \quad R(X(\text{mod } 1), U) \leq \frac{1}{2}V(f).$$

PROOF: Equation (15) follows from

$$\begin{aligned} |E[X(\text{mod } 1)] - \frac{1}{2}| &= \left| \int_0^1 (f_1(x) - 1)x \, dx \right| \\ &\leq \int_0^1 |f_1(x) - 1|x \, dx \\ &\leq \int_0^1 R(X(\text{mod } 1), U)x \, dx \\ &= \frac{1}{2}R(X(\text{mod } 1), U). \end{aligned}$$

For a proof of equation (16), which is due to Kemperman, see Theorem 3.9.c in Engel (1991). *Q.E.D.*

LEMMA A4: 1. Suppose that X and Y are random variables such that $(X|Y=y)$ has a density with finite total variation $V(X|Y=y)$ for all values of y and $E_Y V(X|Y=y)$ is finite. Then X has a density with finite total variation $V(X)$ and $V(X) \leq E_Y V(X|Y=y)$.

2. Let $f(x)$, $f_a(x)$, and $f_c(x)$ denote the densities of the random variables X , aX , and $X + c$, with $a > 0$, and suppose that $f(x)$ has finite total variation $V(f)$. Then $f_a(x)$ and $f_c(x)$ also have finite total variation and $V(f_a) = V(f)/a$; $V(f_c) = V(f)$.

PROOF: The proof of the first statement is analogous to that of Proposition 4.6 in Engel (1991). The proof of the second statement is trivial. *Q.E.D.*

Proof of Proposition 3

Let $R(c_t, U)$ denote the sup-distance between c_t and a distribution uniform on the unit interval, and $V(X)$ denote the total variation of the density of the random variable X . From Lemma A3 it follows that $R(c_t, U) \leq \frac{1}{2}V(c_0 + (v_t/\lambda))$; Lemma A4 and Assumption 2 then imply that $R(c_t, U) \leq \frac{1}{2}V(v_t/\lambda) = \lambda/2V(v_t)$. Assumption 3 now implies that c_t converges—in the sup-distance—to U . Since $J^*(t) = \sup_{s \geq t} R(c_s, U)$ (see Section 4), this is equivalent to having $J^*(t)$ converge to zero. That $Z^*(t)$ also tends to zero follows from the fact that, due to Lemma A3, it is bounded by $\sup_{s \geq t} R(c_s, U)$. *Q.E.D.*

Proof of Proposition 4

We begin by noting that, since in this case c_t only depends on t through the value of $a(t)$, convergence of $Z^*(t)$ and $J^*(t)$ to zero is equivalent to convergence of $Z(t)$ and $J(t)$ to zero. The same holds for Proposition 5.

Let $R(c_t, U)$ denote the sup-distance between c_t and a distribution uniform on the unit interval and $V(X)$ denote the total variation of the density of the random variable X . From Lemma A3 it follows that $R(c_t, U) \leq V(c_0 + (a(t)/\Lambda))$. Using Lemma A4 we then have that $R(c_t, U) \leq E_{c_0} V(\Lambda^{-1}|c_0)/2a(t)$; therefore c_t converges to U and $J^*(t)$ converges to zero at least as fast as $a(t)$.

Theorem 4.4—due to Hopf—in Engel (1991) shows that (c_t, Λ) converges in the weak-star topology to (U, Λ) , with Λ independent from U . It follows that $E(c_t, \Lambda)$ converges to $\frac{1}{2}E\Lambda$, and therefore $Z(t)$ converges to zero. *Q.E.D.*

Proof of Proposition 5

It follows from Lemmas A3 and A4 that $R(c_i, U) \leq k/a(t)$, with $k = E_{c_0} V(f_\theta | c_0)/2$; therefore c_i converges to a distribution uniform on $[0, 1)$ and $Z(t)$ converges to zero.

To show that $J(t)$ converges to zero, we first consider the case where $c_0 \equiv c$. That $J(t)$ converges to zero in this case follows from the expression we derived for $\partial g/\partial a$ in Proposition A3 and the fact that $\sum_k ((k-c)/a^2) f_\theta((k-c)/a)$ converges to $\int \theta f_\theta(\theta) d\theta \equiv E\theta$ because $\theta f_\theta(\theta)$ is Riemann-integrable. Furthermore, since $E\theta = 1$ we have:

$$|J(t)| = \left| \sum_k \frac{k-c}{a^2} f_\theta\left(\frac{k-c}{a}\right) - \sum_{(k-c)/a}^{(k+1-c)/a} \theta f_\theta(\theta) d\theta \right|$$

$$\leq \frac{1}{a} \sum_k \left| \frac{k-c}{a} f_\theta\left(\frac{k-c}{a}\right) - \theta_k f_\theta(\theta_k) \right|,$$

with $(k-c)/a \leq \theta_k \leq (k+1-c)/a$. It follows that $J(t) \leq V(\theta f_\theta(\theta))/a(t)$; therefore the speed of convergence of $J^*(t)$ —and, due to Lemma A3 that of $Z^*(t)$ too—is bounded from above by $1/a(t)$.

The case where c_0 is not equal to a spike follows from the previous argument by conditioning on the value of c_0 and using the hypotheses according to which $E_{c_0} V(\theta f_\theta(\theta) | c_0)$ is finite. Q.E.D.

Generalizations

Propositions 3, 4, and 5 can be extended easily to the case where more than one source of heterogeneity is present using a conditioning argument analogous to the one we used at the end of Section A1. Q.E.D.

PROPOSITION A4: *Suppose that X and Y are independent random variables such that the density of X has bounded variation. Then the sup-distance between (X + Y)(mod 1) and a distribution U uniform on [0, 1] is less than or equal to the sup-distance between X(mod 1) and U.*

PROOF: Let $f_X(u)$ and $f_{X+Y}(u)$ denote the densities of $X(\text{mod } 1)$ and $(X + Y)(\text{mod } 1)$, and $F_Y(u)$ the cumulative distribution function of $Y(\text{mod } 1)$. From Lemma A1 and the independence assumption it follows that $f_{X+Y}(u) = \int f_X(u-v) dF_Y(v)$. Hence:

$$|f_{X+Y}(u) - 1| = \left| \int_0^1 f_X(u-v) dF_Y(v) - 1 \right|$$

$$= \left| \int_0^1 (f_X(u-v) - 1) dF_Y(v) \right|$$

$$\leq \int_0^1 |f_X(u-v) - 1| dF_Y(v)$$

$$\leq R(X(\text{mod } 1), U).$$

The desired conclusion follows by taking the supremum over all u in $[0, 1]$. Q.E.D.

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