Price Theory for Incomplete Markets*

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We provide a price theory for incomplete markets that extends the traditional Walrasian analysis. We derive formulas expressing the consumption response to current and future changes in interest rates and income. Our analysis provides a natural decomposition of these responses into substitution and income effects with structural interpretation, emphasizing statistics such as the marginal propensity to save and local measures of prudence in utility. We handle general uncertainty in a compact and intuitive manner by adjusting probability distributions: a risk-adjusted probability, commonly used in finance, and a novel prudence-adjusted probability, specifically useful for incomplete markets. Our formulas reveal various cross-restrictions implied by the theory on consumer behavior. Numerical explorations show that the new statistics we identify matter significantly to understand aggregate demand in incomplete markets, beyond the impact of heterogeneous marginal propensities to consume or binding borrowing constraints.

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Dear Prudence, won't you come out to play? —The Beatles, The Beatles

1 Introduction

Demand theory provides the framework to understand behavior stemming from rational consumers operating within classical Walrasian Price Theory settings. The central Slutsky equation explains agents' responses to price changes by decomposing them into income and substitution effects. Other well-known results such as Slutksy symmetry, homogeneity, budget exhaustion have been usefully invoked to derive results, limit free parameters, or impose or test the restrictions implied by rationality. However, outside classical settings these results do not necessarily hold. A popular class of models features Walrasian static markets, linked over time by imperfectly and incomplete asset markets, including borrowing constraints and lack of insurance. The goal of this paper is to generalize the classical demand theory framework and flesh out the behavioral predictions and restrictions in such settings.

Our baseline model captures all the essential elements of incomplete market models. Agents care about the discounted sum of expected utility from a single consumption good; we make no special assumption on the utility function. Agents face uncertainty about their future income; we make no special assumption on uncertainty. They can save and borrow in a single risk-free asset and are subject to a borrowing constraint. Markets are incomplete for two reasons: agents cannot insure against future realizations of uncertainty and the borrowing constraint may be binding. Technically, both these frictions imply that the sequence of budget constraints cannot be reduced to a single budget constraint, as in traditional Walrasian Demand theory. We later discuss an extension that allows for multiple goods within each period, multiple risky assets, as well as shocks to the discount factor, the utility function and the borrowing constraint; all our results generalize to these extensions.

Our first set of results provide expressions for the change in current consumption from changes in current and future interest rates and income, along with a decomposition into substitution and an income effects. Naturally, changes in income only produce income effects, while changes in interest rates induce both income and substitution effects. Our formulas provide the primitives and sufficient statistics that shape these responses. They also provide a way to compare the responses to changes in income and interest rates at different time horizon and across states. For example, one might compare a change in the interest today versus one two years from now, or a increase in future income in a low state versus a high state.

It is useful to first consider a Walrasian scenario without uncertainty. In this case, our expressions for the income effect are completely standard, equal to the current marginal propensity to consume times the net present value of income. On the other hand, our expression for the substitution effect is novel and shows that a change in the interest rate at some future date T is discounted by the marginal propensity to save from the current period $t = 0$ to $t = T$, that is, the product of the marginal propensities to save in all intermediate periods. Intuitively, this discounting captures how connected the agent is to this future date. This, in turn, affects the responsiveness in the present of changes in interest rates at this future date. An obvious and extreme case is that of a complete disconnection when the borrowing constraint becomes binds, since then the marginal propensity to save is zero. Our result shows that even when the borrowing constraint does not bind, the marginal propensity to save captures the degree of connection and response to future interest rate changes.

The next challenge is to extend these results to incorporate uncertainty. Our paper show how this can be done in a simple way. Indeed, our expressions are simply weighted averages of the deterministic ones and can be expressed as the expectation using reweighed probabilities. Indeed, we introduce two new probability distributions, or changes of measure, that distort the original objective probability measure to incorporate the agent's preference over both risk and prudence. The first probability is a risk-adjusted one familiar from finance theory: it reweighs states of the world by the marginal utility of income (i.e. first derivative). We show that this is the appropriate measure to compute substitution effects. It also is the appropriate measure to compute welfare impacts. The second probability is a novel prudence-adjusted one that we develop to evaluate income changes. It captures the precautionary effects inherent to incomplete market settings: reweighing states of the world by the curvature of utility with respect to income (i.e. second derivative). Note that this second adjustment is absent in the certainty-equivalent permanent-income model, that is with quadratic utility and absent borrowing constraints. In that case, the probability distribution is the original one, without any adjustments. However, this is a borderline case and we show that an adjustment is generally required for more standard utility functions and borrowing constraints that induce precautionary savings motives.

Let us take a step back and discuss our two changes of measure by focusing on the effects on welfare and consumption of changes in income. In the absence of risk, the effect of a future income change on welfare and consumption is simply summarized by its impact on lifetime wealth. Indeed the consumption response is given by the marginal

	Perfect Foresight	Complete Markets	Incomplete Markets
Consumption	$\frac{\partial c_t}{\partial y_{t+s}} = MPC_t \times \frac{1}{\prod_0^{s-1} R_{t+k}}$	$\frac{\partial c_t}{\partial y(\theta_{t+s})} = MPC_t \times \frac{Q_t(\theta_{t+s})}{\prod_0^{s-1} R_{t+k}}$	$\frac{\partial c_t}{\partial y(\theta_{t+s})} = MPC_t \times \frac{Q_t^I(\theta_{t+s})}{\prod_0^{s-1} R_{t+k}}$
Welfare	$\frac{\partial V_t}{\partial y_{t+s}} = MVW_t \times \frac{1}{\prod_0^{s-1} R_{t+k}}$	$\frac{\partial V_t}{\partial y(\theta_{t+s})} = MVW_t \times \frac{Q_t(\theta_{t+s})}{\prod_0^{s-1} R_{t+k}}$	$\frac{\partial V_t}{\partial y(\theta_{t+s})} = MVW_t \times \frac{Q_t(\theta_{t+s})}{\prod_0^{s-1} R_{t+k}}$

Table 1: Consumption and welfare responses to future state-dependent income changes.

propensity to consume (MPC) times the net present value (NPV) of the marginal income change, while the welfare impact is the marginal value of wealth (MVW) times the NPV of the income change. In this sense, the discount in the NPV can be interpreted as the price, at initial time, of future income.

This can be easily extended to uncertainty if markets are complete by using the Arrow-Debreu prices. In each period, we can consider a probability measure that weighs states in proportion to its price. It is then possible to determine the equivalent wealth change by the expected present value using Q, or simply Q-NPV for short, and with it the consumption and welfare responses. This discussion is summarized in the first two columns of Table 1.

What about incomplete markets? As it turns out, we can still construct a risk-adjusted Q, with the marginal utility (weighted by probability) playing the role of implicit Arrow-Debreu prices. Once again the Q-NPV of future income changes determines the change in welfare. However, it no longer determines the consumption response to future income changes. To see why, imagine an employed agent saving for unemployment. An increase in unemployment benefits makes the agent richer, the income effect, but also relaxes the precautionary motive to save. This precautionary effect is not generally captured by the risk-adjusted Q. This has to do with the fact that prudence and risk aversions are distinct properties of preferences. Indeed, while risk aversion is related to concavity of utility, prudence is related to concavity in marginal utility function (Kimball, 1990).

To compute the response to income changes we construct a new prudence-adjusted probability Q^I and show that it typically overweighs "bad" states compared to Q. Indeed, we provide a formal result for transitory shocks, and confirm it in our numerical explorations for persistent shocks. This implies that an additional dollar in a state where consumption is low generates a larger consumption response under incomplete markets.

As mentioned, the risk- and prudence-adjusted probabilities are key to developing our price theory for incomplete markets. In particular, for interest rate changes the substitution effect is computed using Q while the income effect of these changes uses Q^I, just as it is for income changes. To highlight the role of Q^I versus Q, we decompose the income effect into a welfare effect, determined by the Q-NPV of interest and income changes, and the difference between the Q^I-NPV and the Q-NPV. This gives a decomposition into substitution, wealth and what we call a precautionary effect. The latter effect highlights the impact of incomplete markets.

At the origin of demand theory for Walrasian settings, Slutzky and Hicks provided two alternative compensation schemes. In particular, Slutzky envisioned providing transfers that would allow a previous consumption plan to remain feasible after price changes; the budget constraint then pivots through the original consumption choice. Hicks, instead, envisioned a compensation in terms of income designed to keep utility exactly constant. Although these schemes are different and generally lead to different responses, to a first-oder both schemes deliver exactly the same consumption responses.

Our substitution effect is a defined following the Slutzky compensation scheme, that provides income in real time to compensate for interest rate changes; the budget constraint pivots through the original consumption plan. For comparison sake, we define a Hicks-compensated effect by compensating for current and future interest using initial wealth only. Unlike the Walrasian case, with incomplete markets this compensated effect does not typically equal the substitution effect.[1] Indeed, the compensated effect equals the Slutzky-substitution effect defined earlier, plus the precautionary effect from the interest rate changes defined earlier, captures the precautionary effect present in incomplete markets through the difference between Q^I and Q.

Going beyond the impact on present consumption, we show how to compute the full impulse response of consumption. We also provide further cross-restrictions on behavioral responses relating the current response of consumption to future income and interest rates to the future responses to concurrent changes in these variables.

Finally, we revisit the properties of the Slutsky matrix and show that the substitution effects are symmetric. In this sense, Slutzky symmetry extends to incomplete markets settings. However, it turns out that the Hicks-compensated effect, which does not always equal the Slutzky-substitution effect, is not generally symmetric.

Our individual price theory provides a lens to inspect the mechanisms behind aggregate demand responses. We explore this by dissecting the aggregate response of consumption with heterogeneous agents. The literature has stressed the importance of accounting for the heterogeneity in marginal propensities to consume, as well as their in-

[1] In a Walrasian setting, these two compensation schemes are not identical and they do not have identical results for discrete changes, but they do have the same effect to a first order.

teraction with heterogeneous income shifts. First, we show that the heterogeneity in *risk valuation* Q and Q^I matters beyond MPC heterogeneity. Second, in line with our results for a single agent, the aggregate marginal propensity to save is shown to be crucial to assess the discounting of future interest rate changes.

Finally, we turn to a quantitative exploration in a standard Bewley-Huggett-Aiyagari model. We find that the direct effect of borrowing constraints is rather limited, attenuating the (partial equilibrium) effects of future interest rate cuts by less than 10%. In contrast, the impact of the marginal propensities to save is significant, almost halving the aggregate response to an interest rate cut at a four year horizon. Indeed, because of the tight borrowing constraints, many agents dissave at the margin which hinders their ability to transfer funds in the future, even when they are not constrained. This additional discounting impacts all households and goes largely beyond the direct effect of borrowing constraints. Similarly, the precautionary effects of interest rate changes largely dampens the income effect (or "redistribution channel") of future cuts.

Related Literature. Our work builds on the classical approach to consumer demand. This is a long and important literature with milestone contributions by Marshall (1890), Slutsky (1915), Hicks (1939), Hicks and Allen (1934a) and Hicks and Allen (1934b). Deaton and Muellbauer (1980) provides a thorough review.

While the environment of the classical theory of demand is static, its results can be directly applied to the canonical dynamic complete market model of Arrow and Debreu (1954). Our main contribution is to extend these results to incomplete markets.

Models of incomplete markets and their income fluctuations problems were initially formalized and studied as extensions of Friedman's permanent income model, by Bewley (1977), Schechtman and Escudero (1977), Chamberlain and Wilson (2000) among others. In terms of individual behavior, few properties were obtained beyond existence and monotonicity of the consumption function. Two important later contributions by Kimball (1990) and Carroll and Kimball (1996) studied precautionary motives and established the concavity of the consumption function. Our paper is complementary and continues this line of work, investigating the theoretical predictions of this important class of models.

A quantitative literature has simulated incomplete market models to study its effects on growth Aiyagari (1994), asset pricing Huggett (1993), macroeconomic fluctuations Krusell and Smith (1998), optimal taxation Aiyagari (1995), and may other issues. An extensive empirical literature also emerged, estimating and testing these models, starting with Hall (1978); Flavin (1981); Hall and Mishkin (1982) and continuing with Attanasio and Weber (1995); Carroll (1997); Gourinchas and Parker (2002); Blundell et al. (2008), to name a few. Our paper thus complements this line of work by offering a fuller characterization of the restrictions imposed by the theory and the sufficient statistics involved, also offering a better understanding of the mechanisms at work in consumer behavior and its link to the traditional Walrasian Price theory.

Our work is related to Auclert (2015) which studied the transmission of monetary policy to aggregate demand in heterogeneous agents economies. The main focus in that paper is on an heterogeneous income effect channel that supplements the substitution channel. This channel is based on covariance of heterogenous marginal propensities to consume with asset positions. Auclert (2015) shows that these statistics are sufficient for contemporaneous transitory change to the interest rate, but are no longer sufficient to characterize persistent shocks or changes in future interest rates or income. Our paper provides the necessary characterization and statistics shaping these responses. More broadly, our paper provides a general framework for the growing literature seeking to identify sufficient statistics for partial equilibrium effects (e.g.Kaplan and Violante, 2014, Berger et al., 2017), or to derive the aggregate intertemporal MPCs are in Auclert et al. (2018).

Farhi and Werning (2019) study the response of current consumption to future interest rate changes in an incomplete market setting in general equilibrium. That paper considers both rational expectations outcomes as well as level-k boundedly rational one. The level-k concept employed justifies putting more weight on the partial equilibrium responses, which is precisely the object of study of the present paper. Our paper is also broadly related to a literature extending standard demand theory to allow for boundedly rational agents. Gabaix (2014) considers sparse agents who disregard some of their state variables, Aguiar and Serrano (2017) use Slutsky matrices to characterize deviations from rationality, Farhi and Gabaix (2020) extend the classical framework to revisit optimal taxation theory with bounded rational agents.

2 The Income Fluctuations Problem

For most of the paper we adopt a baseline model that is sufficiently general to encompass traditional incomplete market models. We consider a standard one-good income-fluctuation model with exogenous borrowing constraints. The model is later generalized in Section B to allow for several goods, multiple assets, and shocks to preferences and borrowing constraints.

Shocks, interest rates, and income shifters. We denote the shock process by θ_t, and the history of shocks up to t by $\theta^t = (\theta_0, \theta_1, \cdots, \theta_t)$. We denote by P the probability measure on the history of shocks, by $\mathbb{E}[\cdot]$ the expectation with respect to this measure, and by $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\theta^t]$ the expectation conditional on the realization of θ^t. The agent's income in period t is $y(\theta^t)$. She saves through a short term risk free bond a_t. We denote the gross interest rate between t and $t+1$ by $R_t(\theta^t)$.

Agent's problem. The problem of the agent in period t after history θ^t and with wealth a_t is

$$\max_{\{a_t(\theta^t), c_t(\theta^t)\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t(\theta^t))\right]$$

subject to the sequence of budget constraints

$$c_t(\theta^t) + \frac{a_{t+1}(\theta^t)}{R_t(\theta^t)} = a_t(\theta^{t-1}) + y_t(\theta^t)$$

$$a_t(\theta^t) \geq B_t(\theta^t),$$

with initial condition $a_0(\theta^0) = a_0$ given. Here $B_t(\theta^t)$ is an exogenous borrowing limit.

Individual indirect utility and Marshallian demand functions. For given a_0, the solution to the problem gives a sequence for consumption and assets $c_t(a_0, \theta^t; \{R_t\}, \{y_t\})$ and $a_{t+1}(\theta^t, \{R_t\}, \{y_t\})$ and to denote their dependence on future interest rates processes. Our goal is to study the comparative statics with respect to these processes.

The agent problem is obviously recursive and can be represented by value and consumption functions that depend on current assets. We denote these functions as

$$V_t(a_t, \theta^t; \{R_{t+s}\}, \{y_{t+s}\})$$

$$\hat{c}_t(a_t, \theta^t; \{R_{t+s}\}, \{y_{t+s}\})$$

Note that

$$c_t(\theta^t; \{R_t\}, \{y_t\}) = \hat{c}_t(a_t(\theta^{t-1}, \{R_t\}, \{y_t\}), \theta^t; \{R_{t+s}\}, \{y_{t+s}\}).$$

These functions are generalizations to an incomplete-markets setting of the classical price theory concepts of indirect utility and Marshallian demand functions. To streamline the notation we sometimes leave the dependence on the arguments implicit.

Two key statistics of our analysis are the marginal propensity to consume (MPC) and the marginal propensity to spend (MPS). They are defined as follows:[2]

$$\begin{aligned}\partial_a c(\theta^t) \equiv& \partial_a \hat{c}_t(a_t, \theta^t; \{R_{t+s}\}, \{y_{t+s}\}) \\ MPS_t(\theta^t) \equiv& R_t(\theta^t)(1 - \partial_a c_t(\theta^t)).\end{aligned}$$

In addition, we define the stopping time τ as the first date, starting from $t = 0$, at which the borrowing constraint binds. Note that τ is not generally deterministic and typically depends on the sequence of realized shocks.

Complete Market Benchmark. We will benchmark the incomplete market model against a complete markets model. In that case, agents can purchase, at history θ^t, assets at prices $q_t(\theta^{t+1})$ paying R_t in state θ_{t+1}. Agents are subject to the same exogenous borrowing constraint.The budget constraints are then,

$$c_t(\theta^t) + \sum_{\theta_{t+1}} q_t(\theta^{t+1}) b_{t+1}(\theta^{t+1}) = R_{t-1}(\theta^{t-1}) b_t(\theta^t) + y_t(\theta_t, X_t)$$

$$R_{t-1}(\theta^{t-1}) b_t(\theta^t) \geq B_t(\theta^t).$$

The complete market problem defines additional policy functions, the quantity of assets purchased at θ^t, that are denoted $b_{t+1}(\theta^{t+1}, \{R_t\}, \{y_t\})$ and $\hat{b}_{t+1}(a_t, \theta^{t+1}; \{R_{t+s}\}, \{y_{t+s}\})$. We then define a state-dependent marginal propensity to save

$$MPS(\theta_{t+1}, \theta^t) = R_t(\theta^t) \partial_a \hat{b}_{t+1}(a_t, \theta^{t+1}; \{R_{t+s}\}, \{y_{t+s}\}).$$

We can similarly define a stopping time τ as the first date at which one of the borrowing constraint binds.

3 Main Results: Response to Interest Rate and Income Changes

In this section, we express the derivatives of the Marshallian demand function with respect to interest rate and income changes. Thus, we are examining the responses of consumption to a contemporaneous announcement of interest rate and income changes. We later turn to total derivatives that take into account that wealth at t changes due to the past announcements at $t = 0$.

[2]Our marginal propensity to spend is technically a marginal propensity to increase wealth in the next period.

3.1 Deterministic Case: Discounting with MPS

For presentation purposes it is useful to first consider the case without uncertainty, which effectively puts us in a Walrasian setting. We seek an expression for

$$dc_0 = \sum_{t=0}^{\infty} \left(\frac{\partial c_0}{\partial y(\theta^t)} dy(\theta^t) + \frac{\partial c_0}{\partial R(\theta^t)} dR(\theta^t) \right).$$

We then have the following result.

Proposition 1. *The response of consumption to a change in the sequence of prices and income $\{dR_t\}$ and $\{dy_t\}$ is given by*

$$\frac{dc_0}{c_0} = -\,\epsilon(c_0) \sum_{t=0}^{\tau} \left(\prod_{s=0}^{t} \frac{MPS_s}{R_s} \right) \frac{dR_t}{R_t} + \frac{1}{c_0} \frac{\partial c_0}{\partial a_0} \sum_{t=0}^{\tau} \left(\prod_{s=0}^{t-1} \frac{1}{R_s} \right) \left(\frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t \right),$$

where $\epsilon(c_0) = -c_0 u''(c_0)/u'(c_0)$ is the local elasticity of intertemporal substitution.

Note that there is no reaction to changes beyond τ, the date at which the borrowing constraint binds. This is intuitive, as the borrowing constraint interrupts the connection across periods, segmenting decisions. The first term on the left hand side is the substitution effect, involving the local elasticity of substitution. The second term is an income effect, involving the marginal propensity to consume.

The first term in the expression above is the substitution effects and contains some standard elements and some new elements. As usual, this effect is mediated by the elasticity of substitution. The more novel element is that the expression shows that the substitution effect depends on the sum of discounted price changes, discounting using the agent's marginal propensity to save. The product $\prod_{s=0}^{t} MPS_s/R_s$ represents the marginal propensity to save from the current period to t. Intuitively, this captures how connected the agent is to this future date. This, in turn, affects the responsiveness in the present of changes in interest rates at this future date. Overall, the expression clarifies that the sensitivity of consumption to future change in interest rates may be low for two reasons. First, the planning horizon of the agent may be interrupted at τ, so the agent does not respond to changes in interest rate that happens after τ. Second, the consumption response is further lowered through the product of marginal propensities to save as the MPSs typically fall towards zero as the agent get closer to the borrowing constraint. Note that the first situation can also be thought of as a special case of the second: when the borrowing constraint binds the MPS is equal to zero.

The second term in the expression above is the income effect, which requires little comment, as it is the standard demand theory expression. In period t the income effect is comprised of the change in income and the impact of the change in the interest rate on the budget constraint, which is proportional to the asset position. The impact on consumption is then simply the present value of these income effects times the marginal propensity to consume. The income effect term will become more interesting in the case with uncertainty, to which we not turn.

3.2 Adding Uncertainty: Precautionary Effects

We now consider the case with uncertainty and consider the impact of a change in the path of interest rates and income. The two sequences can be state dependent and are announced at $t = 0$. We now seek an expression for

$$dc_0 = \sum_{t=0}^{\infty} \sum_{\theta^t} \left(\frac{\partial c_0}{\partial y(\theta^t)} dy(\theta^t) + \frac{\partial c_0}{\partial R(\theta^t)} dR(\theta^t) \right).$$

We can also interpret these responses as the partial derivatives of the Marshallian consumption function.

Our next result is one of the main results in the paper and expresses the substitution and income effects as simple weighted averages of the deterministic expression. The weights define probability distributions that can be thought of as risk- and prudence-adjusted probabilities.

Proposition 2. *The response of consumption to a change in the sequence of prices and income $\{dR_s\}$ and $\{dy_s\}$ is given by:*

$$\frac{dc_0}{c_0} = -\epsilon(c_0)\mathbb{E}_0^Q \sum_{t=0}^{\tau} \left(\prod_{s=0}^{t} \frac{MPS_s}{R_s} \right) \frac{dR_t}{R_t} + \frac{1}{c_0}\frac{\partial c_0}{\partial a_0}\mathbb{E}_0^{Q^I} \sum_{t=0}^{\tau} \left(\prod_{s=0}^{t-1} \frac{1}{R_s} \right) \left(\frac{a_{t+1}}{R_t}\frac{dR_t}{R_t} + dy_t \right) \quad (1)$$

for probability distributions Q and Q^I defined by

$$\frac{dQ}{dP}(\theta^{t+1} \mid \theta^t) \equiv \frac{V'(a_{t+1}, \theta^{t+1})}{\mathbb{E}\left(V'(a_{t+1}, \theta^{t+1}) \mid \theta^t\right)} = \frac{u'(c(a_{t+1}, \theta^{t+1}))}{\mathbb{E}\left(u'(c(a_{t+1}, \theta^{t+1})) \mid \theta^t\right)}, \quad (2)$$

$$\frac{dQ^I}{dP}(\theta^{t+1} \mid \theta^t) \equiv \frac{V''(a_{t+1}, \theta^{t+1})}{\mathbb{E}\left(V''(a_{t+1}, \theta^{t+1}) \mid \theta^t\right)} = \frac{u''\partial_a c(a_{t+1}, \theta^{t+1})}{\mathbb{E}\left(u''\partial_a c(a_{t+1}, \theta^{t+1}) \mid \theta^t\right)}. \quad (3)$$

The probability measure Q is a marginal-utility reweighing of the objective probabilities. This is a standard risk-adjustment that is commonly used in finance and other areas of economics. As we show later, this probability also enters the picture under complete markets, since risk adjustment occurs under complete market by way of the Arrow-Debreu prices which are equated to marginal utilities. The probability measure

Q^I is, in contrast, novel and represents a second derivative reweighing of the objective probabilities. Intuitively, this reweighing captures precautionary effects. Indeed, under quadratic utility we have that V'' is constant and so $Q^I = P$. More generally, the local sensitivity of V'', related to V''', has been insightfully defined as a measure of local prudence related to precautionary savings by Kimball (1990). Thus, we can interpret Q^I as a prudence-adjusted probability. We later provide an important result comparing our prudence-adjusted probability Q^I to the standard risk-adjusted Q.

The first term captures the substitution effect allowing for uncertainty. As we can see, future changes in interest rates are valued just as in deterministic case, discounting by the marginal propensity to save, but now taking the expectation using the risk adjusted probabilities Q. Everything else equal, an agent will react more to a change in interest rate in a state with high marginal utility.

The second line in equation (1) shows that income effects are averaged using the prudence-adjusted probability Q^I. Intuitively, under incomplete markets income changes are valued in a risk-adjusted manner, but they affect consumption through precautionary motives. For example, if future income is raised in a low consumption, low income state, then this may affect the desire to "save for a rainy day".

Another intuition is that changes in income affect consumption and this in turn affects the risk-adjustment. The measure Q^I_t takes this change in valuation into account. The substitution effects do not directly incorporate this change in valuation because compensated price effects do not generate any such changes.

The response of consumption to income changes is therefore determined by the potentially binding borrowing constraint—which cuts the planning horizon at τ—and by the prudence-adjustment Q^I.

Note that even when future changes in interest rate or income are deterministic the risk- and prudence-adjustment probabilities can come into play. To see this in a simple case, suppose the path of interest rates is deterministic. First consider a change in future income at some date t that is constant across all states of nature. Then

$$\frac{dc_0}{dy_t} = \partial_a c_0 \frac{1}{\prod_{s=0}^{t-1} R_s} \mathbb{E}_0^{Q^I}(\mathbb{1}((\tau > t)) = \partial_a c_0 \frac{Q^I(\tau > t)}{\prod_{s=0}^{t-1} R_s}.$$

We see that what matters is the prudence-adjusted probability of the borrowing constraint not binding $Q^I(\tau > t)$, which is generally different from the objective probability $P(\tau > t)$. In particular, as we show below for most cases of interest the probability Q^I will overweigh bad states of the world, leading one to expect $Q^I(\tau > t) > P(\tau > t)$. Next consider a future interest rate change at t that is constant across all states of nature. Let us focus on the substitution effect,

$$-\epsilon(c_0)\frac{dR_t}{R_t}\left(\prod_{s=0}^{t}\frac{1}{R_s}\right)\mathbb{E}_0^Q\left[\prod_{s=0}^{t}MPS_s\right]$$

Note that even though R_s is deterministic, the MPS_s are not if the agent faces (idiosyncratic) income uncertainty between 0 and t. Thus, the discounting by $\prod_{s=0}^{t} MPS_s$ is stochastic and the formula uses the expectation with the risk-adjusted probability Q.

Complete Markets. When markets are complete we obtain a similar result

$$\begin{aligned}\frac{dc_0}{c_0} =& -\epsilon(c_0)\mathbb{E}_0^Q\sum_{t=0}^{\tau}\left(\prod_{s=0}^{t}\frac{MPS_s(\theta^{s+1},\theta^s)}{R_s}\right)\frac{dR_t}{R_t} \\ &+\frac{1}{c_0}\frac{\partial c_0}{\partial a_0}\mathbb{E}_0^Q\sum_{t=0}^{\tau}\left(\prod_{s=0}^{t-1}\frac{1}{R_s}\right)\left(\frac{a_{t+1}}{R_t}\frac{dR_t}{R_t}+dy_t\right).\end{aligned} \tag{4}$$

The expression is very similar to incomplete markets. However, now the income effects are evaluated using Q instead of Q^I. In addition, the MPS is defined across states. Intuitively, there is no precautionary effect, so the appropriate response to changes in income is entirely through the risk-adjusted valuation.

Change in Risk. Our framework can accommodate "risk shocks", for example an increase in uncertainty captured by an increase in the variance of shocks, in a simple manner by changing locally the underlying transition kernel π defined by the physical measure P. Changes in the underlying distribution can also be interpreted as "news shocks" or changes in beliefs. We consider absolutely continuous changes with respect to π such that $\mathbb{E}_t\left(d\pi_{t+1}/\pi_{t+1}\right)=0$. The consumption response is then given by

$$\frac{dc_0}{c_0} = -\epsilon(c_0)\mathbb{E}_0^Q\left(\sum_{t=1}^{\tau}\prod_{s=1}^{t-1}\frac{MPS_s}{R_s}\frac{d\pi_t}{\pi_t}\right)$$

The formula makes clear that changes in risk are equivalent to price changes for consumption responses: an increase in the probability of state θ_{s+1} acts as an increase in the virtual Arrow-Debreu price of the state. Since, in partial equilibrium, they do not affect directly the monetary returns of assets, agents respond to "risk shocks" through a substitution channel.

3.3 Risk- and Prudence-Adjusted Probabilities: Origins, Intuition and Results

Our characterization of the derivatives of the Marshallian demand function relies on the probabilities Q and Q^I. Here we discuss the origin of Q and Q^I, provide further intuition and, finally, provide some results comparing them. In particular, we provide conditions ensuring that Q^I overweighs bad state relative to Q.

First, where do these adjusted probabilities come from? They are defined by two martingale conditions, both stemming from the Euler optimality condition equation. The optimal consumption plan satisfies (when the constraint does not bind at t):

$$u'(c_t) = \beta R_t \mathbb{E}_t \left(u'(c_{t+1}) \right), \tag{5}$$

This in turn implies by the Envelope theorem

$$V'(a_t, \theta^t) = \beta R_t \mathbb{E}_t \left(V'(a_{t+1}, \theta^{t+1}) \right)$$

differentiating we obtain

$$V''(a_t, \theta^t) = \beta R_t MPS_t \mathbb{E}_t \left(V''(a_{t+1}, \theta^{t+1}) \right). \tag{6}$$

This implies that discounted version of V' and $-V''$ are positive martingales,[3] which we can use to define the adjusted probabilities.

Next, we provide more intuition for the two adjustments. The first adjusted probability Q, is familiar and related to the standard stochastic discount factor $\beta u'(c_{t+1})/u'(c_t)$. It defines the prices of the shadow Arrow-Debreu securities that would make the consumption plan of the agent optimal in a complete market setting. The prices associated with Q give the valuation, in terms of welfare at time 0 of one more dollar of wealth in the future (when the constraint is not binding on the path). As in the complete market setting, the Arrow-Debreu price $P^{AD}(\theta^{t+1} \mid \theta^t)$ at θ^t of one dollar at θ^{t+1} is given by $Q(\theta^{t+1} \mid \theta^t)/R_t$.

In contrast, the second adjusted probability Q^I gives the compensating variation for consumption at time 0 of one more dollar of wealth in the future. The probability Q allows to compute the standard compensating variation: an additional dollar in state θ^t can be compensated by $Q(\theta^t)$ dollars to keep welfare constant. Q^I allows to perform the same exercise for consumption: an additional dollar in state θ^t has to be compensated by $Q^I(\theta^t)$ dollars to keep consumption constant at time 0. With incomplete markets, the two probabilities Q and Q^I differ since the valuation of future states changes as the agent

[3] More precisely, V' and $-V''$ appropriately discounted and stopped at τ are martingales.

gets richer. The agent consumption plan changes not only because she is richer – in terms of the virtual Arrow-Debreu prices – but also because the Arrow-Debreu prices change with wealth. Q^I incorporates this change and, in this sense, is intimately related to the presence of a precautionary motive.

We formalize this intuition in the following Proposition which relates the twist in valuation introduced by $Q^I$to the degree of prudence (Kimball (1990), Carroll and Kimball (1996)) exhibited in the agent's utility function.

Proposition 3. *We characterize the change in valuation introduced by Q^I as follows.*

a. Assume that $u'''u'/(u'')^2 \geq 1$ and $\beta R < 1$, then:

$$\frac{\partial}{\partial a_{t+1}} \left\{ \frac{Q^I(\theta^{t+1}|\theta^t)}{Q(\theta^{t+1}|\theta^t)} \right\} \leq 0.$$

In particular, if the process $y(\theta_t)$ is i.i.d., we have

$$\frac{Q^I(\theta_{t+1}, \theta^t|\theta^t)}{Q(\theta_{t+1}, \theta^t|\theta^t)} \leq \frac{Q^I(\theta'_{t+1}, \theta^t|\theta^t)}{Q(\theta'_{t+1}, \theta^t|\theta^t)},$$

for any θ'_{t+1} such that $y(\theta'_{t+1}) \leq y(\theta_{t+1})$.

b. In the limiting case where $u'''u'/(u'')^2 = 1$ utility is exponential $u(c) = -e^{-\gamma c}$. Then if borrowing constraints never bind we have $Q^I = Q$.

c. When utility is quadratic $u(c) = c - bc^2$ so that $u'''u'/(u'')^2 = 0$. Then in the absence of borrowing constraints or if they do not bind we have $Q^I = P$

$$\frac{Q^I(\theta_{t+1}, \theta^t|\theta^t)}{Q(\theta_{t+1}, \theta^t|\theta^t)} \geq \frac{Q^I(\theta'_{t+1}, \theta^t|\theta^t)}{Q(\theta'_{t+1}, \theta^t|\theta^t)},$$

for any θ'_{t+1} such that $c(\theta'_{t+1}, \theta^t) \leq c(\theta_{t+1}, \theta^t)$.

The proposition shows that with non-increasing absolute risk aversion, additional wealth in the future reduces the gap between the measure Q and Q^I. This implies in particular that the measure $Q^I$is more volatile than Q with respect to the transitory component of income shocks: the measure Q^I puts relatively more weight on the bad realization of income than the measure Q. Indeed, an increase income in a "bad" state provides more insurance than in a "good" state and this additional benefit is captured by Q^I. In this sense, the measure Q^I is the measure P adjusted for prudence. When the agent is risk averse, the measure Q already puts more weight on states "bad" states compared to the physical measure P, so that the measure Q^I_t amplifies the overweighting of bad states. Note that the Q^Ioverweighting of bad state is also true for some simple persistent income changes. Consider for example an $AR(1)$ income process with persistence ρ and suppose that the borrowing constraint is never binding. In that case, a positive realization of the income shock corresponds to an increase in wealth of $1/(1-(\rho/R))$: the overweighting of bad realization of the income process by Q^I is even stronger than with transitory shocks.

The CARA utility function works as a limiting case. It is well known that, in that case, the degree of precautionary savings does not vary with agents' wealth. Since the precautionary motive is constant, the Q valuation of future states is independent of the agent's initial assets and the measure Q and Q^I coincides.

Finally, the last item in the proposition shows that Q^I is less volatile than Q in the quadratic case. This makes clear that the overweighting of "bad" stated via the prudence-risk adjustment dQ^I/dQ is related to non-increasing absolute risk aversion. Indeed, with quadratic preferences, an increase in the variance of future income reduces welfare (the Q-NPV of the income change is negative) but does not affect consumption. With an increase in variance, the prudence-risk adjustment (through dQ^I/dQ) is positive and exactly offset the negative risk effect (through dQ/dP) and we have $Q^I = P$.

The measure Q^I is specific to the incomplete market setting and has no additional informational content when markets are complete: both welfare and consumption remains constant when an additional dollar in state θ^t is compensated by $Q(\theta^t)$ dollars at 0. Indeed the valuation of future states is then given by exchangeable Arrow-Debreu securities whose prices do not depend on the agent's wealth.

Remark. When markets are complete, the probabilities Q and Q^I are identical:

$$Q(\theta^{t+1} \mid \theta^t) = Q^I(\theta^{t+1} \mid \theta^t),$$

in every state θ^t where the constraint is not binding between θ^t and θ^{t+1}.

When markets are complete, we can again define the measure Q with the stochastic discount factor as $dQ/dP(\theta^{t+1} \mid \theta^t) = \beta_t R_t u'(c(a_{t+1}, \theta_{t+1}))/u'(c(a_t, \theta_t))$. However, since it is possible for the agent to purchase an asset at price $q_t(\theta_{t+1})$ paying R_t in state θ_{t+1}, we have that $dQ/dP(\theta^{t+1} \mid \theta^t) = q_t(\theta^{t+1})$ for all asset levels a_t. It follows that the measure Q^I is also equal to $q_t(\theta^{t+1})$ in all states and at all histories.

3.4 Welfare impacts and Slutzky versus Hicks Compensation

We now investigate the welfare impacts from changes in future interest rates and income and discuss their connection to the substitution effect. In particular, we ask: is the substitution effect is related to a welfare compensated response? Our answer is that, unlike in complete market Walrasian setups, with incomplete markets it depends on how the compensation is carried out.

In particular, it will become apparent that our substitution effect is equal to the response one obtains if compensation is carried out in real time, at each future date and state providing additional income to cancel out the income effects from interest rate changes. This type of compensation ensures that the previously chosen consumption path is still feasible. This is akin to the way Slutzky defined compensation, since it also pivoted the budget set through the original chosen consumption basket. As it turns out, just as in a Walrasian setup this compensation increases welfare for large changes, but the first order effect on welfare is zero.

Another compensation scheme is that of Hicks, who envisioned providing extra income to keep utility unchanged. Here we explore this type of compensation using changes in initial wealth. In a complete market Walrasian setting the Slutzky and Hicks compensation are not the same and produce different results for arbitrary changes, but they lead to the same responses to a first order. We show that in our incomplete market setting this is no longer the case.

Roy's Identity. Let us first consider the welfare impact of changes in income and interest rates. Since shock that occurs beyond the horizon at which the constraint binds have an impact on welfare, we extend the definition of the measure Q to all histories, $Q(\theta^s) = \prod_{k=k}^{s-1} \beta R_k u'(c_s)/u'(c_0) P(\theta^s)$. Note however that Q is not a probability measure and that $\int dQ(\theta^s) \leq 1$, with equality if and only if no constraint binds between t and s.

Proposition 4. *The welfare of an agent with wealth a_t in state θ_t at t is given by $V(a_t, \theta_t, t)$. The change in welfare in response to a change in assets da_0, interest rates $\{dR_s\}$ and income $\{dy_s\}$ is*

$$dV_0 = \partial_a V_0 \left(da_0 + \mathbb{E}_0^Q \left(\sum_{t=0}^{\infty} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t \right\} \right) \right), \tag{7}$$

where $\partial_a V_0 = u'(c_0)$.

Since consumption does not react to shocks occurring beyond τ, we will focus on the welfare impact of a change $\{dR_s\}_{0 \leq s \leq \tau}$ and $\{dy_s\}_{0 \leq s \leq \tau}$,

$$dV_0 = \partial_a V_0 \left(da_0 + \mathbb{E}_0^Q \left(\sum_{t=0}^{\tau} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t \right\} \right) \right). \tag{8}$$

This result is an extension of Roy's identity to the incomplete market setting. Future changes in income and prices are evaluated according to the measure Q since it gives the

Arrow-Debreu prices of future states. Indeed, one dollar in state θ^s is valued at $Q(\theta^s)/\prod_{s'=0}^{s-1} R_{s'}$ of time t dollars and this therefore also gives the welfare equivalent change in wealth at t. Similarly, an increase in the interest rate at s is equivalent to a decrease in the price of consumption at $s' > s$. In the static Roy's identity, we have that $\frac{dV}{dp_i} = -\frac{dV}{da}c_i$: the decrease in welfare resulting of an increase in the price of good i is proportional to the quantity of good i consumed. Here, the quantity of goods consumed after s is given by $\frac{a_{s+1}}{R_s}$ which explains the welfare impact of a change in interest rates.

The formula also allows us to determine the full set of welfare-compensated changes with $dV_0 = 0$ satisfying

$$0 = da_0 + \mathbb{E}_0^Q \left(\sum_{t=0}^{\infty} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t \right\} \right).$$

Once again, however, consumption does not react to changes that occur beyond τ. Thus, we consider only shocks that occur before τ. Their welfare effect is given by equation 8 and so we consider compensated changes to be those satisfying

$$0 = da_0 + \mathbb{E}_0^Q \left(\sum_{t=0}^{\tau} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t \right\} \right).$$

Slutzky Compensation. One simple compensated change is one that happens in "real time" as changes occur. In particular, setting $da_0 = 0$ and suppose

$$0 = \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t$$

at all dates and states of the world. This compensation scheme is in the spirit of the original Slutzky one because it ensures that the agent can maintain the same consumption path $\{c_t\}$ after the change, i.e. $dc_t = 0$ at all dates and states is budget feasible. Of course, it will generally not be optimal to maintain the same consumption path. Indeed, applying Proposition 2, we see that a Slutzky-compensated change will only leave the substitution effect as the predicted change in consumption. This justifies interpreting our substitution effect as a Slutzky-compensated response.

Hicks Compensation. We now explore an alternative compensation scheme. There are two reasons to do so. First, one might argue that the Slutzky-compensation is somewhat at odds with the incompleteness of markets since the implied income transfers are generally state contingent. But then, if such transfers are available, why not use them to improve welfare and complete markets? A counterargument is that the substitution effect is conceptual exercise, even in a Walrasian setting, not a normative one. A second reason for exploring an alternative, however, is that Hicks offered a different form of compensation using income to maintain utility constant. In our context, one can think of this as a transfer of wealth in period 0.

We thus explore the consequences of this simple compensation scheme. To distinguish it from the previously labeled substitution effect (equal to a Slutzky-compensated response) we will denote this response simply as "compensated". As we show next, this response equals the substitution effect plus a term that captures the precautionary effect, present due to the fact that $Q \neq Q^I$.

Proposition 5. *Consider a change in interest rate at t, dR_t. The consumption response when the agent is compensated at initial time is:*

$$dc_0^h = \underbrace{-\epsilon(c_0)c_0\mathbb{E}_0^Q\left(\prod_{s=0}^{t}\frac{MPS_s}{R_s}\frac{dR_{t\wedge\tau}}{R_t}\right)}_{\text{substitution effect}} + \underbrace{\frac{\partial c_0}{\partial a_0}Cov_0^Q\left(\frac{Q^I}{Q},\prod_{s=0}^{t-1}\frac{1}{R_s}\frac{a_{t+1}}{R_t}\frac{dR_{t\wedge\tau}}{R_t}\right)}_{\text{precautionary effect}}. \tag{9}$$

Similarly, a (state-dependent) change in income at t gives the following compensated response:

$$dc_0^h = \underbrace{\frac{\partial c_0}{\partial a_0}Cov_0^Q\left(\frac{Q^I}{Q},\prod_{s=0}^{t-1}\frac{1}{R_s}\,dy_{t\wedge\tau}\right)}_{\text{precautionary effect}}. \tag{10}$$

For comparison, the compensated response in complete markets is:

$$dc_0^h = \underbrace{-\epsilon(c_0)c_0\mathbb{E}_0^Q\left(\prod_{s=0}^{t}\frac{MPS_s(\theta^{s+1},\theta^s)}{R_s}\frac{dR_{t\wedge\tau}}{R_t}\right)}_{\text{substitution effect}}, \tag{11}$$

for a change in interest rate and simply 0 for a change in income.

The first term of the compensated response is simply the *substitution effect*. As in the complete market setting, substitution effects generate no change in welfare. The second term is specific to incomplete markets. As we mentioned before a change in income in state θ^s has two effects: first a *wealth* effect at the initial (pre income change) valuation of the state $Q(\theta^s)$ and second a change in the valuation of the state given by $Q^I(\theta^s) - Q(\theta^s)$.

This second effect is due to the impossibility to make state dependent transfers and is precisely the precautionary motive. Compensating the agent takes care of the first effect and the derivative of Hicksian demand function in incomplete markets is the sum of a *substitution effect* and a *precautionary effect*. The precautionary effect is higher when the income change occurs in the states where the prices Q increase relatively more, that is the states where Q^I/Q is higher. Indeed these correspond to sates where prices increase more with wealth, or equivalently states towards which the agent would transfer more funds if it was feasible to have state dependent savings (at rate R_s). An income change with zero present value according to Q but skewed towards higher Q^I/Q does not change welfare but reduces the need to precautionary save and increases consumption at t. As shown in Proposition 3, when income is i.i.d. and that utility is in the $DARA$ class, the precautionary effects of income change is positive when income increases in a low income state and negative otherwise.

4 Additional Results

This section contains some additional results. First, we discuss the impact of announcing future changes on consumption in subsequent periods. Our previous results described the effect on impact. Here we show that similar formulas and statistics explain the effect on the entire consumption path. Second, we present a result extending the classical Slutzky symmetry to incomplete markets. Third, we show that we can express the cross-relations across elasticities as a Martingale condition. Finally, we draw out some conclusions for aggregate consumption, showing that the heterogeneity in risk- and prudence-adjusted probabilities can affect the aggregate.

4.1 Impulse Response Function

So far, we have considered the consumption responses of agents at the time of the announced changes in current and future interest rates and income. This corresponds to differentiating the Marshallian demand with respect to these variables keeping initial assets fixed. We now characterize the reaction in other periods, after the announcement, sometimes called the impulse response function (IRF).

From a mechanical perspective one can proceed as follows. Our previous characterization tells us how c_0 reacts. We can then use the budget constraint to infer assets for $t = 1$. Once this is done, we can compute consumption at $t = 1$ using this new asset position just as before. Continuing in this way we can compute the entire path. To see this more formally, denote by dc_t the response of expenditure at history θ^t, with $t > 0$, for a change in interest rate and income. Formally we consider the total differential

$$dc_t = \sum_{s \geq t, \theta^s} \left(\frac{\partial \hat{c}_t}{\partial y(\theta^s)} dy(\theta^s) + \frac{\partial \hat{c}_t}{\partial R(\theta^s)} dR(\theta^s) \right) + \frac{\partial c_t}{\partial a_t} da_t,$$

$$da_{t+1} = R_t(dy_t - dc_t + da_t) + \frac{dR_t}{R_t} a_t.$$

To study income and interest rate changes we simply set $da_0 = 0$ and iterate on these equations; our previous analysis provides the determinants of $\frac{\partial \hat{c}_t}{\partial y(\theta^s)}$ and $\frac{\partial \hat{c}_t}{\partial R(\theta^s)}$. The term $\frac{\partial c_t}{\partial a_t} da_t$ captures new effect due to the response of assets that was not present for $t = 0$. The second equation allows us to compute the asset response recursively.

Next, we derive an insightful condition on the response dc_{t+1} in terms of the past response dc_t and other determinants. Our first observation is that, as long as the borrowing constraint is not binding, the log response of consumption along the path, adjusted for the changes in the interest rate, is a martingale under the risk-adjusted probability Q_t,

$$\mathbb{E}_t^Q \left[\frac{1}{\epsilon_{t+1}} \frac{dc_{t+1}}{c_{t+1}} \right] = \frac{1}{\epsilon_t} \frac{dc_t}{c_t} + \frac{dR_t}{R_t}. \tag{12}$$

Although this condition is simply a log-linearization of the agent's Euler equation, it can be viewed as formalizing the "random walk" concept introduced by Hall (1978). However, our martingale condition is obtained for a comparative static and is satisfied by the resulting first-order (log) response around an arbitrary baseline stochastic consumption path.[4]

To interpret the martingale condition let us consider the CRRA case with CRRA case with $\epsilon_t = \epsilon_{t+1} = \epsilon$ constant and suppose there is no current change in the interest rate $dR_t = 0$ (there are possibly future changes dR_{t+s} for $s \geq 1$). Then the martingale condition says that the log response at $t + 1$ is an average of that at t (using the risk-adjusted probabilities). More generally, changes in the response of consumption along the path must be responses to either (i) changes in the interest rates along the path; (ii) changes in the elasticity; (iii) or the arrival of new information.

Our next result goes further and provide an expression for the realized dc_{t+1}, not just its expectation. In other words, for the "error" in the "random walk" above. It helps spell out what arrival of information is relevant.

[4]Indeed, consumption $\{c_t\}$ itself is not a martingale. Our expression is also distinct from standard log-approximation carried out in many models, performed around a deterministic steady-state. Instead, we are studying the first-order response process $\{dc_t\}$ around the original $\{c_t\}$ without approximating $\{c_t\}$ around a steady state. This is why the relevant expectation is $\mathbb{E}^Q$ and not $\mathbb{E}$.

Proposition 6. *Consider a change in the sequence of income $\{dy_t\}$ and interest rates $\{dR_t\}$. If the borrowing constraint does not bind at t, then the response of consumption at $t+1$ is given by*

$$\frac{dc_{t+1}}{\epsilon_{t+1}c_{t+1}} = \frac{dc_t}{\epsilon_t c_t} + \frac{dR_t}{R_t} + \varepsilon_{t+1} \tag{13}$$

where $\mathbb{E}_t^Q[\varepsilon_{t+1}] = 0$ and

$$\varepsilon_{t+1} = \frac{\partial_a c_{t+1}}{\epsilon_{t+1}c_{t+1}} \underbrace{\left(\mathbb{E}_{t+1}^{Q^I}(w_{t+1}) - \mathbb{E}_t^{Q^I}(w_{t+1})\right)}_{\text{Revaluation of Wealth}} - \underbrace{\mathbb{E}_{t+1}^{Q}(\tilde{R}_{t+1}) - \mathbb{E}_t^{Q}(\tilde{R}_{t+1})}_{\text{Revaluation of Interest Rate}} + \underbrace{\left(\frac{Q_t^I}{Q_t} - 1\right)\left(\frac{dc_t}{\epsilon_t c_t} + \frac{dR_t}{R_t} + \mathbb{E}_t^Q(\tilde{R}_{t+1})\right)}_{\text{Revaluation of State Prices}},$$

where $w_{t+1} = \sum_{s=t+1}^{\tau}\prod_{k=t+1}^{s-1}\frac{1}{R_k}\left(\frac{a_{s+1}}{R_s}\frac{dR_s}{R_s} + dy_s\right)$ and $\tilde{R}_{t+1} = \sum_{s=t+1}^{\tau}\prod_{k=t+1}^{s}\frac{MPS_k}{R_k}\frac{dR_s}{R_s}$.

It is worth noting that one can verify that for $t = -1$ this formula replicates Proposition 2 if we were to set $dc_{-1} = \mathbb{E}_{-1}^{Q}(\cdot) = \mathbb{E}_{-1}^{Q^I}(\cdot) = 0$.

To understand the economics in this formula, let us first discuss the case where there are only changes in income. First note that when wealth at s increases, consumption at θ^t increases by $\partial c_t/\partial a_s = \partial c_t/\partial a_t \prod_{k=s}^{t-1} MPS_k$. Developing and slightly rewriting equation (13), we get:

$$dc_t = \frac{\partial c_t}{\partial a_0}\underbrace{\mathbb{E}_0^{Q_I}(w_0)}_{\text{Change in wealth at 0}} + \sum_{s=1}^{t}\frac{\partial c_t}{\partial a_s}\underbrace{\left\{\mathbb{E}_s^{Q_I}(w_s) - \mathbb{E}_{s-1}^{Q_I}(w_s)\right\}}_{\text{Revaluation of wealth}}.$$

The response at t is the sum of two terms. First, there is the initial perceived change in human wealth upon announcement given by the Q^I-NPV of the income change. Second, over time the agent as the history of shocks θ^s unfolds the agent gets information and reevaluates the change in human wealth. Note that the reevaluation of wealth may occur because the changes in income are stochastic or because τ is stochastic. The consumption change dc_t may also be stochastic even without any wealth revaluations, due to the fact that $\frac{\partial c_t}{\partial a_0}$ depends on the history of shocks θ^t. Returning to equation (13), these two possibilities must be captured by uncertainty in Q_t^I/Q_t or ϵ_t along the path.

Note that the revaluation term is specific to incomplete markets, as the agent cannot freely transfer income across states. By contrast, with complete markets, the agent can freely reallocate the initial wealth change across states. The revaluation term cancels and the response is given by:[5]

$$dc_t = \frac{\partial c_t}{\partial a_0}\mathbb{E}_0^Q(w_0).$$

It is worth noting that the consumption response to a positive income shock is (weakly) positive at all dates in complete markets. Because of the revaluation of wealth, this not necessarily the case in incomplete market, even on average.

Finally, if the income change happens at 0, then there is no revaluation effect and the average consumption response at t takes a simple form:

$$dc_t/dy_0 = \partial_a c_t \Pi_{k=0}^{t-1} MPS_k.$$

It is easy to see that the discounted sum of the responses is exactly one, and that the stronger the marginal propensities to consume, the more front loaded the response is.

Turning to changes in interest rates, we have a similar revaluation of perceived interest rate changes along the IRF path. It should be noted that, even if the change is deterministic, the revaluation may occur due to the stochasticity of τ and the MPSs. Intuitively, as the agent moves closer to a borrowing constraint she becomes more disconnected from the future and less sensible to future interest rate changes. In this sense, the perceived interest rate change becomes larger after a "good" shock (e.g. higher income) and smaller after a "bad" one (e.g. lower income), even if the actual interest change is deterministic.

Why is it that even without revaluation of perceived interest rate or human wealth changes, the consumption response $dc_t/(\epsilon_t c_t)$ may still be disturbed along the path? This is related to precautionary effects in that it requires shocks to Q_t^I/Q_t. Once again we can also link this to a revaluation of the implicit Arrow-Debreu state prices under incomplete markets. To see this, consider first the complete market case where there is no such revaluation. Assuming that state prices remain constant and considering an interest rate change far in the future, we would have:

$$\frac{dc_{t+1}}{\epsilon_{t+1}c_{t+1}} = \frac{dc_t}{\epsilon_t c_t}.$$

Along the IRF path, in complete markets, consumption responds proportionally more in states where the value of consumption is more elastic. Note that this requires state

[5]The formula is only true if the constraint does not bind between 0 and t on the history θ^t. If it does the formula becomes $dc_t = \frac{\partial c_t}{\partial a_{\tau_0}}\mathbb{E}_{\tau_0}^Q\left(\sum_{k=\tau_0}^{T\wedge\tau}\prod_{k'=\tau_0}^{k-1}\frac{1}{R_{k'}}dy_k\right)$, where τ_0 is the last date at which the constraint binds between 0 and t.

dependent transfers. When the agent transfers funds only through a risk free asset, a dollar of saving at t instead translates to an increase in consumption at $t+1$ of $\partial_a c_{t+1} \neq \epsilon_{t+1} c_{t+1}$. Note that the ratio $\partial_a c_{t+1}/(\epsilon_{t+1} c_{t+1})$ is exactly proportional to Q^I/Q: when she saves more, the agent unexpectedly becomes richer, in "bad" states where the MPC is large or consumption low (compared to the complete market benchmark). Equivalently, this means that a decrease in the shadow Arrow-Debreu price at $t+1$ would be needed in states where the agent values wealth more to maintain the complete market relationship. Denoting by $q_t(\theta^{t+1})$ the price of θ_{t+1} (at θ^t) the revaluation of the state price is:

$$\frac{dq_t}{q_t} = \frac{dc_t}{\epsilon_t c_t} - \frac{dc_{t+1}}{\epsilon_{t+1} c_{t+1}} \propto 1 - \frac{Q_t^I}{Q_t}.$$

The relationship translates once again the fact that shadow Arrow-Debreu prices depend on the agent's wealth (when the curvature of the marginal value of wealth is not constant) in incomplete markets.

4.2 Slutsky Symmetry

A central result of price theory is that the price derivatives of the compensated demand function are symmetric. With our definition of the compensated responses in incomplete market, the symmetry breaks because of the precautionary effect. Indeed, a change in price at the initial time t creates no precautionary response since $Q_t = Q_t^I = 1$: there is no risk in contemporaneous changes and therefore no precautionary effects and no response of consumption (besides the substitution effect) at future dates. However a change in price at $s > t$ creates a precautionary effect at t: the precautionary effect is inherently asymmetric. We focus here on the substitution effect and show that a modified version of Slutsky symmetry exists with incomplete markets. To formally introduce changes in the price of consumption at history θ^{t+s}, we modify the budget constraint of the agent to:

$$\frac{a_{t+1}(\theta^{t+1})}{R_t(\theta^t)} = a_t(\theta^t) - p_t(\theta^t)(c_t(\theta^t) - y_t(\theta_t)).$$

In that case the compensated response to a price change $dp_{t_1}(\theta^{t_1})$ is defined as:

$$\frac{dc^S_{t_0}(\theta^{t_0})}{dp_{t_1}(\theta^{t_1})} = \frac{dc_{t_0}(\theta^{t_0})}{dp_{t_1}(\theta^{t_1})} + \frac{dc_{t_0}(\theta^{t_0})}{dy_{t_1}(\theta^{t_1})}(c(\theta^{t_1}) - y_{t_1}(\theta_{t_1}))$$

In words, the pure substitution effect is the total effect, including the implied change in wealth at t_0, of a price change at t_1, where the price change is compensated for at t_1, so that the precautionary effects are canceled.

Proposition 7. *The substitution effects satisfy for any s, $s' \geq 0$:*

$$\frac{Q_t(\theta^t)}{\prod_{k=0}^{t-1} R_k(\theta^k)} \frac{dc_t^S(\theta^t)}{dp_{t'}(\theta^{t'})} = \frac{Q_t(\theta^{t'})}{\prod_{k=0}^{t'-1} R_k(\theta^k)} \frac{dc_{t'}^S(\theta^{t'})}{dp_t(\theta^t)}, \tag{14}$$

In particular we have:

$$\mathbb{E}_0^Q \left(\frac{1}{\prod_{k=0}^{t-1} R_k(\theta^k)} \frac{dc_t^S}{dp_{t'}} \right) = \mathbb{E}_0^Q \left(\frac{1}{\prod_{k=0}^{t'-1} R_k(\theta^k)} \frac{dc_{t'}^S}{dp_t} \right). \tag{15}$$

As mentioned before, the measure Q is related to the price of the fictitious Arrow-Debreu securities that would make the consumption plan of the agent optimal in a complete market setting. More precisely, we have that $P^{AD}(\theta^t) = p_t(\theta^t)Q(\theta^t) / \prod_{k=0}^{t-1} R_k$, with $P^{AD}(\theta^t)$ the Arrow-Debreu price of state θ^t in complete markets. In complete markets, we can rewrite the equation as :

$$\frac{dc_t^S(\theta^t)}{dP^{AD}(\theta^{t'})} = \frac{p_{t'}(\theta^{t'})}{P^{AD}(\theta^{t'})} \frac{dc_t^S(\theta^t)}{dp_{t'}(\theta^{t'})} = \frac{p_t(\theta^t)}{P^{AD}(\theta^t)} \frac{dc_{t'}^S(\theta^{t'})}{dp_t(\theta^t)} = \frac{dc_{t'}^S(\theta^{t'})}{dP^{AD}(\theta^t)}$$

And we obtain the standard symmetry. This implies, in particular, that in incomplete market, we would recover the standard slutsky symmetry if we were to use the shadow Arrow-Debreu prices defined by $p_t(\theta^t)Q_t(\theta^t) / \prod_0^{t-1} R(\theta^k)$. In incomplete market, the relation expresses the fact that consumption reacts too much to changes in prices in states that are overpriced ($P^{AD}(\theta^t) < p_t(\theta^t)P_t(\theta^t) / \prod_0^{t-1} R(\theta^k)$) and too little in states that are underpriced. This comes from the fact that the agent cannot make state dependent transfers: in response to an increase in price at 0 for example, the agent would like to substitute towards states with higher Q but cannot do so because she only has access to a risk free bond.

4.3 Cross-Restrictions on Elasticities

Our previous formulas and decomposition were expressed in terms of the present value of price and income changes. As we now show, one can alternatively the consumption responses recursively in terms of the sensitivity of income to contemporaneous changes in prices and income. The following result characterizes the sensitivities of consumption to future income and interest rate changes.

Proposition 8. *For all $t \geq 0$, the sensitivities of consumption at t with respect to income and* *interest rate changes at T are given by*

$$\frac{\frac{y(\theta^t)}{c_0}\frac{\partial \hat{c}_0}{\partial y(\theta^t)}}{\frac{1}{c_0}\frac{\partial \hat{c}_0}{\partial a}} = \mathbb{E}_0^{Q^I}\left[\frac{\mathbb{1}(t \leq \tau)}{\prod_{s=0}^{t-1} R_s}\frac{\frac{y(\theta^t)}{c_t}\frac{\partial \hat{c}_t(\theta^t)}{\partial y(\theta^t)}}{\frac{1}{c(\theta^t)}\frac{\partial \hat{c}_t(\theta^t)}{\partial a}}\right], \tag{16}$$

$$\frac{-\frac{R_t(\theta^t)}{c_0}\frac{\partial \hat{c}_0}{\partial R_t(\theta^t)}}{\frac{1}{c_0}\frac{\partial \hat{c}_0}{\partial a}} = \mathbb{E}_0^{Q^I}\left[\frac{\mathbb{1}(t \leq \tau)}{\prod_{s=t}^{t-1} R_s}\frac{-\frac{R_t(\theta^t)}{c_t(\theta^t)}\frac{\partial \hat{c}_t(\theta^t)}{\partial R_t(\theta^t)}}{\frac{1}{c_t(\theta^t)}\frac{\partial \hat{c}_t(\theta^t)}{\partial a}}\right], \tag{17}$$

Proposition 8 relates the sensitivity of consumption to future changes in income or interest rates at 0 to the contemporaneous sensitivity of consumption at t to income and interest rates at t. In this way it characterizes a cross-restriction implied by the theory on different elasticities. It establishes the property that $[(y(\theta^t)/c_s)\partial\hat{c}_s/\partial y(\theta^t)]/[(1/c_s)(\partial\hat{c}_s/\partial a)]$ and $[-(R_t/c_s)(\partial\hat{c}_s/\partial R_t)]/[(1/c_s)(\partial\hat{c}_s/\partial a)]$ are discounted (at the prevailing interest rates) martingales under the probability measure Q^I, as long as no borrowing constraints binds, otherwise they become sub-martingales.

The response to interest rate change is here written in terms of Q^I while Proposition 2 expressed the response in terms of both Q and Q^I. In our setting, the two probabilities are related: for a an arbitrary $dR_t(\theta^t)$, we have

$$\mathbb{E}_0^{Q}\left(\prod_0^{t-1}\frac{MPS_s}{R_s}\frac{dR_t}{R_t}\mathbb{1}_{t\leq\tau}\right) = \mathbb{E}_0^{Q^I}\left(\prod_0^{t-1}\frac{1}{R_s}\frac{\partial_a c_0}{\partial_a c_t}\frac{\epsilon_t c_t}{\epsilon_0 c_0}\frac{dR_t}{R_t}\mathbb{1}_{t\leq\tau}\right).$$

This means in particular that in a deterministic setting, the time 0 response to a change in R_t can be written $R_t dc_0/dR_t = -\partial_a c_0/\partial_a c_t(c_t\epsilon_t MPS_t - \partial_a c_t a_{t+1})/\prod_0^t R_s$.[6]

Equations (16) and (17) can be understood as Euler equations for behavior. Just like the standard Euler equation, these are optimality conditions involving endogenous objects which characterize the objective that the agent is trying to achieve (smoothing his consumption). They cannot directly be used to solve exactly for the path of consumption, but they provide a unified framework and set of intuitions for the different effects at work.

4.4 Extensions

In the appendix, we generalize our benchmark model by allowing the agent to purchase multiple goods and assets in every period. This can be done in a straightforward manner: intertemporal decisions are fully determined by the per period expenditure function

[6]This formula is consistent with results for a deterministic setting in the working paper version of Auclert (2015), where the MPS discounting is implicitly contained in the MPC and consumption at t.

and the static indirect utility function. We can therefore characterize the probabilities Q and Q^I in terms of the per period indirect utility function and the marginal propensities to spend which replace the per period utility function and the marginal propensities to consume in the simple model. Having defined the relevant adjusted probabilities Q and Q^I, we can decompose the spending response in terms of income, substitution and precautionary effects, evaluate the welfare impact of income and price changes and explore the properties of the response, in particular the symmetry of the substitution matrix. The response of good demand is then characterized by the total spending response and the static elasticities: the price elasticities derived from the static Hicksian demand function and the static Engel curves.

We focus in particular on the comparative statics of expenditure with respect to good price changes. It can be summarized, as for interest rate changes, by an income, precautionary and substitution channels. As in the static Slutsky equation, the income effect of a change in the price of good i is proportional to expenditure share (at the time of the change) of good i. It is weighted by the state price Q for pure income effects while precautionary effects depend on the distance between Q and Q^I. To characterize the substitution effects, we show that a change in the vector of commodity prices at t, $\{dp_t^i\}$ generates a change in the price index at t given by $\partial c_t / \partial e_t \cdot dp_t$, where $\partial c_t / \partial e_t$ are the static Engel curves. Intuitively, a price matters more for the price index if agents consume more of it at the margin. Since an increase in the price index at t is equivalent to increasing the interest rate at t and lowering it at $t-1$, we obtain the same characterization for the substitution effects of commodity prices as for the interest rate.

In parallel, we show how a richer asset structure can improve agents' insurance. In particular, we show, on the one hand, how it can extend the planning horizon of the agent by bypassing the state where the constraint binds and, on the other mute, the precautionary effects, by characterizing the subspace on which the measure Q and Q^I coincide. Finally we derive the response of asset demand to changes in the sequence of price and income which allows us to characterize the full impulse response of expenditure.

In a more technical appendix, we explain how to extend our result to a continuous time environment.

4.5 Implications for Aggregate Consumption

So far, we have focused on individual consumption functions and characterized their sensitivities to income and interest rate changes. These individual consumption functions can be combined into an aggregate consumption function and our results can then be used to compute the sensitivity of the aggregate consumption function to income and interest rate changes.

To do so, assume that the initial joint distribution of assets a_t and productivity histories θ^t at date t is given by Ψ_t. The aggregate consumption function is given by :

$$C_t(\Psi_t; \{R_t\}, \{X_t\}) = \int \hat{c}_t(a_t, \theta^t; \{R_t\}, \{X_t\}) d\Psi_t(a_t, \theta^t).$$

To simplify exposition, we focus on the standard case were risk – captured by the process θ_t – is idiosyncratic. In that case the interest rate is deterministic. To streamline notation, we drop the stopping time τ which is implicitly captured, for example in Q^I, since by definition $Q^I(\theta^t) = 0$ for $t > \tau$.

Corollary 1. *When shocks are idiosyncratic, the aggregate sensitivity to aggregate income changes is given by:*

$$\frac{\partial C_0}{\partial X_t} = \partial_a C_0 \frac{\mathcal{Q}_t^I}{\prod_{k=0}^{t-1} R_k} \frac{\partial Y_t}{\partial X_t} \left\{ 1 + \mathrm{Cov}_0^{\Psi} \left(\frac{dQ^I/dP}{\mathcal{Q}_t^I}, \frac{\partial y_t/\partial X_t}{\partial Y_t/\partial X_t} \right) + \mathrm{Cov}_t^{\Psi} \left(\frac{\partial_a c_0}{\partial_a C_0}, \frac{\mathbb{E}_0^{Q^I}(\partial y_t/\partial X_t)}{\mathcal{Q}_t^I \partial Y_t/\partial X_t} \right) \right\}$$

where $\partial_a C_0 = \mathbb{E}_0^{\Psi}(\partial_a c_0)$ is the average marginal propensity to consume, $\partial Y_t/\partial X_t = \mathbb{E}_0^{\Psi}(\partial y_t/\partial X_t))$ is the impact on aggregate income and $\mathcal{Q}_t^I = \mathbb{E}^{\Psi}(Q^I(t \leq \tau))$ is the average Q^I probability that the constraint does not binds at horizon t.

The formula expresses the aggregate sensitivity to income changes by isolating three channels. When MPCs are constant across the distribution and that the impact of the X shifter is state independent[7] and constant across the distribution Ψ, the formula reduces to $\partial_a C_0 \, \mathcal{Q}_t^I / \prod R_k \, \partial Y_t/\partial X_t$. This term corresponds to a quasi representative agent response. However compared to the perfect foresight/representative agent case, the aggregate income change is discounted by $\mathcal{Q}_t^I$. In a perfect foresight environment, the response would be given by $\partial_a C_0 \, 1/\prod R_k \, \partial Y_t/\partial X_t$ and the income change is simply discounted by the gross interest rate R_t. Risk and binding borrowing constraints introduce an additional discounting $\mathcal{Q}_t^I$ which captures the average probability, taken with the probability Q^I, that agents' horizon is cut off before t. As discussed in Proposition 2, this discounting is larger than the average physical probability that the constraint binds: $\mathcal{Q}_t^I \leq \mathbb{E}_t^{\Psi}(\mathbb{1}(s \leq \tau))$. When the constraint never binds, we have $\mathcal{Q}_t^I = 1$ and we recover the perfect foresight response.

The second term in the formula, $\mathrm{Cov}_0^{\Psi}\left(dQ^I/dP/\mathcal{Q}_t^I, \partial_X y_t/\partial_X Y_t\right)$, introduces the first distributional impact of the aggregate shifter X_t. Keeping the marginal propensities to

[7] That is $\partial y_s/\partial X_s = \partial Y_s/\partial X_s$ in all states θ_s.

consume constant, the aggregate response deviates from the quasi representative agent one when the impact of X varies across states or the income/wealth distribution. The Q^I value of the change then differs from PQ_t^I since on the one hand, Q^I varies across states and on the other, agents value differently future states across the distribution Ψ. Incomplete markets therefore generate a new form of cross-sectional heterogeneity, through the risk valuation Q^I, that matters to understand aggregate demand beyond heterogeneous marginal propensities to consume. When the effect of X is state independent, but not constant across the distribution, this covariance term is positive when $\partial_X y_t$ is positively correlated with initial wealth and income. Indeed, income/wealth poor households have a higher chance to be constrained and value future income less. The impact is however more ambiguous when the incidence of X is state dependent. If the constraint never binds, Q^I is larger in "bad" states and heterogeneity in Q^I further dampens the response to the shifter X_t when the individual shocks are higher in good states. For example, for a "wage" shock given by $\partial_X y_t = y_t$, the covariance $\mathrm{Cov}_0^\Psi \left(dQ^I/dP, \partial_X y_t\right)$ would be negative when the constraint never binds. When the constraint binds however, this not as clear since the probability that the agent is constrained in a "bad" state is larger, in which case Q^I is zero. In our quantitative analysis of a wage shock, we see that the impact of Q^I is strongest in the middle of the distribution. At median income, agents are not constrained but face the highest variance of future income, so that the first effect dominates: heterogeneous risk valuations dampen the aggregate response.

Finally the third term introduces heterogeneity in marginal propensities to consume. For contemporaneous shocks, this is the only form of heterogeneity which matters and our formula reduces to:

$$\frac{\partial C_0}{\partial X_0} = \partial_a C_0 \frac{\partial Y_0}{\partial X_0} \left\{ 1 + \mathrm{Cov}_t^\Psi \left(\frac{\partial_a c_0}{\partial_a C_0}, \frac{\partial y_0 / \partial X_0}{\partial Y_0 / \partial X_0} \right) \right\},$$

which has been extensively used in the literature. For contemporaneous changes, heterogeneous *MPCs* amplifies the aggregate consumption response when income changes are largest at the bottom of the income/wealth distribution where the *MPCs* are largest. The impact of heterogeneous *MPCs* is however different for future income changes for the same reasons highlighted above. High *MPCs* agents are typically constrained and have therefore a low Q^I value of future income changes. The Q^I valuation of future states can therefore be anti correlated with the marginal propensities to consume. We now turn to the aggregate response to interest rate changes.

Corollary 2. *When shocks are idiosyncratic, the aggregate sensitivity to interest rate changes is* *given by:*

$$\frac{R_t}{C_0}\frac{\partial C_0}{\partial R_t} = \partial_a C_0 \frac{\mathcal{Q}_t^I}{\prod_{k=0}^{t-1} R_k}\frac{A_{t+1}}{C_0}\left\{1+\mathrm{Cov}_0^\Psi\left(\frac{dQ^I/dP}{\mathcal{Q}_t^I},\frac{a_{t+1}}{A_{t+1}}\right)+\mathrm{Cov}_0^\Psi\left(\frac{\partial_a c_0}{\partial_a C_0},\frac{\mathbb{E}_0^{Q^I}(a_{t+1})}{\mathcal{Q}_t^I A_{t+1}}\right)\right\}$$
$$-\bar{\epsilon}_0\frac{\mathcal{Q}_t}{\prod_0^t R_k}\mathcal{MPS}_0^t\left\{1+\mathrm{Cov}_0^\Psi\left(\frac{dQ/dP}{\mathcal{Q}_t},\frac{MPS_0^t}{\mathcal{MPS}_0^t}\right)+\mathrm{Cov}_t^\Psi\left(\frac{\epsilon_0 c_0}{\bar{\epsilon}_0 C_0},\frac{\mathbb{E}_0^Q(MPS_0^t)}{\mathcal{Q}_t\mathcal{MPS}_0^t}\right)\right\},$$

where $A_t = \mathbb{E}_0^\Psi(a_t)$ *denotes aggregate wealth,* $\bar{\epsilon}_0 = \mathbb{E}_0^\Psi(\epsilon_0 c_0/C_0))$ *is the average elasticity of intertemporal substitution and* $\mathcal{MPS}_0^t = \mathbb{E}_t^\Psi(\prod_{k=0}^t R_k(1-\partial_a c_k))$ *is the average marginal propensity to save between* 0 *and* t.

As in Proposition 2, the wealth effect of interest rate changes corresponds to an income change of $\partial_X y_t = a_{t+1}/R_t$. The impact of heterogeneous Q^I valuation is again ambiguous: across the distribution, higher income households have higher future wealth and higher average Q^I valuation of future states, but across states, Q^I is higher in "bad" states where wealth is lower. In our simulation, the second effect dominates and heterogeneous valuations overall mute the aggregate response. Note that in the case where there is no aggregate liquidity, the wealth response is purely explained by distributional effects.

We now turn to substitution effects. The quasi representative agent response is now given by $-\bar{\epsilon}_0\mathcal{Q}_t/\prod_0^t R_k\mathcal{MPS}_0^t$. In the perfect foresight case, the marginal propensities to save are uniformly equal to 1 – agents only consume the capital gains generated by an additional dollar of wealth – and the substitution effect is given by $-\bar{\epsilon}_0 1/\prod_0^t R_k$. Incomplete markets therefore introduce two forms of discounting. As for income effect an effective fraction $\mathcal{Q}_t$ of agents are constrained between t and s and do not respond to interest rate changes. Second, the quasi representative agent has a lower marginal propensity to save. This second discounting largely dominates the first. Indeed, being constrained depends on agents' wealth which is quite persistent: notwithstanding a sequence of particularly bad shocks, agents move on average slowly to the constraint. The effective fraction of constrained household at horizon t, $\mathcal{Q}_t$, is then typically almost constant. By contrast, risk affects all agents: it increases the average *MPC* and lowers the average marginal propensities to save. In the extreme case where *MPCs* are constant, we would have $\mathcal{MPS}_0^t = (R(1-\partial_a C))^{t+1}$, the discounting by the marginal propensities to save would at least be exponential. For an average annual *MPC* of 0.2 and an interest rate $R = 1.05$, the substitution effect at a 4 years horizon is at least halved compared to the perfect foresight case, simply through the *MPS* discounting.

The two distributional effects are similar to the wealth effect ones. Note however that when the elasticity of intertemporal substitution is constant across the distribution, the second distributional effect is given by $\text{Cov}_0^{\Psi}\left(c_0/C_0, \mathbb{E}_0^Q(MPS_0^s)/\mathcal{Q}_t\mathcal{MPS}_0^t\right)$. Since current consumption is correlated with marginal propensities to save, this second distributional effect amplifies the quasi representative agent response.

5 Illustrative Examples

We now present some simple examples to illustrate the different forces at work. In the interest of space, we focus on sensitivities to interest rates. We imagine that there is a one-time unanticipated announcement at t of a change in interest rate at $t+s \geq t$, and we characterize the response of consumption at t.

5.1 Borrowing Constraints Without Uncertainty

We first consider the case with no idiosyncratic risk. To simplify we consider a constant interest rate equal to the inverse of the discount factor $R_t = R = \beta^{-1}$. We also assume that utility is iso-elastic so that $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ if $\gamma \neq 1$ and $u(c) = \log(c)$ otherwise. We denote by $\epsilon = 1/\gamma = -u'(c)/(cu''(c))$ the intertemporal elasticity of substitution, which is the inverse of the coefficient of relative risk aversion. We assume a binding borrowing constraint between τ and $\tau+1$. Our course, without uncertainty the change of measure is trivial $Q = Q^I = P$. The point of this first example is therefore illustrates the impact of binding borrowing constraints while abstracting away from the risk- and prudential-adjustment effects.

Consumption is constant only up to τ with

$$c_t = c = \frac{1-R^{-1}}{1-R^{-1-\tau}}\left(a_0 + \sum_{s=0}^{\tau} R^{-s} y_s\right)$$

for $t \leq \tau$ while assets are not constant in general before τ and depend on the income path.[8] In this example the marginal propensities to consume can be shown to equal

$$\frac{\partial c_t}{\partial a_t} = \frac{1-R^{-1}}{1-R^{-1-\tau+t}}$$

which is increasing in t and decreasing in τ (only depending on the difference $\tau - t$).

For $t > \tau$, we trivially have $-\frac{R_t}{c_0}\frac{\partial c_0}{\partial R_t} = 0$, because the horizon is interrupted at τ.

[8] Indeed $a_t = (1-1/R^{\tau+1-t})/(1-1/R^{\tau+1})[a_0 + \sum_{k=0}^{\tau} y_k/R^k] - \sum_{k=0}^{\tau-t} y_{t+k}/R^k$.

Focusing on $t < \tau$ we have

$$-\frac{R_t}{c_0}\frac{\partial c_0}{\partial R_t} = \frac{1}{R^{t+1}}\left(\frac{R^{\tau+1} - R^{t+1}}{R^{\tau+1} - 1}\epsilon - a_{t+1}\right)$$

Without the borrowing constraint, the substitution effect at horizon t would be ϵ/R^{t+1}. With the borrowing constraint, the agent becomes more disconnected with the future: the MPCs increases over time as the borrowing constraint is approached; by implication, the MPSs fall over time towards zero. This introduces an additional discount in the substitution term given by

$$\frac{R^{\tau+1} - R^{t+1}}{R^{\tau+1} - 1} < 1$$

which is decreasing in t and reaches 0 at $t = \tau$. Thus, the substitution response decreases towards 0 as the horizon of the interest change t approaches the borrowing constraint. At a given horizon t, the discount is stronger as the distance to the constraint $\tau - t$ decreases. Conversely, note that as $\tau \to \infty$ this discount converges to 1.

The income effect from the interest rate change is standard. The impact of an interest rate change at t depends on whether the agent is a net borrower or a net saver at t ($a_t + y_t - c \leq 0$ or $a_t + y_t - c \geq 0$), which depends on the particular sequence of y_t.

Regarding income changes, for $t > \tau$, we have $\frac{\partial c_0}{\partial y_t} = 0$ and $\frac{\partial c_0}{\partial y_t} = \frac{1}{R^t}$ for $t \leq \tau$. The difference is simply that the horizon relevant for the computation of permanent income stops at τ when the borrowing constraint binds, as is well understood.

5.2 CARA utility: Uncertainty Without Borrowing Constraints

We consider here a simple stochastic environment. Agents have Constant Absolute Risk Aversion utility over consumption $u(c_t) = -1/\gamma e^{-\gamma c_t}$ and income y_t is iid with mean 0. Consumption is linear in wealth and income $c_t = (1 - 1/R)(a_t + y_t) + R\Gamma$, with $\Gamma = -1/\gamma(R-1)\left[ln(\beta R\mathbb{E}(exp(-\gamma(1-1/R)y))\right]$. We choose R such that the distribution of wealth is stationary, which gives $\Gamma = 0$.

As in the deterministic environment, marginal propensities to consume are constant $\partial_a c_t = (1 - 1/R)$ and marginal propensities to save are 1 everywhere. However the change of measure is non trivial, $Q(y^t) = Q^I(y^t) = exp(-\gamma(1-1/R)\sum_{k=1}^t y_k)/\mathbb{E}(exp(-\gamma(1-1/R)\sum_{k=1}^t y_k)$. This fundamentally changes the response to state dependent income shocks. For example with $\partial y_t/\partial X_t = y_t$, we have $\mathbb{E}(\partial y_t/\partial X_t) = 0$, that is permanent income is unaffected, but:

$$\frac{\partial c_0}{\partial X_t} = \frac{\partial_a c_0}{R^t}\mathbb{E}^{Q^I}\left(\frac{\partial y_t}{\partial X_t}\right)$$
$$= \frac{\partial_a c_0}{R^t}\mathbb{E}\left(\frac{exp(-\gamma(1-1/R)y)}{\mathbb{E}(exp(-\gamma(1-1/R)y)}y\right)$$

The change of measure overweighs bad realizations (y<0) and underweights good ones so the response of consumption is negative. When $y \sim \mathcal{N}(0,\sigma)$, this simplifies to:

$$\frac{\partial c_0}{\partial X_t} = -\frac{1}{R^t}\gamma[(1-1/R)\sigma]^2$$

Consumption decreases with risk aversion and with the variance of the income process.

From the budget constraint, asset after an history y^t is given by $a_t = a_0 + \sum_0^{s-1} y_k$: shocks are directly passed to assets. We can then easily specialize our formula for interest rate changes to get:

$$R\frac{\partial c_0}{\partial R_t} = -\epsilon(c_0)c_0\mathbb{E}_0^{Q}\left(\prod_{s=0}^{t}\frac{MPS_s}{R_s}\right) + \frac{\partial c_0}{\partial a_0}\mathbb{E}_0^{Q^I}\left(\frac{a_{t+1}}{R^{t+1}}\right)$$
$$= -\frac{1}{\gamma}\frac{1}{R^{t+1}} + \frac{1-1/R}{R^{t+1}}\left[a_0 + y_0 + t\mathbb{E}\left(\frac{exp(-\gamma(1-1/R)y)}{\mathbb{E}(exp(-\gamma(1-1/R)y)}y\right)\right]$$

The change of measure dampens the income effects of interest rate changes compared to the perfect foresight case. It can be seen more clearly when shocks are normally distributed, the expression is then simplified to:

$$R\frac{\partial c_0}{\partial R_t} = -\frac{1}{R^{t+1}}\left(\frac{1}{\gamma} - (1-1/R)(a_0+y_0) + s\gamma[(1-1/R)\sigma]^2\right)$$

Since shocks have a full pass-through to assets and assets is a random walk shocks accumulate quickly over time (hence the multiplication of the risk term by t). Therefore, even though the change in permanent income is given by $(1-1/R)(a_0+y_0)$, the change of measure makes the response negative at long horizons t for all levels of risk aversion and initial asset.

5.3 CRRA with Precautionary Effects

We consider here a simple stochastic environment. In period 0, agents have an income of $\bar{y}$. In period 1, a shock ϵ is realized and the agent receives $y_h = \bar{y} + \epsilon_h$ for all remaining periods with probability π_h and $y_l = \bar{y} + \epsilon_l$ with with complementary probability. We note by ϵ_i the deviations from average income $\bar{y}$ so that $\mathbb{E}(\epsilon) = 0$. The agent starts with wealth $a \geq \bar{y}$ in period 0 to insure that the borrowing constraint set at $a_t \geq 0$ never binds. Utility is logarithmic in consumption. We take $\beta = 1/R$ so that $c_t = (1 - 1/R)(a_t + y_t)$ for all $t \geq 1$. To simplify the formulas, we take $\epsilon_l = -\epsilon_h$ – so that $\pi_h = \pi_l = 1/2$. Consumption in period 0 is then given by:[9]

$$c_0 = \frac{R-1}{R} y_p - \frac{\kappa}{R} \quad with \quad \kappa = \frac{y_p}{2} \left(\sqrt{1 + \frac{4R}{R-1} \frac{Var(\epsilon)}{y_p^2}} - 1 \right)$$

The first term gives the perfect foresight consumption, where the agent consumes a fraction $1 - 1/R$ of permanent income $y_p = a + R/(R-1)\bar{y}$. Perfect foresight consumption is dampened by the term κ which increases with the variance of shocks and decreases with permanent income. The marginal propensities to consume and save are given in 0 by $\partial_a c_0 = 1 - 1/R + \lambda$ and $MPS_0 = 1 - \lambda/R$. The perfect foresight MPC and MPS (given by $1 - 1/R$ and 1) are increased and decreased respectively by $\lambda = 1/2(1 - 1/\sqrt{1 + 4R/(R-1)\, Var(\epsilon)/y_p^2})$, where λ is larger the larger the variance of the shocks is.[10] In all future periods, since uncertainty is resolved, the MPCs and MPSs are the same as in the perfect foresight case and wealth is constant given by $a_t = a + \kappa$. Because of this, the probabilities Q and Q^I do not play a role for the response to interest rate which is given by:

$$-\frac{R}{c_0} \frac{\partial c_0}{\partial R_t} = \frac{1}{R^{t+1}} \left(\sigma(1 - \frac{\lambda}{R}) - (1 + \frac{R}{R-1} \lambda) \frac{a + \kappa}{y_p - \frac{\kappa}{R-1}} \right).$$

When $Var(\epsilon) = 0$, we have $\lambda = \kappa = 0$ and we recover the formulas of 5. The first term is the substitution effect and shows how risk affects the discounted by the marginal propensities to save highlighted in Proposition 2: as risk increases, MPS_0 decreases and the substitution effect is dampened. Since risk increases savings and MPC while decreasing consumption at time 0, the income effect is increased which overall decreases the consumption response.[11] While the probabilities Q and Q^I do not matter, in this example, for interest rate changes, they do for state dependent income changes. They are given

[9]Without the additional assumption, we have $\kappa = \frac{y_p}{2} \left(\sqrt{\left[1 + \frac{R(\epsilon_h + \epsilon_l)}{(R-1)y_p}\right]^2 + 4\frac{R-1}{R} \frac{Var(\epsilon)}{y_p^2}} - \left[1 + \frac{R(\epsilon_h + \epsilon_l)}{(R-1)y_p}\right] \right)$.

[10]Additionally, λ decreases with permanent income.

[11]The income effect enters negatively in the consumption response.

by:

$$\frac{dQ}{dP}(\epsilon_{i,t}) = \frac{y_p - \frac{\kappa}{R-1}}{y_p + \kappa + \tilde{\epsilon}_i}$$
$$\frac{dQ^I}{dP}(\epsilon_{i,t}) = \frac{1 - \frac{\lambda}{R}}{1 + \frac{R}{R-1}\lambda}\left(\frac{y_p - \frac{\kappa}{R-1}}{y_p + \kappa + \tilde{\epsilon}_i}\right)^2.$$

Where to simplify notation we note $\tilde{\epsilon} = R/(R-1)\epsilon$. The measure Q overweighs the low state ϵ_l: we have $Q(\epsilon_l)/Q(\epsilon_h) = (y_p + \kappa + \tilde{\epsilon}_h)/(y_p + \kappa + \tilde{\epsilon}_l)P(\epsilon_l)/P(\epsilon_h)$. In terms of Q, the odds of $\tilde{\epsilon}_l$ happening increase with $\tilde{\epsilon}_h - \tilde{\epsilon}_l$[12] and decrease with permanent income. Similarly we have $Q^I(\epsilon_l)/Q^I(\epsilon_h) = (y_p + \kappa + \tilde{\epsilon}_h)/(y_p + \kappa + \epsilon_l)Q(\epsilon_l)/Q(\epsilon_h)$ and Q^I further increases the odds of ϵ_l compared to Q. The consumption response to a increase in income in the low state is then given by:

$$R\frac{\partial c_0}{\partial y_l} = \frac{(1 - R^{-1} + \lambda)}{\pi_l + (1 - \pi_l)\left(\frac{y_p + \kappa + \tilde{\epsilon}_l}{y_p + \kappa + \tilde{\epsilon}_h}\right)^2}\pi_l.$$

Where $1 - R^{-1} + \lambda$ is the initial MPC. By comparison a sate independent increase in income $dy = \pi_l$ which has the same present value as the y_l increase gives:

$$R\frac{\partial c_0}{\partial y_l} = (1 - R^{-1} + \lambda)\pi_l.$$

The impact of risk is twofold. First compared to 5, the MPC is increased by λ. Second the measure Q^I further increases the response by overweighting the odds of ϵ_l by $((y_p + \kappa + \tilde{\epsilon}_h)/(y_p + \kappa + \tilde{\epsilon}_l))^2$ compared to the physical measure. We can use the decomposition of Corollary 5 to explicit the channels of the response. First the income effect is given by:

$$\frac{(1 - R^{-1} + \lambda)}{\pi_l + (1 - \pi_l)\left(\frac{y_p + \kappa + \tilde{\epsilon}_l}{y_p + \kappa + \tilde{\epsilon}_h}\right)}\pi_l.$$

The pure income effect is lower than the full response: the discrepancy between the two is due to a precautionary effect. Agents save in period 0 to increase consumption in the low state but since they can only do so with a safe bond, some of it is wasted in the high state. An income increase in the low state therefore increases consumption beyond the

[12] Both directly and through κ which increases with the variance of ϵ.

income effect. The ratio of the precautionary effect and the income effect is given by:

$$\frac{\frac{|\tilde{\epsilon}_l|}{c_l}}{1+2\frac{\pi_l}{1-\pi_l}\frac{|\tilde{\epsilon}_l|}{c_l}+\frac{\pi_l}{(1-\pi_l)^2}\left[\frac{|\tilde{\epsilon}_l|}{c_l}\right]^2} \quad with \quad c_l = y_p+\kappa+\tilde{\epsilon}_l$$

In the relevant range, the ratio of the precautionary effect to the income effect increases with $\tilde{\epsilon}_l/c_l$: when $\tilde{\epsilon}_l$ contributes more to c_l, the consumption in the low state, the amplification of the consumption response through the precautionary effect is larger.

6 Quantitative Illustration

In this section, we consider a standard Bewley-Aiyagari-Huggett model of incomplete markets. This model features not only occasionally binding borrowing constraints but also precautionary savings. As a result, individual consumption functions are no longer linear but are instead concave so that marginal propensities to consume depend on the position in the asset distribution. Moreover, precautionary savings activates the change of measure terms in our formulas. We explore the responses of aggregate and individual consumption to change in interest rates and to several types of income shocks. We perform two exercises: first we quantify the contribution of substitution, income and precautionary effects using the results of Proposition 2 and Corollary 5. Second, we compare the responses to the standard perfect foresight responses. In particular we examine how the borrowing constraint and the adjusted probabilities Q and Q^I contributes to the dampening of the perfect foresight responses.

6.1 Calibration

There is a unit mass of infinitely-lived agents. Time is discrete with a period taken to be a quarter. Agents have logarithmic utility $\sigma = 1$ and discount factor β.

There is a unit supply of Lucas trees capitalizing the flow of dividends δY_t, where Y_t is aggregate output. Agents face idiosyncratic non-financial income risk $y_t(1-\delta)Y_t$. The idiosyncratic income process is $\log(y_t) = \rho_\theta \log(y_{t-1}) + \theta_t$, where θ_t is i.i.d. over time, independent across agents and follows a normal distribution with variance σ_θ^2 and mean $\mathbb{E}[\theta_t] = -\sigma_\theta^2/2$ so that $\mathbb{E}[e^{\theta_t}] = 1$.

Agents can borrow and lend subject to borrowing constraints. We assume that the borrowing contracts take the same form as the Lucas trees. We also assume that the borrowing constraints take a simple form, namely that agents cannot have a negative position.

We denote by Ψ_t the wealth distribution.

This model cannot be solved analytically, and so we rely on simulations instead. We consider a steady state $\{Y, R, \Psi\}$ of the model with a 2% annual interest rate and a corresponding quarterly interest rate of $R = 1.005$. We take $\rho_\theta = 0.966$, and $\sigma_\theta^2 = 0.017$ for the idiosyncratic income process as in McKay et al. (2015) and Guerrieri and Lorenzoni (2015). For our baseline economy, we take $\frac{V}{Y} = 1.44$ for the fraction of outside liquidity to output, exactly as in McKay et al. (2015).[13] The values of $\beta = 0.988$ and $\delta = 0.035$ are calibrated to deliver these values of R and $\frac{V}{Y}$. The fraction of borrowing-constrained agents in the steady state is then 14.7%.

6.2 Interest Rate Responses

We first focus on the aggregate effects of interest rates. The total response of an interest rate shock at t at the time of the announcement is given, as in Proposition 2 and Corollary 5 by:

$$\frac{R}{c_0}\frac{dc_0}{dR_t} = \underbrace{-\epsilon(c_0)\mathbb{E}_0^Q\left(\prod_0^t \frac{MPS_s}{R}\mathbb{1}_{\tau>t}\right)}_{\text{substitution effect}}$$
$$+\underbrace{\frac{1}{c_0}\frac{\partial c_0}{\partial a_0}\mathbb{E}_0^Q\left(\mathbb{1}_{\tau>t}\frac{a_{t+1}}{R^{t+1}}\right)}_{\text{welfare effect}} + \underbrace{\frac{1}{c_0}\frac{\partial c_0}{\partial a_0}\mathrm{Cov}_0^Q\left(\frac{Q^I}{Q}, \mathbb{1}_{\tau>t}\frac{a_{t+1}}{R^{t+1}}\right)}_{\text{precautionary effect}}$$

We aggregate these individual consumption responses as in Section 4.5, according to the steady state distribution of income and assets Ψ. For example the aggregate substitution effect is simply given by:

$$-\mathbb{E}_0^\Psi\left(\epsilon(c_0)c_0\mathbb{E}_0^Q\left(\prod_0^t \frac{MPS_s}{R}\frac{\mathbb{1}_{\tau>t}}{R}\right)\right).$$

We plot in Figure 1 the aggregate consumption response at time 0 of an interest rate change at different time horizons t, and examine the contribution of the substitution, income and precautionary effect to the total response of consumption.

Due to our logarithmic specification, the substitutions effects (in red) largely dominate. The wealth effect (in yellow) slightly counterbalances the substitution effects. The precautionary effects (in purple) while small compared to the full response, significantly

[13]This value for the fraction of outside liquidity to output $\frac{V}{Y} = 1.44$ is meant to capture the value of liquid (as opposed to illiquid) wealth in the data.

dampens the income effects (by approximately 15% at long horizons). In the next subsections, we explore how the borrowing constraints and the risk adjusted probabilities Q and Q^I contributes to the substitution and income/precautionary effects and we will see that borrowing constraints and risk significantly dampens the standard risk-less response.

6.2.1 Attenuation of Substitution Effects

We first explore how market incompleteness and borrowing constraints affect the substitution effects. As explained in Proposition 2, the substitution effects are given by the elasticity of substitution at time t (here equals to 1) times the present value of the change in prices. For a change dR_t at t, the effect is given by:

$$\left[\frac{dc_0}{dR_t}\right]_{subs} = -\epsilon(c_0)c_0\mathbb{E}_0^Q\left(\prod_0^t \frac{MPS_s}{R}\frac{\mathbb{1}_{\tau>t}}{R}\right)$$

We benchmark our simulated response against the standard risk-less response. As described in Section 5, the standard response is characterized by $Q = Q^I = P$ – the agent valuations Q and Q^I are given by the physical measure precisely because there is no risk –, by constant marginal propensities to save $R(1-\partial_a c) = 1$, and by never binding borrowing constraints $\tau = \infty$:

$$\left[\frac{dc_0}{dR_t}\right]_{riskless} = -\epsilon(c_0)c_0\prod_0^t \frac{1}{R_s}\frac{1}{R_t}$$

The difference between the riskless response and the simulated one, is explained by three elements that we introduce sequentially: the binding borrowing constraints summarized by the stopping time τ, the discounting introduced by the marginal propensities to save $\prod_0^t MPS_s/R_s$ and finally the risk valuation Q. We introduce each of these elements to understand how they contribute to the dampening of the risk-less response.

We first consider the direct effect of borrowing constraints, namely that the agent does not respond to shocks occurring after the constraint binds. Future state are still valued according to the physical measure P and we keep the marginal propensities to save equal to 1 but cut the planning horizon of the agent at τ

$$\left[\frac{dc_0}{dR_t}\right]_{constraint} = -\epsilon(c_0)c_0\mathbb{E}_0\left(\frac{1}{R^{t+1}}\frac{\mathbb{1}_{\tau>t}}{R}\right) = -\epsilon(c_0)c_0\frac{1}{R^{t+1}}\frac{P(\tau>t)}{R}$$

The ratio of the riskless response and the constrained response $[dc_0/dR_t]_{constraint}$ directly gives the probability that the agent is constrained before t and provides the first source of attenuation.

Second we account for the stochastic discounting encompassed in the MPS_s which allows us to quantify more precisely the dampening of the riskless response generated by the agent's inability to perfectly transfer funds across states. Formally, future states are still valued according to P, but the planning horizon of the agent is cut at τ and the future is discounted according to $\prod_0^t MPS_s/R_s$.

$$\left[\frac{dc_0}{dR_t}\right]_{discount} = -\,\epsilon(c_0)c_0\mathbb{E}_0\left(\prod_0^t \frac{MPS_s}{R}\,\frac{\mathbb{1}_{\tau>t}}{R}\right)$$

Finally, we consider the full response:

$$\left[\frac{dc_0}{dR_t}\right]_{subs} = -\,\epsilon(c_0)c_0\mathbb{E}_0^Q\left(\prod_0^t \frac{MPS_s}{R}\,\frac{\mathbb{1}_{\tau>t}}{R}\right)$$

The ratio of the full response to the discounted response $[dc_0/dR_t]_{discount}$ allows us to quantify how much of the attenuation is due to the risk measure Q. Again, the measure Q gives the agents' "shadow" valuation of future states and translates, when it differs from P their lack of insurance.

Figure 2 pictures the aggregate substitution effects at time t depending on the horizon at which the change in interest rate happens. The direct effect of the borrowing constraints is rather limited, attenuating the substitution effects by less than 10%. This is due to two facts: first, agents who are initially constrained have lower consumption and second, agents who are unconstrained move, on average, rather slowly to the constraint. Indeed, note that the slope of the $[dc_0/dR_t]_{constraint}$ curve barely differs from the $[dc_0/dR_t]_{riskless}$ slope, indicating that if agent are not initially constrained, the probability that they will be constrained at a future date increases slowly. In contrast, the impact of the marginal propensities to save is significant, almost halving the aggregate response to an interest rate change at a four year horizon. Because of the tight borrowing constraints, agents dissave at the margin which hinders their ability to transfer funds in the future, even when they are not constrained. This additional discounting impact all households and goes largely beyond the direct effect of borrowing constraints. Finally, the measure Q introduce relatively little discounting. Quantitatively, the variance of the Radon-Nikodym dderivative dQ/dP is small, which indicates that agents are relatively well insured, on average, conditional on their initial state.

To get a better understanding of these aggregate effects, we examine in Figures 3, 4 and 5, the individual responses at different level of income. At the disaggregated level, the direct effect of the borrowing constraint is strongest at the bottom of the asset distribution for low income agents. However, even at the lowest income level, agents with median asset holdings are not be constrained at a 4 years horizon and still barely respond to interest rate changes. This is again due to their low marginal propensities to save, as they are below their future expected income but cannot consume at their permanent income level. The same patterns are present at medium and high income levels. The risk measure Q plays a slightly more important role at medium income level since the variance of their future income is larger, but its role in attenuating consumption responses remains very limited.

6.2.2 Attenuation of Income Effects

We now turn to wealth and precautionary effects. As in Proposition 2, those effects are given by the marginal propensity to consume at time t (here equals to 1) times the present value of the change in income valued according to Q^I. For a change dR_t at t, the response is given by:

$$\left[\frac{dc_0}{dR_t}\right]_{income} = \partial_a c_0 \mathbb{E}_0^{Q^I}\left(\frac{1}{R^{t+1}}\mathbb{1}_{\tau>T}\frac{a_{t+1}}{R}\right)$$

We then again quantify the effects of the borrowing constraints and of the effect of the adjusted probabilities Q and Q^I. To do this, we benchmark the simulated response against a "virtual" riskless response. The standard riskless response is given as in Section 5 by $Rdc_0/dR_t = \partial_a c_0 a_0/R^{t+1}$: since the agent does not save or dissave, assets remain constant at their initial value a_0. In the simulated response however, assets of course fluctuate so that the flow of income changes generated by a change in the path of interest rate $\{a_{t+1}/RdR_t/R\}_{risky}$ differs from the true riskless response. To better quantify the effect of the change of valuation and of the borrowing constraint, we therefore define a virtual riskless response where the fluctuating flow of income change $\{dy_t^R\} = \{a_{t+1}/RdR_t/R_t\}_{risky}$ is valued according to the measure P and where the borrowing constraint does not bind, that is:

$$\left[\frac{dc_0}{dR_t}\right]_{riskless} = \partial_a c_0 \mathbb{E}_0\left(\frac{1}{R^{t+1}}\frac{a_{t+1}}{R}\right)$$

$[dc_0/dR_t]_{riskless}$ gives the consumption response of a fictitious agent with the same marginal propensity to consume and flow of assets as the simulated Bewley-Aiyagari-Huggett agent but who is risk neutral and values the future according to the physical measure P.

Second we introduce the borrowing constraints:

$$\left[\frac{dc_0}{dR_t}\right]_{constraint} = \partial_a c_0 \mathbb{E}_0 \left(\frac{1}{R^{t+1}} \mathbb{1}_{\tau>t} \frac{a_{t+1}}{R_t}\right)$$

$[dc_0/dR_t]_{constraint}$ only incorporates the direct effect of the borrowing constraint, that is the the flow of income changes $\{dy_t^R\} = \{a_{t+1}/RdR_t/R\}_{risky}$ is cut at the first time the borrowing constraint binds given by the stopping time τ. The ratio of $[dc_0/dR_t]_{constraint}$ to $[dc_0/dR_t]_{riskless}$ therefore gives the fraction of the total income change that can be transferred at time t without violating the borrowing constraint.

Third, we consider the change of measure Q:

$$\left[\frac{dc_0}{dR_t}\right]_{wealth} = \partial_a c_0 \mathbb{E}_0^Q \left(\frac{1}{R^{t+1}} \mathbb{1}_{\tau>t} \frac{a_{t+1}}{R}\right)$$

$[dc_0/dR_t]_{wealth}$ represent the consumption response when future states are valued according to their Arrow-Debreu prices. It represents the wealth effect of the interest rate change. In addition, $\mathbb{E}_0^Q \left(\frac{1}{R^{t+1}} \frac{a_{t+1}}{R}\right)$ is exactly the compensating variation for the shock arising at the planning horizon of the agent, that is $\mathbb{1}_{\tau>t} dR_t$. The ratio of $[dc_0/dR_t]_{wealth}$ to $[dc_0/dR_t]_{constraint}$ is then the ratio of the wealth value to the physical value of the income change.

Finally, we consider the complete response in which the future is valued according to the measure Q^I:

$$\left[\frac{dc_0}{dR_t}\right]_{income} = \partial_a c_0 \mathbb{E}_0^{Q^I} \left(\frac{1}{R^{t+1}} \mathbb{1}_{\tau>t} \frac{a_{t+1}}{R_t}\right)$$

As explained in section 3.4, the difference between $[dc_0/dR_t]_{income}$ and $[dc_0/dR_t]_{wealth}$ gives the precautionary effect of the interest rate change.

Figure 6 pictures the aggregate wealth and precautionary effects at time t depending on the horizon at which the change in interest rate happens. The welalth and precautionary effects are small compared to the substitution effects since, with our *log* specification of utility, agents have a high elasticity of intertemporal substitution. The first term of our decomposition, the riskless response $[dc_0]_{riskless}$, is upward slopping as agent with low initial wealth have higher wealth, on average, in the future and high marginal propensities to consume – and vice versa for agents with high initial wealth. The presence of borrowing constraints cancels this shifting of expected wealth to high MPCs agents. Note again that the slope of $[dc_0/dR_t]_{constraint}$ is small as agent become constrained at a very slow rate. The change of measure Q plays a very limited role initially, as with substitution effects, but becomes more significant at longer horizon. Indeed future wealth is very anti correlated with the valuation of future states. Finally, the precautionary effect plays a significant role in explaining the attenuation of the interest rate change, especially at horizon shorter than two years, where the effect of the borrowing constraints and the change of measure Q are negligible.

At the disaggregated level (Figures 7, 8 and 9), the channels through which the interest rate change is attenuated are very different depending on the initial income level of the agent. At low income levels, virtually all of the attenuation is due to the presence of borrowing constraint. Agents without assets would still respond to a future change in interest rate when we shut down the constraint, since they have positive wealth in expectation at future dates. For medium and high income households, most of the attenuation is due to a precautionary effect. This is particularly strong for agents with medium income. Even though the wealth effect is close to the riskless effect in the bottom two quartiles of the asset distribution, the precautionary effect completely cancels it.

6.3 Aggregate Income Changes

We focus here on the aggregate response of consumption to income shocks. The total response of an interest rate shock at $t+s$ at the time of the announcement is given, as in Proposition 2 and Corollary 5 by:

$$\frac{dc_0}{dX_t} = \underbrace{\frac{\partial c_0}{\partial a_0}\mathbb{E}_0^Q\left(\mathbb{1}_{\tau>t}\frac{\partial y_t/\partial X_t}{R^t}\right)}_{\text{wealth effect}} + \underbrace{\frac{\partial c_0}{\partial a_0}\mathrm{Cov}_0^Q\left(\frac{Q^I}{Q}, \mathbb{1}_{\tau>t}\frac{\partial y_t/\partial X_t}{R^t}\right)}_{\text{precautionary effect}}$$

We consider three types of of income shocks. The first one is state independent: all agents will receive one more dollar at a future date and $\partial y_t/\partial X_t = 1$. The second is a wage increase: the income change is proportional to agents' income: $\partial y_t/\partial X_t = y_t$. The third one is a transitory increase in the productivity of high types: $\partial y_t/\partial X_t = 1$ for $y_t = \max y_t'$ and 0 otherwise. We then aggregate the responses linearly according to the steady state distribution of wealth Ψ. To make the shocks comparable, we normalize their size so that the aggregate contemporaneous response is 1. The results are plotted in Figure 10.

As in the previous section, we benchmark the responses against the standard perfect foresight response where the borrowing constraint never binds given by:

$$\left[\frac{dc_0}{dX_t}\right]_{riskless} = \partial_a c_0 \mathbb{E}_0\left(\frac{1}{R^t}\frac{\partial y_t}{\partial X_t}\right)$$

As before, $[dc_0/dX_t]_{riskless}$ gives the consumption response of a risk neutral agent who values the future according to the physical measure P.

We then consider the change of measure Q:

$$\left[\frac{dc_0}{dX_t}\right]_{wealth} = \partial_a c_0 \mathbb{E}_0^Q \left(\frac{1}{R^t} \mathbb{1}_{\tau>t} \frac{\partial y_t}{\partial X_t}\right),$$

which gives the first term of our decomposition, the wealth effect.

Finally, we consider the complete response in which the future is valued according to the measure Q^I:

$$\left[\frac{dc_0}{dX_t}\right]_{income} = \partial_a c_0 \mathbb{E}_0^{Q^I} \left(\frac{1}{R^t} \mathbb{1}_{\tau>t} \frac{\partial y_t}{\partial X_t}\right).$$

The difference between $[dc_0/dX_t]_{income}$ and $[dc_0/dX_t]_{wealth}$ gives the second term of our decomposition, the precautionary effect.

The blue line represents the "riskless response": each agent is given at t_0 the present discounted value of their future income change. The response incorporates the heterogeneity in marginal propensities to consume but not the heterogeneity in risk valuation. When future change are valued with respect to Q, we get the wealth effect. The aggregate wealth effect is one fifth the riskless response for a state independent shock. Finally the total response is given when future income is valued with Q^I. The shaded area between the wealth effect and the total response gives the precautionary effects. Precautionary effects are strongest for shocks that occur in good states, that is when agents are highly productive and earn y^h. Indeed, if we consider a median agent, she should consume more initially in expectation of a shock arising in a good state. This would however decrease consumption in bad states which explains the large precautionary effect.

Finally, we examine more precisely how the precautionary effect contributes to the dampening of income shocks. Formally, in Figure 11 we consider the ratio $1 - [dC_0]_{income} / [dC_0]_{wealth}$, which expresses the precautionary effect as a percentage of the wealth effect. While precautionary are negligible at an horizon of one or two quarters they rapidly become large: at a two year horizon, they reduce wealth effects by 10% to 35% depending on the nature of the shock.

6.4 Taking Stock

As in the representative agent model, with logarithmic utility, the aggregate response to interest rate changes is mostly explained by substitution effects. However, the effect is quite muted compared to the complete market-representative agent case: in our standard Bewley-Aiyagari-Huggett specification, the substitution effects are 10% lower for a contemporaneous change and 50% lower at a 4 years horizon. This dampening is mostly explained by the *MPS* discounting highlighted in Proposition 2. Indeed, because of the tight borrowing constraints, agents dissave at the margin which hinders their ability to transfer funds in the future, even when they are not constrained. The discounting affects all households responses and goes largely beyond the direct effect of borrowing constraints. By contrast the measure Q plays a limited role: examining the disaggregated responses, this stems from the fact that the marginal propensities to save are quite persistent so that the average marginal propensities to save according to the measure Q or P are almost identical. Similarly, the adjusted probabilities Q and Q^I are almost irrelevant for state independent income change. This indicates that the probability of being constrained taken with the physical measure P or the measure Q and Q^I are quasi identical. However they are fundamental to understand state dependent income changes, stemming for example from interest changes or wage changes. In these cases, the income change is lower in "bad" states which triggers a precautionary effect through the measure Q^I. The aggregate consumption response is largely dampened, an effect that goes well beyond the impact of heterogeneous marginal propensities to consume. The strength of the precautionary effect across the distribution is in fact uncorrelated with the marginal propensities to consume and is in this sense an independent channel to understand aggregate responses. Indeed, precautionary effects are small at the bottom and top of the asset and income distribution because income/wealth poor agents are constrained while income/wealth rich agents are sufficiently insured. They are strongest in the middle of the distributions where the *MPC*s are close to the average. This is because household in the middle of the distribution have the highest variance of future income and wealth. Therefore, changes correlated with future income and wealth such as changes in interest rate or wage creates a large precautionary effect for these agents.

7 Conclusion

We have characterized the sensitivity of consumption to income and price changes in incomplete markets. In the presence of uncertainty, our characterization relied on two adjusted probabilities. These probabilities, together with contemporaneous elasticities, elasticity of intertemporal substitution, marginal propensity to spend and the static elasticity of substitution and Engel curves, provide sufficient statistics to characterize consumption responses. The probability Q captures the welfare value of wealth shock and allows us to explicit the income and substitution channel of price changes thereby extending their complete market definition to an incomplete market setting. The probability Q therefore provides a direct link between agents' consumption responses and the welfare impact of shocks. The probability Q^I is specific to incomplete markets and determines the precautionary effect of shocks. The difference between the probabilities encapsulates the effect of market incompleteness and has an economics interpretation as the maximal strength of the precautionary effects of shocks.

Quantitatively, our simulations show that the probability Q and Q^I matter well beyond the heterogeneity in marginal propensities to consume or the direct impact of borrowing constraint. Since contemporaneous elasticities (e.g. marginal propensities to consume, elasticity of intertemporal substitution) do not provide enough information to evaluate the impact of policies, it is important for future research to incorporate in their analysis estimates of households' risk valuations. Our framework provides the tools to recover these valuations from reduced form consumption responses and to connect them structurally to the agent's decision problem.

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A Figures

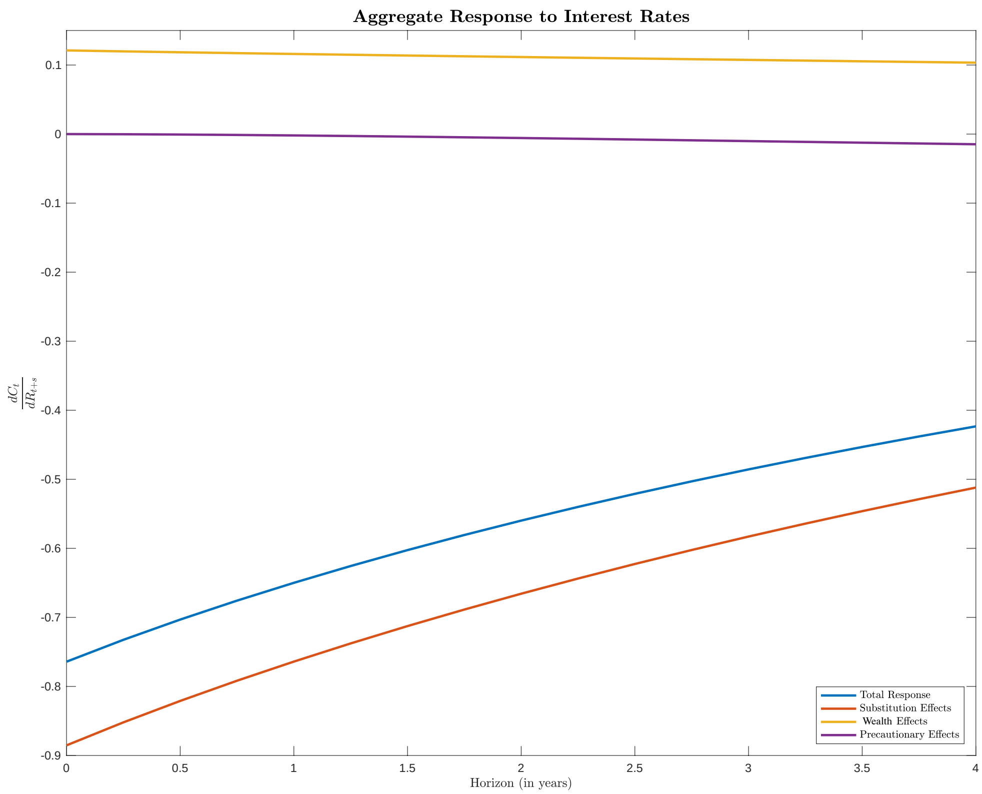

Figure 1: Response to interest rate change and decomposition.

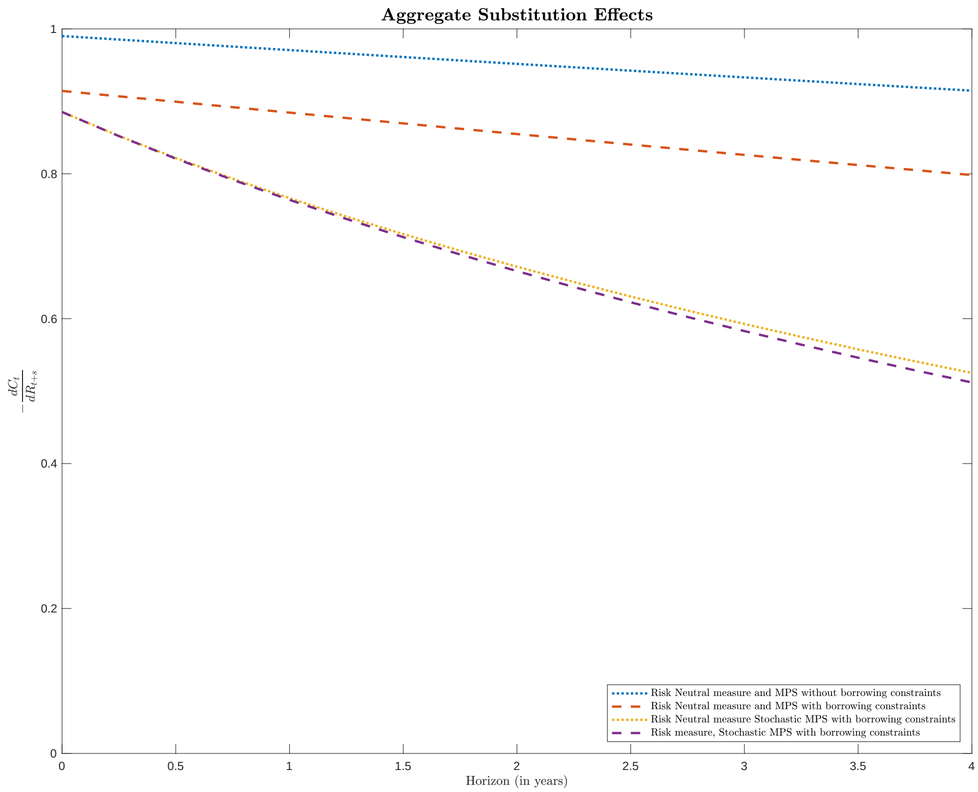

Figure 2: Aggregate substitution effects and its decomposition.

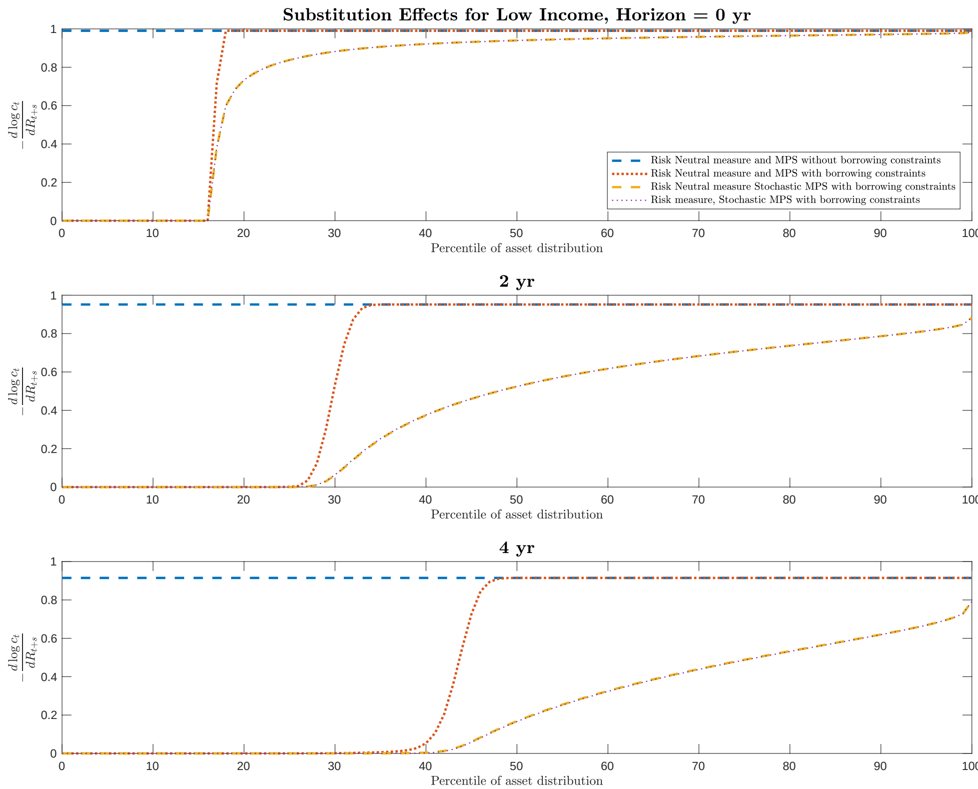

Figure 3: Substitution effects for low income agents.

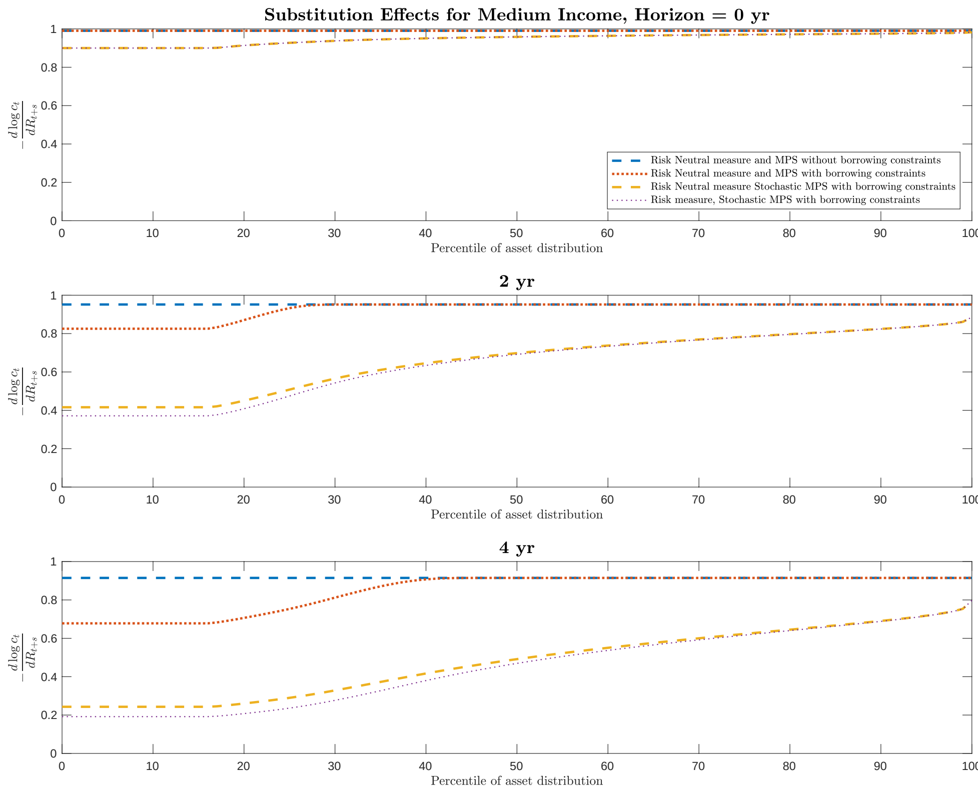

Figure 4: Substitution effects for medium income agents.

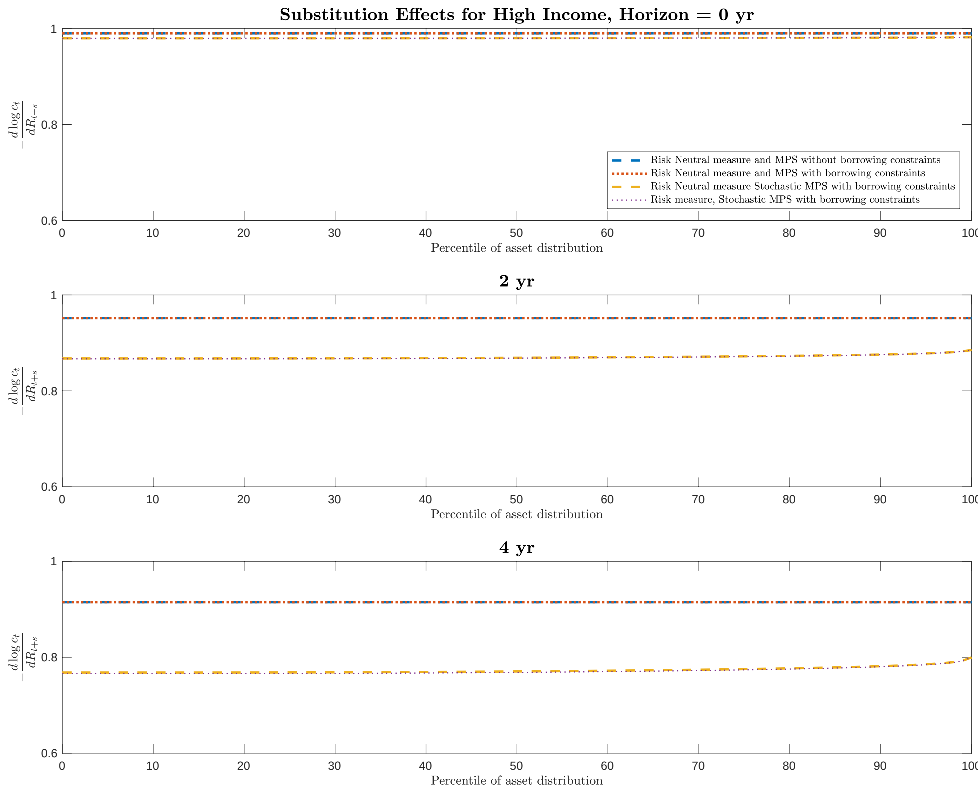

Figure 5: Substitution effects for high income agents.

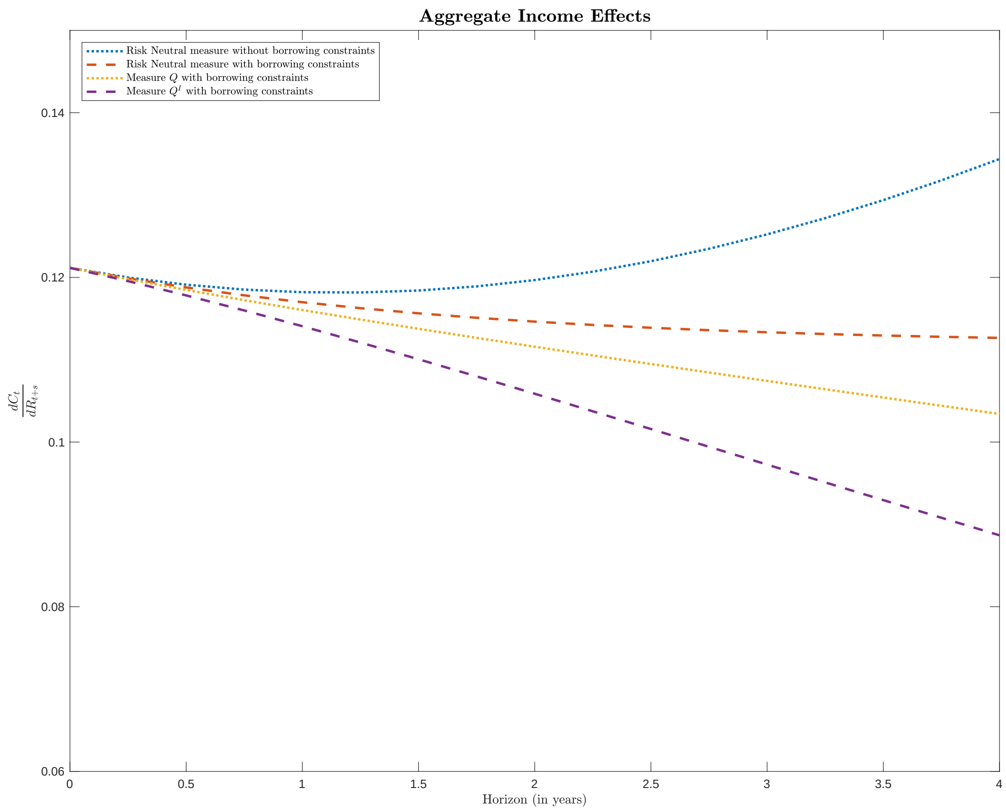

Figure 6: Aggregate income effect of interest rate changes and its decomposition.

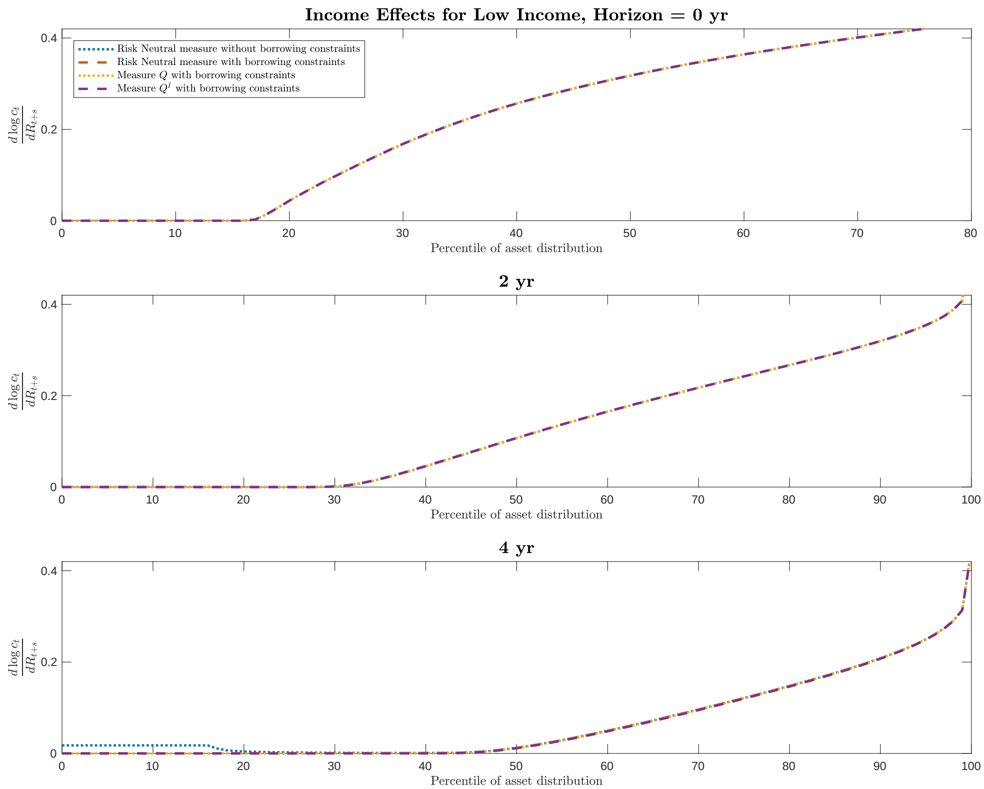

Figure 7: Income effects for low income agents.

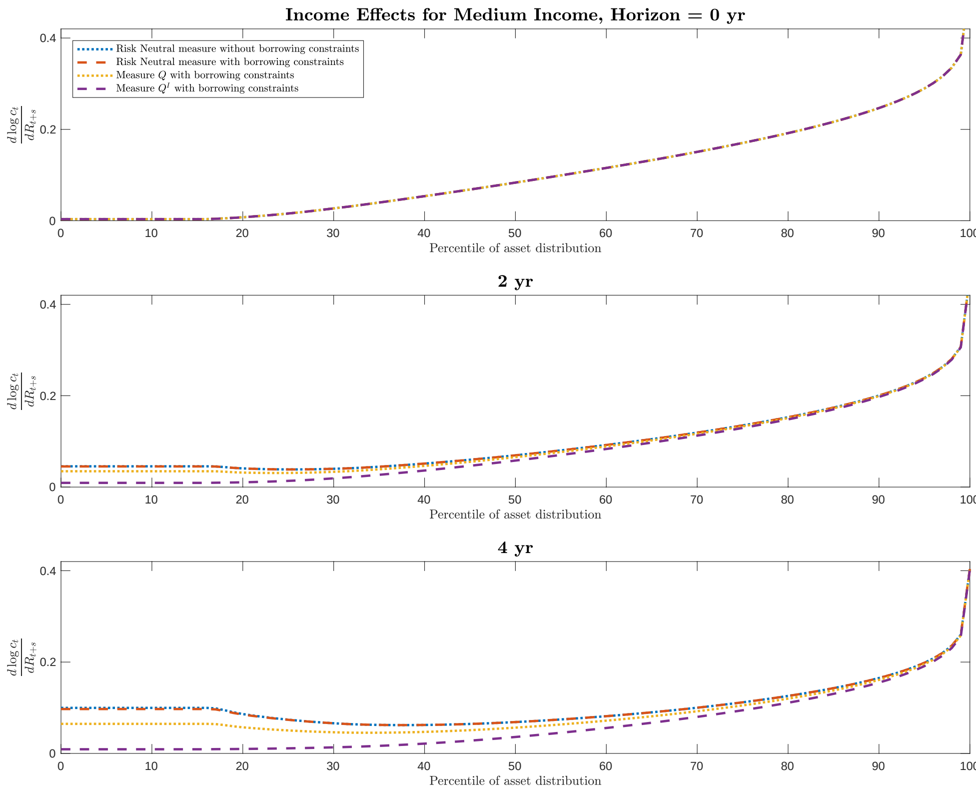

Figure 8: Income effects for medium income agents.

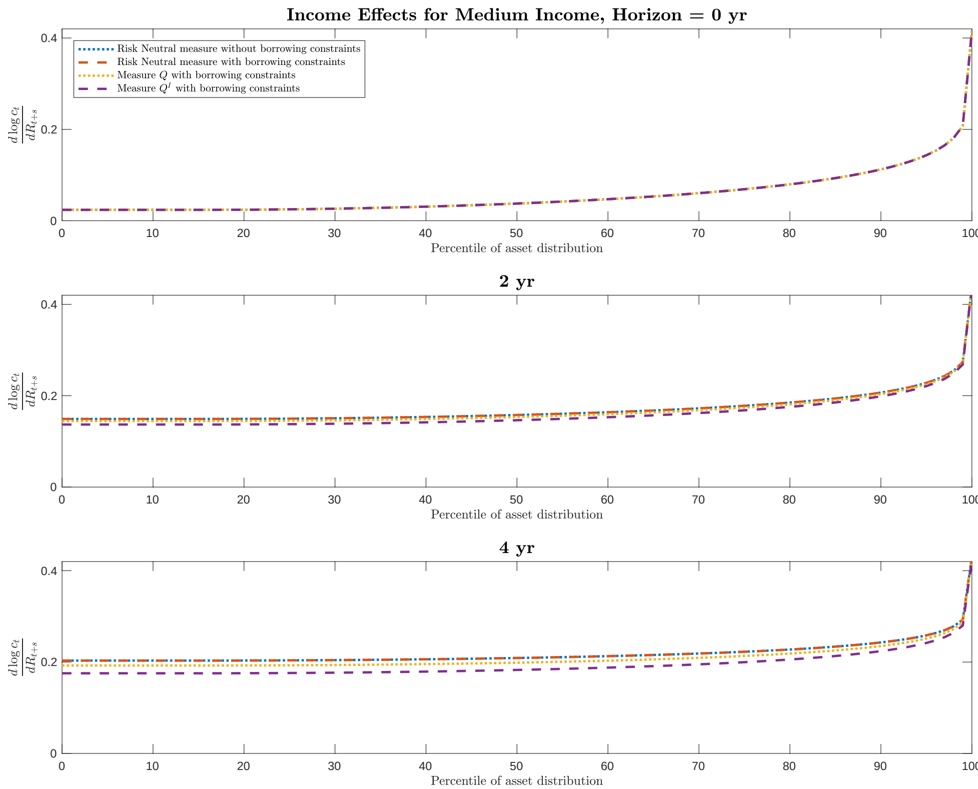

Figure 9: Income effects for high income agents.

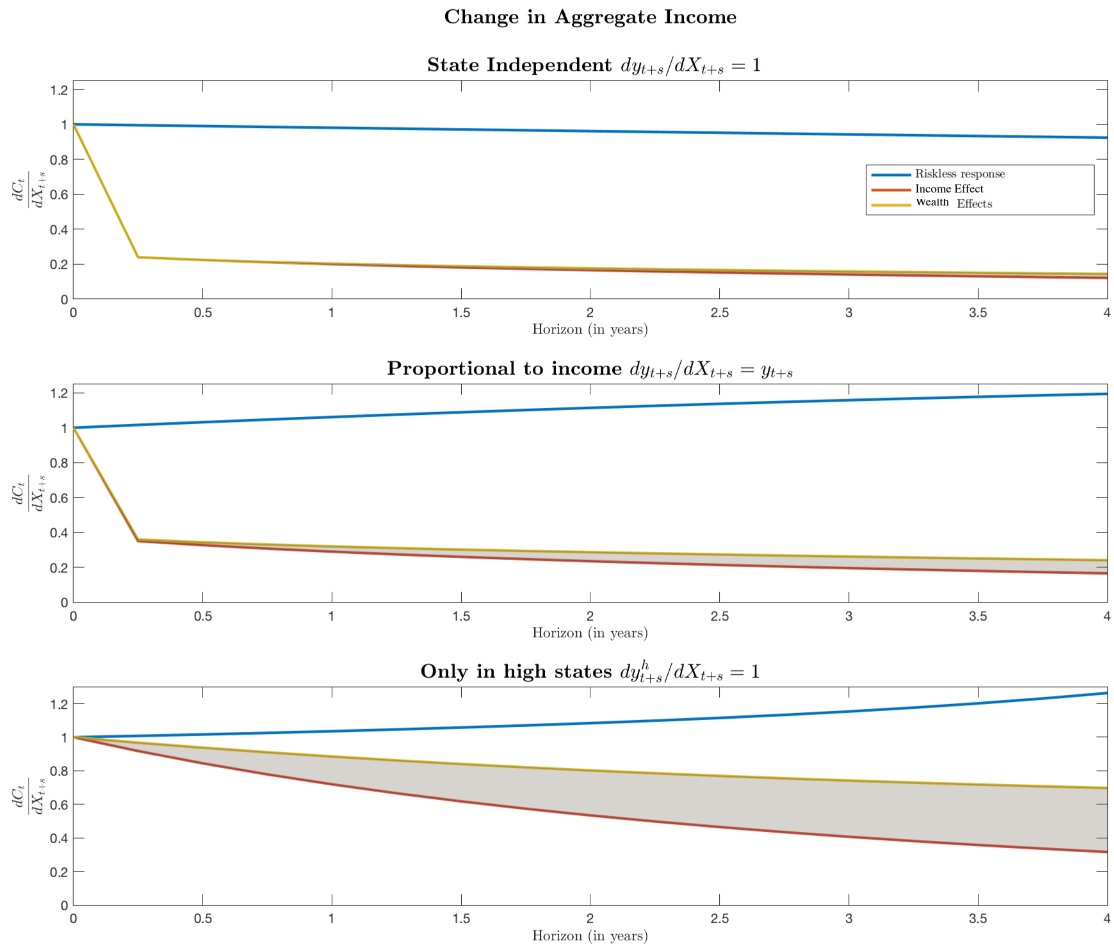

Figure 10: Change in aggregate income.

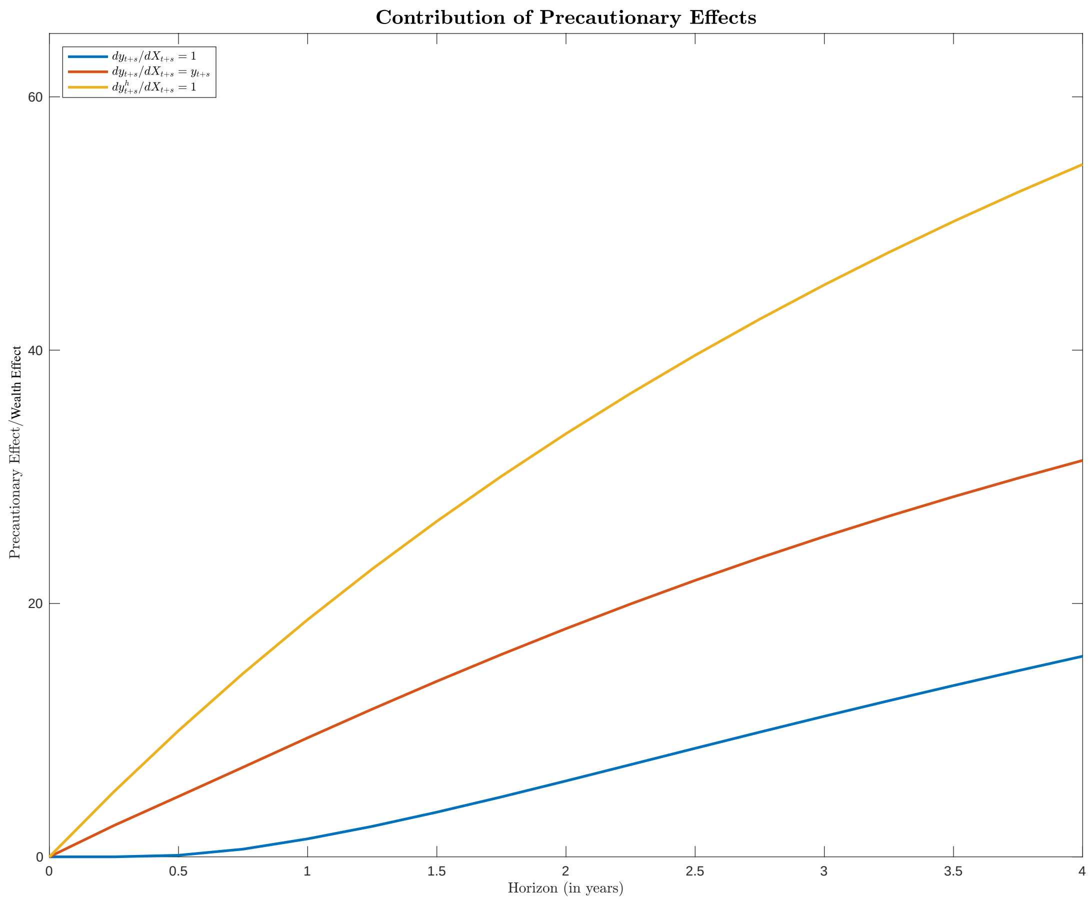

Figure 11: Precautionary effects relative to welfare effect.

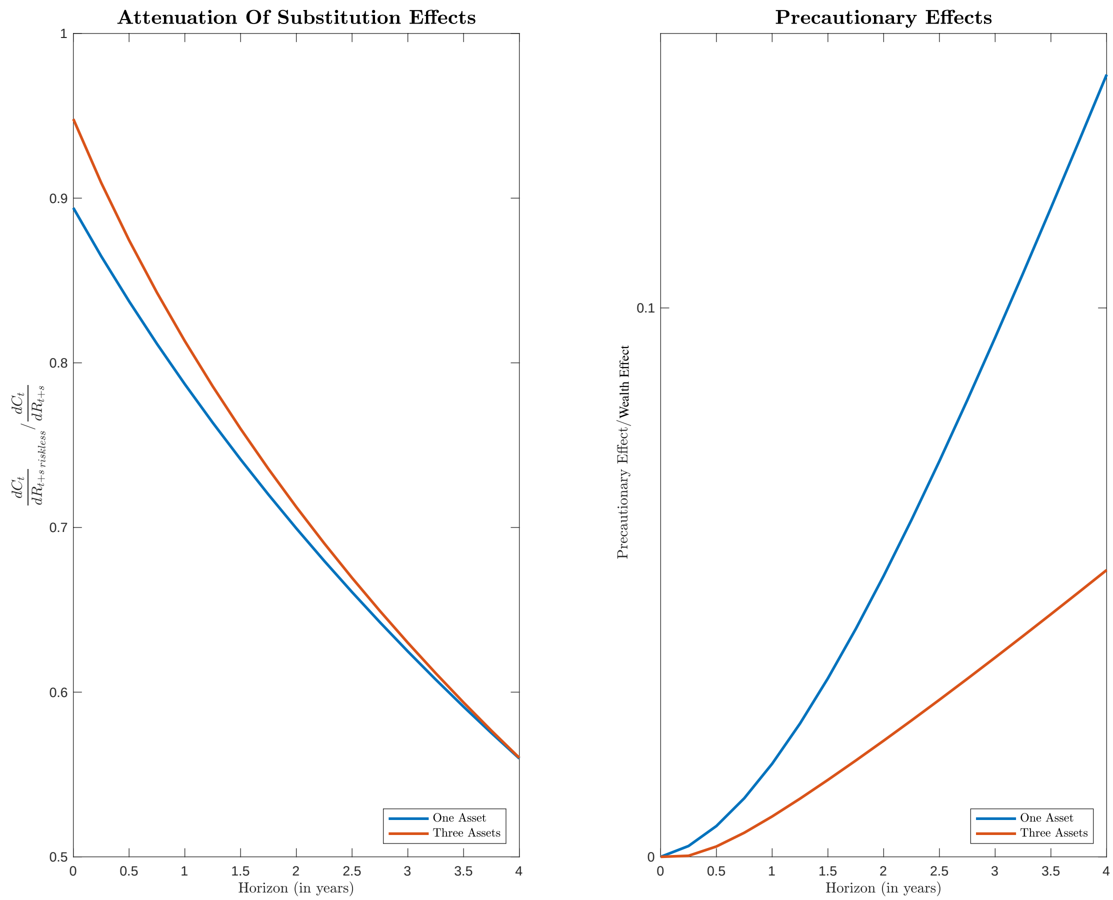

Figure 12: Three Assets Model.

B General Model

We generalize the model of the main text by allowing for multiple goods and multiple assets in every period.

B.1 Model and State-Price Densities

In each state θ_t, the agent can purchase N goods $\{c_1, ..., c_N\}$ at prices $\{p_t^1(\theta^t), ..., p_t^N(\theta^t)\}$. We denote by c_t and p_t the vectors of goods and prices and by $v(e_t, p_t, \theta_t)$ the within period value function as which depends on net expenditures e_t, prices p_t, and can also depend on the state θ_t:

$$v(e_t, p_t, \theta_t) = \max_{c_t} u(c_t, \theta_t)$$

s.t.

$$p_t \cdot c_t = e_t.$$

We denote by $c(e_t, p_t, \theta_t)$ the corresponding static consumption function.

Let us first consider a simple portfolio choice. The agent has access to M assets $b_1, ..., b_M$. One unit of b_i pays $R_t(\theta^t)$ in state $\theta_{t+1} \in \Theta_{t+1}^i$ where $\{\Theta_{t+1}^i\}_{1 \le i \le M}$ are a partition of Θ_{t+1}, the state space at $t = 1$. Asset prices are denoted by $q_t^1(\theta^t), ..., q_t^M(\theta^t)$. b_i, with $q_t^1 + ... + q_t^M = 1$.

The problem of the agent can then be written as

$$\max_{\{a_t(\theta^t), e_t(\theta^t)\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \prod_{k=0}^{t-1} \beta(\theta_k) v(e_t(\theta^t), p_t, \theta_t) \right]$$

subject to the sequence of budget constraints

$$\sum_{k=0}^{N} q_t^k(\theta^t) b_{t+1}^k(\theta^t) = a_t(\theta^t) + y_t(\theta_t) - e_t(\theta^t) \quad \forall t \ge 0,$$

$$\frac{a_{t+1}(\theta^{t+1})}{R_t(\theta_t)} = b_{t+1}^k(\theta^t) \quad \theta_{t+1} \in \Theta_{t+1}^k,$$

$$a_t \ge B_t \quad \forall t \ge 0,$$

with initial condition

$$a_t(\theta^0) = a_0,$$

where B_{t+s} is an exogenous borrowing limit. The optimal expenditure plan is characterized by the M Euler equations associated with each of the assets b_k:

$$\begin{cases} q_t^k v_e(e_t) = \beta_t R_t \mathbb{E}_k\left(v_e(e_{t+1})\right) \\ q_t^k v_e(e_t) > \beta_t R_t \mathbb{E}_k\left(v_e(e_{t+1})\right) \quad \textit{and}\; R_t b_{t+1}^k = B_{t+1} \end{cases}$$

with $\mathbb{E}_k(\,\cdot\,) = \mathbb{E}(\,\cdot \mid \theta_{t+1} \in \Theta_{t+1}^k) P(\Theta_{t+1}^k|\theta^t)$. Because agents have access to a larger set of assets, a binding borrowing constraint in state $\theta_{t+1} \in \Theta_{t+1}^k$ does not hamper their ability to transfer funds from states $\theta_{t+1} \in \Theta_{t+1}^{-k}$. The planning horizon is then only partially interrupted in state θ_t and we modify our definition of the stopping time τ accordingly.

Definition 1. The hitting time τ at a given initial date t, level of asset a_t and state θ_t is defined for any continuation history ${}^t\theta = \{\theta_{t+k}\}_{k\geq 1}$ as;

$$\tau(t, a_t, \theta_t, {}^t\theta) = \inf\{s > t \mid q_{s-1}^k v_e(e(a_{s-1}, \theta_{s-1}) > \beta_s R_s(\theta^s)\mathbb{E}_{s-1,k}\left(v_e(e(a_s, \theta_s'))\right) \; \textit{with}\, k\, \textit{s.t.}\, \theta_s \in \Theta_s^k\},$$

with $\tau(t, a_t, \theta_t, {}^t\theta) = \infty$ if the constraint does not bind on the path ${}^t\theta$. We denote by $k(\tau)$ the index of the constrained asset at history ${}^t\theta$.

By contrast, the hitting time defined in the previous section corresponds to the first date at which the risk free asset is constrained. We now denote it by τ^{rf}. Note that $\tau^{rf} \leq \tau$ and it more precisely relates to τ as:

$$\tau^{rf}(\theta^{t+s}) = t+s \Leftrightarrow \exists\, \theta_{t+s+1} \quad \textit{s.t.} \quad \theta^{t+s+1} = \{\theta^{t+s}, \theta_{t+s+1}\} \quad \textit{and} \quad \tau(\theta^{t+s+1}) = t+s+1$$

In words, the hitting time of the previous section stops at the first date any of the assets is constrained. In that sense, since agents have access to a richer class of assets to transfer funds from future states, their planning horizon is expanded.

With the two hitting times in hand, we can rederive the two Euler equations of Proposition **??**. The equations for consumption c_t now apply to expenditure e_t, with the marginal utility of consumption $u'(c_t)$ replaced by the marginal utility of income $v_e(e_t, p_t, \theta_t)$.

Proposition 9. *The optimal expenditure plan satisfies between any date t and T the two Euler equation stopped at the first time the risk free asset is constrained:*

$$v_e(e_t) = \mathbb{E}_t\left(\prod_{s=t}^{T\wedge\tau^{rf}-1} \{\beta_s R_s\}\, v_e(e_{T\wedge\tau^{rf}})\right) \tag{18}$$

and

$$v_{ee}(e_t)\,\partial e_t/\partial a_t = \mathbb{E}_t\left(\prod_{s=t}^{T\wedge\tau^{rf}-1}\{\beta_s R_s MPS_s(\Theta^k_{s+1},\theta^s)\}\, v_{ee}(e_{T\wedge\tau^{rf}})\,\partial e_{T\wedge\tau^{rf}}/\partial a_{T\wedge\tau^{rf}}\right), \quad (19)$$

with $\partial e_s/\partial a_s$ the marginal propensity to spend at s and $MPS_s(\Theta^k_{s+1},\theta^s) = R_s\partial_a b^k(\theta^s)$ for $\theta_{s+1}\in\Theta^k_{s+1}$,the marginal propensity to transfer funds in states Θ^k_{s+1} at s.

In addition, the two Euler equations can be extended using the hitting time τ in the following way:

$$\begin{aligned} v_e(e_t) =& \mathbb{E}_t\left(\mathbb{1}_{T\leq\tau-1}\prod_{s=t}^{T-1}\{\beta_s R_s\}\, v_e(e_T)\right) \\ &+\mathbb{E}_t\left(\mathbb{1}_{T>\tau-1}\prod_{s=t}^{\tau-2}\{\beta_s R_s\}\, q^{\tau-1}_{k(\tau)} v_e(e_{\tau-1})\right) \end{aligned} \quad (20)$$

and

$$\begin{aligned} v_{ee}(e_t)\,\partial e_t/\partial a_t =& \mathbb{E}_t\left(\mathbb{1}_{T\leq\tau-1}\prod_{s=t}^{T-1}\{\beta_s R_s MPS_s(\Theta^k_{s+1},\theta^s)\}\, v_{ee}(e_T)\,\partial e_T/\partial a_T\right) \\ &+\mathbb{E}_t\left(\mathbb{1}_{T>\tau-1}\prod_{s=t}^{\tau-2}\{\beta_s R_s MPS_s(\Theta^k_{s+1},\theta^s)\}\, q^{\tau-1}_{k(\tau)}\, v_{ee}(e_{\tau-1})\,\partial e_{\tau-1}/\partial a_{\tau-1}\right) \end{aligned} \quad (21)$$

Since utility is additively separable in time, the intratemporal allocation in state θ_t only depends on good prices and the total amount spent at θ_t, e_t. As a result, the indirect utility v is sufficient to determine the allocation of income across states and the intertemporal problem is characterized as in Proposition **??**: the first equation constrains the discounted flow of marginal value of income to be constant while the second constrains the marginal propensities to spend across states.

The first set of equations shows that, as in the previous section, the stopped processes $v_{e,t\wedge\tau^{rf}}$ and $v_{ee,t\wedge\tau^{rf}}\partial_a e_{t\wedge\tau^{rf}}$ discounted by $\beta_t R_t$ and $\beta_t R_t MPS_t$ are martingales. However only part of the agent portfolio might be constrained at θ_t. We denote by $\tilde{q}_t$ the indicator that takes the value one if no constraint binds at θ^t and $\sum q^{\tau-1}_{k(\tau)}$ – the sum of the prices of the constrained assets – otherwise. The second set of equation shows that $[\tilde{q}v_e]_{t\wedge\tau-1}$ and $[\tilde{q}v_{ee}\partial_a e]_{t\wedge\tau-1}$ are also martingales, where $\tilde{q}$ serves to indicate which fraction of the portfolio is constrained at θ_t. We use this fact to extend the definition of the adjusted probabilities Q and Q^I associated with this martingales on the space $\{\theta^{t+s} \mid \tau(\theta^{t+s}) > t+s\}$.

Definition 2. The probabilities Q and Q^I are given by their Radon-Nikodym derivatives at each point θ^{t+1} of the state space, with $\theta_{t+1} \in \Theta^k_{t+1}$ where the constraint does not bind:

$$\frac{dQ}{dP}(\theta^{t+1} \mid \theta^t) = \frac{\beta_t R_t \, v_e(e_{t+1}(a_{t+1}, \theta_{t+1}))}{v_e(e_t(a_t, \theta_t))} = q^k_t \frac{v_e(e_{t+1}(a_{t+1}, \theta_{t+1}))}{\mathbb{E}_{k,t}\left(v_e(e_{t+1}(a_{t+1}, \theta_{t+1}))\right)}, \tag{22}$$

and

$$\frac{dQ^I}{dP}(\theta^{t+1} \mid \theta^t) = \frac{\beta_t R_t MPS_t(\Theta^k_{t+1}, \theta^t) v_{ee} \partial_a e(a_{t+1}, \theta_{t+1})}{v_{ee} \partial_a e(a_t, \theta_t)} = q^k_t \frac{v_{ee} \partial_a e(a_{t+1}, \theta_{t+1})}{\mathbb{E}_{k,t}\left(v_{ee} \partial_a e(a_{t+1}, \theta_{t+1})\right)}. \tag{23}$$

The interpretation of the two probabilities remains the same: $Q_{t+s} / \prod_t^{t+s-1} R_k$ is the price of the virtual Arrow-Debreu security of state θ^{t+s} while Q^I is the time 0 consumption value of future wealth and incorporates the change in Arrow-Debreu prices as the agent gets richer. Note that $Q(\cdot \mid \theta^t)$ and $Q^I(\cdot \mid \theta^t)$ coincide on Θ^k_{t+1}: we have $Q(\{\theta^t, \Theta^k_{t+1}\} \mid \theta^t) = Q^I(\{\theta^t, \Theta^k_{t+1}\} \mid \theta^t) = q^k_t$. Indeed one additional dollar in Θ^k_{t+1} is equivalent to one additional unit of the k^{th} asset at θ^t and its price is fixed at q^k_t independently of the agent's wealth. Since the M assets are available, funds in Θ^k_{t+1} and θ^t are fungible: this neutralizes the corresponding wealth effects on Arrow-Debreu prices and makes the price densities Q and Q^I equivalent on $\{\Theta^k_{t+1}\}_k$. The logic can be extended to the payoffs of any feasible investment as shown in the following lemma.

Lemma 1. *An investment consistent with the stopping time τ is given by the sequences $\{b^k_{s+1}(\theta^s\}_{0 \leq s \leq T-1}$ and $\{d_s(\theta^s\}_{0 \leq s \leq T}$, with T potentially equal to infinity, such that:*

$$\begin{cases} \sum q^k_s b^k_{s+1}(\theta^s) = b_s(\theta^s) - d_s(\theta^s) \\ b_{s+1}(\theta^s) = R_s b^k_{s+1}(\theta^s) \quad for \quad \theta_{s+1} \in \Theta^k_{s+1} \\ b_{s+1}(\theta^s) = 0 \quad if \quad \tau \leq s+1 \\ d_T(\theta^T) = b_T(\theta^T). \end{cases}$$

We then have:

$$\begin{aligned} \mathbb{E}^Q \left(\sum_0^T \frac{d_s}{\prod_0^{s-1} R_k} \right) =& \mathbb{E}^{Q^I} \left(\sum_0^T \frac{d_s}{\prod_0^{s-1} R_k} \right) \\ =& \mathbb{E} \left(\sum_0^T \prod_0^{s-1} \frac{q_k}{R_k} d_s \right) = b_0 \end{aligned}$$

B.2 Uncompensated Responses

We start by deriving the expenditure responses to contemporaneous changes in income, interest rate and good prices, we then extend this derivation to future changes in prices and income.

Proposition 10. *The sensitivities of expenditure at t with respect to contemporaneous income, interest rate changes and good prices at t are given by*

$$\frac{y_0}{e_0}\frac{\partial e_0}{\partial y_0} = \frac{y_0}{e_0}\frac{\partial e_0}{\partial a_0}, \tag{24}$$

$$-\frac{R_0}{e_0}\frac{\partial e_0}{\partial R_0} = \epsilon(e_0)\frac{MPS_0}{R_0} - \frac{1}{e_0}\frac{\partial e_0}{\partial a_0}\frac{a_1}{R_0}, \tag{25}$$

$$-\frac{p_0^i}{e_0}\frac{\partial e_0}{\partial p_0^i} + \frac{p_0^i c_0^i}{e_0} = \epsilon(e_0)\frac{MPS_0}{R_0}\frac{\partial c_0^i}{\partial e_0}p_0^i + \frac{1}{e_0}\frac{\partial e_0}{\partial a_0}p_0^i c_0^i, \tag{26}$$

where all the functions are evaluated at the optimum and $\epsilon(e_0) = -v_e(e_0)/e_0 v_{ee}(e_0)$ denotes the elasticity of intertemporal substitution, and $MPS_0 = R_0 \sum q_0^k \partial_a b_1^k = R_0\left(1 - \partial_a e_0\right)$ denotes here the total marginal propensity to save.

Proposition 10 translates the results of Proposition **??** in terms of expenditure. The sensitivity of expenditures to present income changes $(y_0/e_0)(\partial e_0/\partial y_0)$ is the product of the marginal propensity to spend $\partial e_0/\partial a_0$ and the ratio of income to total spending at 0, y_0/e_0. The sensitivity of contemporaneous interest rate changes is the sum of a substitution effect $\epsilon(e_0)\frac{MPS_0}{R_0}$ – where $\epsilon(e_0)$ denotes again the inverse of the curvature of indirect utility – and an income effect which depends on wealth at 1. Note that the effect of binding borrowing constraints on the agent's ability to substitute between periods is implicit in MPS_0: when the k^{th} asset is constrained we have $\partial_a b_1^k = 0$ which reduces the marginal propensity to save and therefore the substitution effect. The definition of $MPS_0 = R_0 \sum q_0^k \partial_a b_1^k$ makes clear that with multiple assets, the substitution effect is only zero when *all* the assets are constrained, as the remaining free assets can still be used to transfer future funds following an interest rate decrease.

The sensitivity to current good price changes is the sum of three terms. The first one, $\frac{p_0^i c_0^i}{e_0}$, is the mechanical effect of an increase in price i: if the agent does not change her consumption of good i, expenditure increases proportionally to the share of good i in total spending. The behavioral responses correspond to an income effect $-\frac{\partial e_0}{\partial a_0}\frac{p_0^i c_0^i}{e_0}$, which is the product of the marginal propensity to spend times the effective loss of income given by the share of good i consumption, and a substitution effect given by $-\epsilon(e_0)\frac{MPS_0}{R_0}\frac{\partial c_0^i}{\partial e_0}p_0^i$. The term $\frac{\partial c_0^i}{\partial e_0}p_0^i$ is the change in price index in response to an increase in the price of good i.

To see this more clearly, let us consider the case of a CES aggregator, where the aggregate good consumption C, the price index P and good i demand c^i are given by:

$$C = \left(\sum_1^N \omega_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$
$$P = \left(\sum_1^N \omega_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$
$$c_i = e\frac{\omega_i p_i^{-\sigma}}{\sum \omega_i p_i^{1-\sigma}}.$$

We then directly have $\frac{d\,ln\,P}{d\,ln\,p_i} = \frac{\omega_i p_i^{1-\sigma}}{\sum \omega_i p_i^{1-\sigma}} = \frac{\partial c^i}{\partial e} p^i$. The substitution effect can then be interpreted as $-\epsilon(e_0)\frac{MPS_0}{R_0}\frac{d\,ln\,P_0}{d\,ln\,p_0^i}$ where the change in the price index is given by the marginal propensity to spend on good i. Since the real interest rate is given $\frac{P_0 R_0}{P_1}$, a change in the price index P_0 is equivalent to a change in R_0 which explains the form taken by the substitution effect generated by an increase in price p_0^i.

We now consider the impact of a change in the path of interest rates $\{dR_s\}_{0\leq s}$, income $\{dy_s\}_{0\leq s}$ and good prices $\{dp_s\}_{0\leq s}$ announced at time 0, on expenditure at 0. Formally we again have

$$de_0 = \sum_{\theta^t,\,0\leq t}\left(\frac{\partial e_0}{\partial y(\theta^t)}dy(\theta^t) + \frac{\partial e_0}{\partial R(\theta^t)}dR(\theta^t) + \frac{\partial e_0}{\partial p(\theta^t)}dp(\theta^t)\right),$$

so that we can still interpret these responses as the partial derivatives – according to the announced change $\{dR_s\}_{0\leq s}$, $\{dy_s\}_{0\leq s}$ and $\{dp_s\}_{0\leq s}$ – of the Marshallian expenditure function keeping assets constant.

Proposition 11. *The response of expenditure to a change in the sequence of prices and income $\{dR_s\}_{0\leq s}$, $\{dy_s\}_{0\leq s}$, $\{dp_s\}_{0\leq s}$ is given by:*

$$\frac{de_0}{e_0} - \frac{c_0}{e_0}dp_0 = -\epsilon(e_0)\mathbb{E}_0^Q\left(\sum_{t=0}^{(\tau-1)}\prod_{s=0}^{t}\frac{MPS_s(\Theta_{s+1}^k,\theta^s)}{R_s}\left\{\frac{dR_t}{R_t} + \frac{\partial c_t}{\partial e_t}dp_t - \frac{\partial c_{t+1}}{\partial e_{t+1}}dp_{t+1}\right\}\right)$$
$$+\frac{1}{e_0}\frac{\partial e_0}{\partial a_0}\mathbb{E}_0^{Q^I}\left(\sum_{s=0}^{(\tau-1)}\prod_{s=0}^{t-1}\frac{1}{R_s}\left\{\frac{a_{t+1}}{R_t}\frac{dR_t}{R_t} + dy_t - c_t dp_t\right\}\right), \tag{27}$$

where $\epsilon(e_0) = -v_e(e_0)/e_0 v_{ee}(e_0)$ *is the elasticity of intertemporal substitution and* $a_{t+1}(\theta^s)/R_t = \sum q_t^i b_{t+1}^i$ *is wealth at* $t+1$ *from* t *perspective.*

As in Proposition 2, the first line of the decomposition defines the substitution effects, where future interest rate changes are valuated with the virtual Arrow-Debreu price density Q and discounted by the product of marginal propensities to save, while the second defines the income effects, valuated according to the measure Q^I which takes into account the wealth effect on the implied prices of future states. The insurance provided by the M assets is implicit in the formula. First, the horizon of the agent is determined by the new stopping time τ. Given an expenditure plan $\{e_{t+s}\}_{s\geq 0}$, the horizon of the agent, when she only has access to a risk free asset to readjust her spending following a change in prices and income, is characterized by the stopping time $\tau^{rf} \leq \tau$. As explained in the previous subsection, τ^{rf} stops at the first date at which *any* of the M assets is constrained, and, in that, sense, the planning horizon of the agent is expanded. Second, the agent's ability to transfer funds across states is now characterized by the marginal propensity to save $MPS_s(\Theta_{s+1}^k, \theta^{s'})$. When the agent can only saves with a risk free asset, the marginal propensity to saves is constrained to be equal across the Θ_s^k subspaces. The richer asset structure allows the agent to access funds from the "good" states bypassing the "bad" ones: the finer the partition $\{\Theta^k\}_k$, the more flexibility the agent has to transfer wealth and the stronger the substitution effects. Third and finally, income changes which correspond to the payoffs of feasible investments, as described in lemma 1, have the same valuation under the measure Q and Q^I and the insurance effects of these income changes are muted. To see this, consider an income change in 1, the expenditure response at 0 is then given by:

$$\begin{aligned}
&\frac{de_0}{e_0} = \frac{1}{e_0}\frac{\partial e_0}{\partial a_0}\left\{\mathbb{E}_0^Q\left(\frac{dy_1}{R_1}\right) + \mathbb{E}_0^{Q^I}\left(\frac{dy_1}{R_1} - \frac{d\bar{y}_1}{R_1}\right)\right\} \\
&\textit{with} \quad d\bar{y}_1 = \mathbb{E}_0^Q\left(dy_1 \mid \theta_1 \in \Theta_1^k\right).
\end{aligned}$$

The valuation Q^I only matters for the fraction of the income change that deviates from the average change on Θ_1^k. More generally, using Lemma 1, we can rewrite the income effects, for an income shock at t, as:

$$
\begin{aligned}
de_0 =& \frac{\partial e_0}{\partial a_0} \mathbb{E}_0^Q \left(\frac{dy_t}{\prod_0^{t-1} R_k} \right) \\
&+ \frac{\partial e_0}{\partial a_0} \mathbb{E}_0^{Q^I} \left(\sum_{s=1}^{t} \frac{\mathbb{E}_s^Q \left(\frac{dy_t}{\prod_s^{t-1} R_k} \right) - \mathbb{E}_{s-1}^Q \left(\frac{dy_t}{\prod_s^{t-1} R_k} \mid \theta_s \in \Theta_s^k \right)}{\prod_0^{s-1} R_k} \right)
\end{aligned}
$$

The finer the partition $\{\Theta^k\}_k$, the less the measure Q^I matters, as income changes can approximately be insured against through the available assets. In the limit, we recover the complete market response where future changes are valued according to Q.

In terms of good prices, the expenditure response differs for contemporaneous and future changes. First, since the expenditure function $e_t = \sum p_t^i c_t^i$ directly depends on current prices only, there is obviously no "mechanical" effect of future price changes. Second since the real interest rate at 0 is $P_0 R_0 / P_1$, and that $d\, ln\, P_t / d\, ln p_t^i = \partial_e c_t^i p_t^i$, a price change at t is equivalent to an increase in interest rate at t and an equal decrease at $t-1$, for $t > 0$. Since there is clearly no substitution towards past consumption at 0, only the equivalent increase in interest rate is present for contemporaneous changes, which explains the somewhat different response to present and future price changes.

Equivalently, we can characterize the sensitivities of expenditure to future prices, interest rate changes and income recursively:

Corollary 3. *For all $t \geq 1$, the sensitivities of expenditure at 0 with respect to prices and interest rate changes at t are given by*

$$
\frac{\frac{\partial e_0}{\partial p_t}}{\frac{\partial e_0}{\partial a_0}} = \mathbb{E}_0^{Q^I} \left[\frac{1 - \frac{\partial e_t}{\partial a_t}}{\Pi_{k=0}^{t-1} R_k} \frac{\frac{\partial e_t}{\partial p_t}}{\frac{\partial e_t}{\partial a_t}} \mathbb{1}(\tau > t) \right], \tag{28}
$$

$$
\frac{\frac{\partial e_0}{\partial R_t}}{\frac{\partial e_0}{\partial a_0}} = \mathbb{E}_0^{Q^I} \left[\frac{1}{\Pi_{k=0}^{t-1} R_k} \frac{\frac{\partial e_t}{\partial R_t}}{\frac{\partial e_t}{\partial a_t}} \mathbb{1}(\tau > t) \right], \tag{29}
$$

$$
\frac{\frac{\partial e_0}{\partial y_t}}{\frac{\partial e_0}{\partial a_0}} = \mathbb{E}_0^{Q^I} \left[\frac{1}{\Pi_{k=0}^{t-1} R_{t+u}} \frac{\frac{\partial e_t}{\partial y_t}}{\frac{\partial e_t}{\partial a_t}} \mathbb{1}(\tau > t) \right], \tag{30}
$$

where all functions are evaluated at the optimum.

As before, the Corollary states that $\frac{\partial e_t}{\partial R_t} / \frac{\partial e_t}{\partial a_t} \mathbb{1}(\tau > T)$ and $\frac{\partial e_t}{\partial y_t} / \frac{\partial e_t}{\partial a_t} \mathbb{1}(\tau > T)$ discounted by $1/\prod_0^{t-1} R_s$ are martingales with respect to the measure Q^I. Note however that it is not the case for the sensitivity of expenditure to prices change. This is due to the discrepancy between the response to present and future price changes described above and the term $1-\frac{\partial e_t}{\partial a_t}$ in 28 precisely corrects for it. In other words, because of the discontinuity in the expenditure response the discounted sensitivity to future price changes is a submartingale rather than martingale, we have:

$$\frac{\frac{\partial e_0}{\partial p_t}}{\frac{\partial e_0}{\partial a_0}} = \mathbb{E}_0^{Q^I}\left[\frac{1}{\Pi_{k=0}^{t'-1} R_k}\frac{\frac{\partial e_{t'}}{\partial p_t}}{\frac{\partial e_{t'}}{\partial a_{t'}}}\mathbb{1}(\tau > t')\right] \quad for \quad t' < t,$$
$$\frac{\frac{\partial e_0}{\partial p_t}}{\frac{\partial e_0}{\partial a_0}} \leq \mathbb{E}_0^{Q^I}\left[\frac{1}{\Pi_{k=0}^{t-1} R_k}\frac{\frac{\partial e_t}{\partial p_t}}{\frac{\partial e_t}{\partial a_t}}\mathbb{1}(\tau > t)\right]$$

From the expenditure response, we can easily express the individual good response at t using the intratemporal hicksian demand function.

Corollary 4. *The response of good consumption to a change in the sequence of prices and income* $\{dR_s\}_{0\leq s}$, $\{dy_s\}_{0\leq s}$, $\{dp_s\}_{0\leq s}$ *is given b*

$$dc_0^i = \sum_j \frac{\partial c_0^{i,h}}{\partial p_0^j} dp_0^j + \frac{\partial c_0^i}{\partial e_0}\{de_0 - c_0 dp_0\}$$

where $\frac{\partial c_0^{i,h}}{\partial p_0^j}$ *is the price derivative of the static hicksian demand for good* i.

The corollary simply generalizes the standard Slutsky equation. The change in good i demand works through an intratemporal substitution channel, which is exactly the same as in the static case, and an income channel, which incorporates all intertemporal effects through the change in total expenditure de_0 derived in Proposition 12.

Responses to Past Announcements

We now derive the expenditure response when the announcement of the price and income was made in the past. We denote by $d_{t_0}e_t$ the response of expenditure at history θ^t, for a change in interest rate $\{dR_s\}_{0\leq s}$, income $\{dy_s\}_{0\leq s}$ and prices $\{dp_s\}_{0\leq s}$ announced at 0

Formally we have:

$$de_t = \sum_{\theta^s,\, t\leq s} \left(\frac{\partial e_t}{\partial y(\theta^s)} dy(\theta^s) + \frac{\partial e_t}{\partial R(\theta^s)} dR(\theta^s) + \frac{\partial e_t}{\partial p(\theta^s)} dp(\theta^s) \right)$$
$$+ \frac{\partial e_t}{\partial a_t} \sum_{\theta^s,\, 0\leq s} \left(\frac{da_t}{dy(\theta^s)} dy(\theta^s) + \frac{da_t}{dR(\theta^s)} dR(\theta^s) + \frac{da_t}{dp(\theta^s)} dp(\theta^s) \right),$$

Past changes in prices and income affect spending indirectly, via a change in asset position at t, while contemporaneous and future changes have a direct and indirect impact.

Corollary 5. *The expenditure response at history θ^t for a change in the sequence of income $\{dy_s\}_{t_0\leq s\leq T}$ announced in $t_0 < t$ is given by:*

$$de_t = \frac{\partial e_t}{\partial a_t} \sum_{s=0}^{t} MPS_s(\theta^t) \left\{ \mathbb{E}_s(w_s) - \mathbb{E}_{s-1}\left(w_s \mid \Theta_s^{k_s} \right) \right\},$$

with $w_s = \sum_{k=s}^{(\tau-1)} \prod_{s'=s}^{k-1} \frac{1}{R_{s'}} dy_k$and $k_{s+1} = k$ if $\theta_{s+1} \in \Theta_{s+1}^k$ along the history θ^t. For a change in in the sequence of interest rates $\{dR_s\}_{0\leq s}$,

$$de_t = \frac{\partial e_t}{\partial a_t} \sum_{s=0}^{t} MPS_s(\theta^t) \left\{ \mathbb{E}_s(w_s) - \mathbb{E}_{s-1}\left(w_s \mid \Theta_s^{k_s} \right) \right\}$$
$$- \epsilon(e_t) e_t \sum_{s=0}^{t} \left\{ \frac{Q_s^I(\theta^t)}{Q_s(\theta^t)} \mathbb{E}_s^Q(\tilde{R}_s) - \frac{Q_{s-1}^I(\theta^t)}{Q_{s-1}(\theta^t)} \mathbb{E}_{s-1}^Q \left(\frac{R_{s-1}}{MPS_{s-1}(\Theta_s^{k_s}, \theta^{s-1})} \tilde{R}_{s-1} \mid \Theta_s^{k_s} \right) \right\}$$

With $w_s = \sum_{k=s}^{(\tau-1)} \prod_{s'=k}^{k-1} \frac{1}{R_{s'}} \frac{a_{k+1}}{R_k} \frac{dR_k}{R_k}$ and $\tilde{R}_s = \sum_{k=s}^{(\tau-1)} \prod_{s'=s}^{k} \frac{MPS_{s'}(\theta^{s'+1}, \theta^{s'})}{R_{s'}} \frac{dR_k}{k}$.Where to simplify notation, we denote by $MPS_s(\theta^t)/R_s^t = \prod_s^{t-1} MPS_k(\theta^{k+1}, \theta^k)/R_k$ the total discounted marginal propensity to save in state θ^t from θ^s state and use $Q_{-1} = Q_{-1}^I = 0$ and $MPS_s(\theta^s)/R_s^s = 1$.

Remark 1. To simplify the formula, we did not include the stopping times τ_s. Note however that the formulas are still consistent with the stopping times: if the constraint binds between θ^t and θ^s, then $MPS_s(\theta^t) = 0$. Similarly, from the definition of Q^I, $Q_s^I(\theta_y^{t'}) = 0$ if the constraint binds between θ^s and $\theta_y^{t'}$.

The Corollary shows that the responses to past announcements feature the same misallocation terms as in the one asset case. However, as the agent can more freely relocate funds across states, the effect is subdued. Indeed suppose that income increases in state $\theta_{t+1} \in \Theta_{t+1}^k$. If the agent can save only through a risk free bond, the misallocation term

$1 - Q_t^I(\theta_{t+1})$ in state θ_{t+1} and $-Q_t^I(\theta_{t+1})$ otherwise. By contrast here the terms is 0 on Θ_{t+1}^l for $l \neq k$, and $1 - Q_t^I(\theta_{t+1})/Q_t^I(\Theta_{t+1}^l)$ at θ_{t+1}. As the partition $\{\Theta_{t+1}^k\}_k$ becomes finer, the agent can more freely allocate wealth across states and and the misallocation terms becomes smaller. In the limit we obtain the complete market responses where there is no misallocation.

General Asset Structure

We now briefly show how to generalize our results with a more general portfolio choice problem. The agent has access to M independent assets $b^1, ..., b^M$. One unit of b^i pays $R_{t+1}^i = R_t(\theta_t) r_t^i(\theta_{t+1})$ in state θ_{t+1}, where R_t is a component of returns common to all assets , and r_t^i is asset and state specific. Asset prices are denoted by $q_t^1(\theta_t), ..., q_t^M(\theta_t)$, with $q_t^1 + ... + q_t^M = 1$.
When the constraint binds at history θ^t, the set of feasible local change in portfolio is restricted in the following way: we denote by $C_t(\theta^t) = \{\theta_{t+1} \mid R_t \sum_i r_t^i(\theta_{t+1}) b_{t+1}^i(\theta^t) = B_{t+1}\}$ the set of states for which the constraint binds at θ^{t+1}, and the set of feasible local change to the agent portfolio by $\mathcal{B}_t(\theta^t) = \{(db_1, ..., db_M) \mid R_t \sum_i r_t^i(\theta_{t+1}) db_i = 0 \, for \, \theta_{t+1} \in C_t(\theta^t)\}$. From $\mathcal{B}_t$ we can define the set unconstrained states $\mathcal{U}_t = \{\theta_{t+1} \mid \exists (db_1, ..., db_M) \in \mathcal{B}_t \, with \sum_i r_t^i(\theta_{t+1}) db_i \neq 0\}$ and the set of constrained stated as the complementary $\mathcal{C}_t = \mathcal{U}_t^c$. Note that $C_t \subset \mathcal{C}_t$, but that the reverse is not true: if the asset returns are proportional in two states, for example, even if the constraint binds in only one of them, the agent will be unable to transfer funds from the other. With these notations, we can redefine the stopping time τ:

Definition 3. The hitting time τ at a given date t, level of asset a_t and state θ_t is defined for any continuation history ${}^t\theta = \{\theta_{t+k}\}_{k \geq 1}$ as:

$$\tau(t, a_t, \theta_t, {}^t\theta) = \inf\{s > t \mid \theta_{t+1} \in \mathcal{C}_t(\theta^t)\},$$

with $\tau(t, a_t, \theta_t, {}^t\theta) = \infty$ if the constraint does not bind on the path ${}^t\theta$.

To simplify exposition, we redefine the asset structure. We denote by $\tilde{b}^1, ..., \tilde{b}^M$ the modified assets – with prices $\tilde{q}^1, ..., \tilde{q}^M$ and returns $\tilde{R}^1, ..., \tilde{R}^M$ – that satisfy the following properties. First, the M assets are divided in two subgroup: $\tilde{b}_t^1, ..., \tilde{b}_t^{\imath_t}$ form a basis of $\mathcal{B}_t$, where $\imath_t$ is its dimension, while $\tilde{b}_t^{\imath_t}, ..., \tilde{b}_t^M$ spans its complimentary. Therefore, at any history, the first $\imath_t$ assets can be freely purchased and the remaining $M - \imath_t$ are binding. Second, the new basis satisfies $\mathbb{E}_t(v_{ee} \partial_a e_t \tilde{R}_{t+1}^i \tilde{R}_{t+1}^j) = 0$ for all $1 \leq i, j \leq \imath_t$. This can simply be obtained via Gram-Schmidt orthogonalization and allows us to recover the simple asset structure of the previous section. Finally, we rescale the prices and returns of the assets. The price of the i^{th} asset is scaled according to its share in the marginal portfolio, so that the new price is $\tilde{q}^i_t \frac{\partial_a \tilde{b}^i_{t+1}}{1-\partial_a e_t}$. This scaling is valid at any history where $\mathcal{B}_t$ is non empty and the corresponding return in state θ_{t+1} is $\tilde{R}^i_t \frac{\partial_a \tilde{b}^i_{t+1}}{1-\partial_a e_t}$. To avoid adding in notations, $\tilde{q}^i_t$ and $\tilde{R}^i_{t+1}$ refer to the rescaled prices and returns from now on. The advantage of the rescaling is that prices sum to one and that the total return in state θ_{t+1}, $\tilde{R}_{t+1} = \sum_k \tilde{R}^k_{t+1} = \frac{\sum_k R^k_{t+1} \partial_a b^k_{t+1}}{1-\partial_a e_t}$ does not depend on the construction of the basis $\tilde{b}^1, ..., \tilde{b}^M$. The measures Q_t and Q^I_t are then defined as before with $\tilde{R}_{t+1}$ replacing R_t.

Definition 4. The measures Q_t and Q^I_t are given by their Radon-Nikodym derivatives in each state $\theta_{t+1} \notin \mathcal{C}_t(\theta^t)$:

$$\frac{dQ}{dP}(\theta^{t+1} \mid \theta^t) = \frac{\beta_t \tilde{R}_{t+1}\, v_e(e(a_{t+1}, \theta_{t+1}))}{\tilde{R}_{t+1} v_e(e(a_t, \theta_t))} = \frac{\tilde{R}_{t+1} v_e(\theta^{t+1})}{\mathbb{E}\left(\tilde{R}_{t+1} v_e(e(a_{t+1}, \theta_{t+1})) \mid \theta^t\right)},$$

and

$$\begin{aligned}\frac{dQ^I}{dP}(\theta^{t+1} \mid \theta^t) &= \frac{\beta_t \tilde{R}_{t+1} MPS_t(\theta^{t+1}, \theta^t) v_{ee} \partial_a e(a_{t+1}, \theta_{t+1})}{v_{ee} \partial_a e(a_t, \theta_t)} \\ &= \frac{\tilde{R}_{t+1} MPS_t v_{ee} \partial_a e(a_{t+1}, \theta_{t+1})}{\mathbb{E}\left(\tilde{R}_{t+1} MPS_t v_{ee} \partial_a e(a_{t+1}, \theta_{t+1}) \mid \theta^t\right)}.\end{aligned}$$

with $MPS_t(\theta^{t+1}, \theta^t) = \sum_i \tilde{R}_{t+1} \partial_a \tilde{b}^i_{t+1} = \sum_i R_{t+1} \partial_a b^i_{t+1}$

Since neither our definition of the marginal propensities to save nor the total return $\tilde{R}_{t+1}$ depend on the way we redefine $\tilde{b}^1, ..., \tilde{b}^M$, the measure Q and Q^I are also independent of the choice of the basis. We can then rederive the expenditure response to a contemporaneous announcement of a change in prices and income as in the previous sections.[14]

Proposition 12. *The response of expenditure to a change in the sequence of prices and income* $\{dR_s\}_{0\leq s}$, $\{dy_s\}_{0\leq s}$, $\{dp_s\}_{0\leq s}$ *annouced at t is given by:*

$$\begin{aligned}\frac{de_0}{e_0} - \frac{c_0}{e_0} dp_0 = -\epsilon(e_0) \mathbb{E}^Q_0 &\left(\sum_{t=0}^{(\tau-1)} \prod_{s=0}^{t} \frac{MPS_s(\theta_{s+1}, \theta^s)}{\tilde{R}_{s+1}} \left\{ \frac{dR_t}{R_t} + \frac{\partial c_t}{\partial e_t} dp_t - \frac{\partial c_{t+1}}{\partial e_{t+1}} dp_{t+1} \right\} \right) \\ &+ \frac{1}{e_0} \frac{\partial e_0}{\partial a_0} \mathbb{E}^{Q^I}_0 \left(\sum_{s=0}^{(\tau-1)} \prod_{s=0}^{t-1} \frac{1}{\tilde{R}_{s+1}} \left\{ \frac{a_{t+1}}{\tilde{R}_{t+1}} \frac{dR_t}{R_t} + dy_t - c_t dp_t \right\} \right),\end{aligned} \tag{31}$$

[14] The response to past announcements can be derived in a similar fashion.

The formulas are essentially identical to the simpler case, but the risk free return R_t is replaced by $\tilde{R}_t$. Indeed, if the risk free asset is constrained at t but not the risky one, wealth is tranferred at $t+1$ through the later. The effective return between t and $t+1$ would then given by the risky returns. $\tilde{R}_t$ generalizes this idea by taking the return of the marginal portfolio.

Finally we restate the result of Lemma 1 with this general asset structure. The payoffs of any investment, where the investment is cashed out before the borrowing constraint binds, has the same present value under the measures Q and Q^I and is the initial wealth needed to invest. The Lemma will be useful when we discuss the compensated responses.

Lemma 2. *An investment consistent with the stopping time τ is given by the sequences $\{b^k_{s+1}(\theta^s)\}_{t\leq s\leq T-1}$ and $\{d_s(\theta^s)\}_{t\leq s\leq T}$, with T potentially equal to infinity, such that:*

$$\begin{cases} \sum q^k_s b^k_{s+1}(\theta^s) = b_s(\theta^s) - d_s(\theta^s) \\ b_{s+1}(\theta^{s+1}) = \sum_k R^k_s b^k_{s+1}(\theta^s) \\ b_{s+1}(\theta^{s+1}) = 0 \quad if\ \theta_{s+1} \in \mathcal{B}_t(\theta^t) \\ d_T(\theta^T) = b_T(\theta^T). \end{cases}$$

We then have:

$$\begin{aligned} \mathbb{E}^Q\left(\sum_0^T \frac{d_s}{\prod_0^{s-1}\tilde{R}_{k+1}}\right) =& \mathbb{E}^{Q^I}\left(\sum_0^T \frac{d_s}{\prod_0^{s-1}\tilde{R}_{k+1}}\right) \\ =& b_0 \end{aligned}$$

B.3 Compensated Responses

We now consider the welfare impact of income and price changes and re-derive the compensated expenditure responses. We keep the notations of the simple asset structure, but the results are also true in the general case.

Proposition 13. *The welfare of an agent with wealth a_0 in state θ_0 at t is given by $V_0(a_0, \theta_0)$. The change in welfare in response to a change in the sequence of prices and income $\{dR_s\}_{0\leq s}$, $\{dp_s\}_{0\leq s}$ and $\{dy_s\}_{0\leq s}$ is given by*

$$dV_0 = \mathbb{E}\left(\sum_0^\infty \beta^t v_e(e_t)\left\{\frac{a_{t+1}}{R_t}\frac{dR_t}{R_t} + dy_t - \sum c^i_t\, dp^i_t\right\}\right)$$

If we only consider changes that happen at the planning horizon of the agent, that is $\{dR_s\}_{0\leq s\leq(\tau-1)}$, $\{dp_s\}_{0\leq s\leq(\tau-1)}$ and $\{dy_s\}_{0\leq s\leq(\tau-1)}$, the welfare impact is given by:

$$
\begin{aligned}
dV_0 &= \partial_a V_0 \mathbb{E}_0^Q \left(\sum_{t=0}^{(\tau-1)} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t - \sum c_t^i \, dp_t^i \right\} \right) \\
&= v_e(e_0) \mathbb{E}_0^Q \left(\sum_{t=0}^{(\tau-1)} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t - \sum c_t^i \, dp_t^i \right\} \right),
\end{aligned} \tag{32}
$$

at the horizon τ at which the agent responds to the change.

Remark 2. If we extend the definition of the measure Q to all histories, $Q(\theta^s) = \prod_0^{s-1} \beta R_k v_e(e_s) / v_e(e_t)$, we have:

$$
dV_0 = v_e(e_0) \mathbb{E}^Q \left(\sum_{0}^{\infty} \prod_{s=0}^{t-1} \frac{1}{R_s} \left\{ \frac{a_{t+1}}{R_t} \frac{dR_t}{R_t} + dy_t - \sum c_t^i \, dp_t^i \right\} \right)
$$

Note however that Q is not a probability measure and that $\int dQ(\theta^t) \leq 1$, with equality if and only if no constraint binds before t.

The formula extends our dynamic Roy's identity to settings with multiple goods and assets. In the static formula, the welfare impact of change in price is given by $p^i \partial V / \partial p^i = \partial_I V \, p^i c^i$, where $\partial_I V$ is the derivative of indirect utility to income. In the dynamic setting, the price of good i at history θ^s is ${}^t p_s^i(\theta^s) = Q(\theta^s) / \prod_t^{s-1} R_k p_s^i$ from time t perspective. The dynamic formula is essentially the same as the static one with ${}^t p_s^i \partial V / \partial \, {}^t p_s^i = \partial_a V \, {}^t p_s^i c_s^i$ and the marginal value of wealth replacing the marginal value of income.

As before, the formula gives the equivalent change in wealth at t, in terms of welfare, of a change in prices and income. For changes arising at the planning horizon of the agent (that is $\{dR_s\}_{t \leq s < \tau}$, $\{dp_s\}_{t \leq s < \tau}$ and $\{dy_s\}_{t \leq s < \tau}$), the equivalent change in wealth is

$$
W_0 = \mathbb{E}^Q \left(\sum_{t}^{T \wedge (\tau-1)} \prod_{s'=t}^{s-1} \frac{1}{R_{s'}} \left\{ \frac{a_{s+1}}{R_s} \frac{dR_s}{R_s} + dy_s - \sum c_s^i \, dp_s^i \right\} \right)
$$

and any transfers $\{t_s(\theta^s)\}_{0 \leq s}$, with present value $\mathbb{E}^Q \left(\sum_0^T t_s / \prod_{s'=t}^{s-1} R_{s'} \right)$ equal to W_0 exactly compensate agents for these shocks. To be consistent with the market incompleteness defined by the asset structure and the borrowing constraints, we consider transfers that satisfy the description of lemma 1 and 2. More precisely, the transfers satisfy:

$$
\begin{cases}
b_0 = W_0 \\
\sum q_s^k b_{s+1}^k(\theta^s) = b_s(\theta^s) - t_s(\theta^s) \\
b_{s+1}(\theta^{s+1}) = R_s b_{s+1}^k(\theta^s) \quad for \quad \theta_{s+1} \in \Theta_{s+1}^k \\
b_{s+1}(\theta^{s+1}) = 0 \quad if \quad \tau \leq s+1.
\end{cases}
$$

In words, the transfers are made through the assets available to the agents and do not bypass the borrowing constraints. If transfers were unrestricted, a social planner could replicate the complete market allocation without borrowing constraints, which would be at odds with our setting. Note that the cost of those transfer is precisely $b_0 = W_0$ and that there discounted value with respect to the measure Q is also W_0 by lemma 1 and 2. They therefore compensate agents for shocks emerging at the planning horizon of agents, which are the only ones they respond to. We can now define the change in hicksian (or compensated) demand e_t^h as:

$$de_0^h = \sum_{\theta^{t+s},\, 0 \le t \le T} \left(\frac{\partial e_0}{\partial y(\theta^t)} \left(dy(\theta^t) - t_t(\theta^t) \right) + \frac{\partial e_0}{\partial R(\theta^t)} dR(\theta^t) + \frac{\partial e_0}{\partial p(\theta^t)} dp(\theta^t) \right),$$

This hicksian demand derivative is well defined as it does not depend on the precise timing of the transfers (as long as they satisfy our above description).

Corollary 6. *Consider a change in interest rate at t, dR_t. The compensated response of consumption at 0 where the compensation is done through transfers satisfy lemma 1 and 2 with present value $b_0 = W_0$ – so that the change is welfare neutral – is given by:*

$$\begin{aligned} de_0^h = &- \epsilon(e_0) e_0 \mathbb{E}_0^Q \left(\prod_{s=0}^{t} \frac{MPS_s}{R_s} \frac{dR_{t\wedge\tau-1}}{R_t} \right) \\ &+ \frac{\partial e_0}{\partial a_0} Cov_0^Q \left(\frac{Q^I}{Q}, \prod_{s=0}^{t-1} \frac{1}{R_s} \frac{a_{t+1}}{R_t} \frac{dR_{t\wedge\tau}}{R_t} \right). \end{aligned} \tag{33}$$

A (state-dependent) change in income at t gives the following compensated response:

$$de_0^h = \frac{\partial e_0}{\partial a_0} Cov_0^Q \left(\frac{Q^I}{Q}, \prod_{s=0}^{t-1} \frac{1}{R_s} dy_{t\wedge\tau} \right). \tag{34}$$

Finally, a (state-dependent) change in the price of good i at $t > 0$ gives:

$$\begin{aligned} de_0^h = &\epsilon(e_0) e_0 \mathbb{E}_0^Q \left(\prod_{s=0}^{t-1} \frac{MPS_s}{R_s} \frac{\partial c_t^i}{\partial e_t} dp_{t\wedge\tau} - \prod_{s=0}^{t} \frac{MPS_s}{R_s} \frac{\partial c_t^i}{\partial e_t} dp_{t\wedge\tau}^i \right) \\ &+ \frac{\partial e_0}{\partial a_0} Cov_0^Q \left(\frac{Q^I}{Q}, \prod_{s=0}^{t-1} \frac{1}{R_s} c_t^i\, dp_{t\wedge\tau}^i \right). \end{aligned} \tag{35}$$

The terms $Cov_0^Q(\frac{Q^I}{Q}, \cdot)$ again represent the precautionary effect of shocks. They are also well defined and do not depends on the precise timing of transfers as long as there present value is W_0 and that they are consistent with the asset structure and the borrowing constraints Note that if the income change can themselves be replicated through some asset investment consistent with lemma 1 and 2, this precautionary effects are nil. This is natural since the precautionary effect is precisely generated by income changes that cannot be insured through the assets available to the agent.

We only wrote the compensated effect of price changes for $t > 0$ since at 0 the expression is slightly different and given by:

$$\frac{de_0^h}{dp_0^i} = -\epsilon(e_0)e_0(1-\partial_a e_0)\frac{\partial c_0^i}{\partial e_0} + c_0^i$$

The first term is the substitution effect while the second is the mechanical effect. In the static decision problem, the compensated effect of a price change is exactly equal to the mechanical effect. In this dynamic environment, a price change at t also corresponds to a decrease in future prices, since it is equivalent to an increase in real interest rate, the substitution effect is therefore added to the mechanical effect. For goods consumption, the mechanical effect disappears and the response is:

$$\frac{dc_0^{i,h}}{dp_0^j} = \frac{\partial c_0^{i,h}}{\partial p_0^j} - \frac{\partial c_0^i}{\partial e_0}\left\{\epsilon(e_0)e_0(1-\partial_a e_0)\frac{\partial c_0^j}{\partial e_0}\right\}$$

where the first term is the static price derivative of hicksian demand. Note that the response is symmetric: we have $\frac{dc_0^{i,h}}{dp_0^j} = \frac{dc_0^{j,h}}{dp_0^i}$. This is because the price effect at t do not have a precautionary component. As in the previous section, this result extends to all substitution effects. We finish this section by restating the Slutsky symmetry result in the general case. First let us redefine formally the substitution effects:

$$\frac{dc_{t_0}^{i,S}(\theta^{t_0})}{dp_{t_1}^j(\theta^{t_1})} = \frac{dc_{t_0}^i(\theta^{t_0})}{dp_{t_1}^j(\theta^{t_1})} + \frac{dc_{t_0}^i(\theta^{t_0})}{dy_{t_1}(\theta^{t_1})}c^j(\theta^{t_1})$$

In words, the pure substitution effect is the total effect, including the implied change in wealth at t_0, of a price change at t_1, where the price change is compensate for at t_1, so that the precautionary effects are canceled.

Proposition 14. *The substitution effects satisfy for any dates t_0, t_1 and any goods i, j:*

$$\frac{Q(\theta^{t_0})}{\prod_0^{t_0-1} R_k}\frac{dc^{i,S}(\theta^{t_0})}{dp^j(\theta^{t_1})} = \frac{Q(\theta^{t_1})}{\prod_0^{t_1-1} R_k}\frac{d^s c^{j,S}(\theta^{t_1})}{dp^i(\theta^{t_0})}. \tag{36}$$

where we use the extended definition of Q.

C Continuous Time Derivation

In this appendix, we recast the simple model in continuous time and show how to obtain our main results in that case. The multidimensional stochastic state is given by θ_t. It is assumed to be Markov and follows a general jump-diffusion process defined by the infinitesimal operator $\mathcal{A}_t$. The agent problem at t is given by:

$$\begin{aligned} &\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_t \int_t^{\infty} e^{-\rho t} u(c_t, \theta_t, t) dt \\ &\text{s.t. } \dot{a}_t = y_t(\theta_t) + r_t a_t - c_t \\ &\qquad a_t \geq \underline{a}. \end{aligned}$$

Both income y_t and utility are stochastic and depend on the state θ_t. u is a strictly increasing and concave utility function that can be time (and state) dependent, c_t is consumption, $\rho > 0$ is the discount factor, a_t are bond holdings, $\{r_t\}_{t\geq 0}$ is the exogenous time path of interest rates. To streamline notations we sometime drop the dependency on θ and t. When we write u' and u'', we mean the derivatives of utility with respect to consumption $\partial_c u$ and $\partial_{cc} u$.
From the sequential problem, we derive the standard Hamilton-Jacobi-Bellman equation:

$$\rho V(a, y, \theta, t) = \max_c \; u(c, \theta, t) + \partial_a V(a, y, \theta, t)(y + r_t a - c) + \mathcal{A}_t V(a, y, \theta, t) + \partial_t V(a, y, \theta, t)$$

with a state constraint $a \geq \underline{a}$.The first order condition gives $u'(c) = \partial_a V$ and the boundary condition when the constraint is binding imposes $c(\underline{a}, \theta, t) = r_t \underline{a} + y(\theta, t)$

As before, we define by τ the stopping time at which the agent hits the borrowing constraint, we can then rederive our two Euler equations:

Lemma 3. *Marginal utility and its derivative satisfy, for any initial time t, state θ and asset level a at which the constraint does not bind and final time T, the following Euler equations :*

$$u'(c) = \mathbb{E}_t \left(e^{-\int_t^{T\wedge\tau} (\rho - r_{t'}) dt'} u'_{T\wedge\tau} \right) \tag{37}$$

$$(\partial_a c) u''(c) = \mathbb{E}_t \left(e^{-\int_t^{T\wedge\tau} (\rho - 2r_{t'} + \partial_a c) dt'} (\partial_a c) u''_{T\wedge\tau} \right) \tag{38}$$

The stochastic discount factors $e^{-\int_t^{T\wedge\tau} (\rho - r_{t'}) dt'}$ and $e^{-\int_t^{T\wedge\tau} (\rho - 2r_{t'} + \partial_a c) dt'}$ are the continuous time analogs of $\prod_t^{T\wedge\tau} \beta R_k$ and $\prod_t^{T\wedge\tau} \beta R_k MPS_k$

Proof. Differentiating the HJB equation and using the first order condition we get :

$$(\rho - r_t)V_a = \partial_a V_a s(a, y, c, t) + \mathcal{A}_t V_a + \partial_t V_a, \tag{39}$$

where $s(a, y, c, t) = \dot{a}_t = r_t a + y - c$ is the instantaneous saving. A second differentiation gives :

$$(\rho - 2r_t + \partial_a c)V_{aa} = \partial_a V_{aa} s(a) + \mathcal{A}_t V_a + \partial_t V_a \tag{40}$$

$T \wedge \tau$ is a finite stopping time, the first exit time from a bounded set, we can then apply Dynkin's formula to $e^{-\int_t^s (\rho - r_t)} u'_s$ and $e^{-\int_t^s (\rho - 2r_t + \partial_b c)} (\partial_a c) u''_s$, which give respectively :

$$\begin{aligned}
&\mathbb{E}_t \left(e^{-\int_t^{T\wedge\tau} (\rho - r_t) dt} u'_{T\wedge\tau} \right) - u'(c_t) \\
&= \mathbb{E}_t \left(\int_t^{T\wedge\tau} e^{-\int_t^s (\rho - 2r_{t'} + \partial_a c) dt'} \left(\partial_a V_{aa} s(a,) + \mathcal{A}_t V_{aa} + \partial_t V_{aa} - (\rho - 2r_t + \partial_a c) V_{aa} \right) ds \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_t \left(e^{-\int_t^{T\wedge\tau} (\rho - 2r_t + \partial_a c) dt} (\partial_a c) u''_{T\wedge\tau} \right) - (\partial_a c) u''(c) \\
&= \mathbb{E}_t \left(\int_t^{T\wedge\tau} e^{-\int_t^s (\rho - r_{t'}) dt'} \left(\partial_a V_a s(a) + \mathcal{A}_s V_a + \partial_t V_a - (\rho - r_s) V_a \right) ds \right) \\
&= 0
\end{aligned}$$

Where the third line makes use of (39) and (40) respectively. □

The two discount rates $e^{-\int_t^{T\wedge\tau} (\rho - r_{t'}) dt'}$ and $e^{-\int_t^{T\wedge\tau} (\rho - 2r_{t'} + \partial_a c) dt'}$ makes the stopped processes $u'_{t\wedge\tau}$ and $u''_{t\wedge\tau} \partial_a c_{t\wedge\tau}$ martingales. We can redefine the associated measures Q and Q^I by their Radon-Nykodim derivatives with respect to the physical measure P:

$$\begin{aligned}
\frac{dQ}{dP} &= \frac{e^{-\int_t^{\tau} (\rho - r) dt'} u'_{t\wedge\tau}}{\mathbb{E}_t \left(e^{-\int_t^{\tau} (\rho - r) dt'} u'_{t\wedge\tau} \right)} \\
\frac{dQ^I}{dP} &= \frac{e^{-\int_t^{\tau} (\rho - 2r + \partial_a c) dt'} u''_{t\wedge\tau} \partial_a c_{t\wedge\tau}}{\mathbb{E}_t \left(e^{-\int_t^{\tau} (\rho - 2r + \partial_a c) dt'} u''_{t\wedge\tau} \partial_a c_{t\wedge\tau} \right)}
\end{aligned}$$

We rederive the response of consumption to a local change in the path of interest rate and income $\{dr_t\}_{t\geqslant 0}$ and $\{dy_t\}_{t\geqslant 0}$. As a preliminary step,we show how marginal value of wealth respond to these changes.

Lemma 4. *Consider a perturbation of the path of interest rate and income* $\{\mathbf{dr_t}\}_{t\geqslant 0}$ *and* $\{\mathbf{dy_{t,i}}\}_{t\geqslant 0}$*. The change in the marginal value of wealth* $\partial_a V$ *is given by :*

$$dV_a(a,\theta_t,t) = \mathbb{E}_t\left(\int_t^{\tau} e^{-\int_t^s\left(\rho - r_{t'} + \partial_a c\right)dt'}\left\{dr_s V_a + \partial_a V_a\left(dr_s a + dy_s\right)\right\}ds\right) \tag{41}$$

Proof. Differentiating (39) at an arbitrary point in the interior of the state space, we have :

$$\begin{aligned}(\rho - r_t)\, dV_a = {} & \partial_a dV_a s(a) + \mathcal{A}_t dV_a + \partial_t dV_b \\ & + dr_t V_a + \partial_a V_a\left(dr_t a + dy_t - dc\right)\end{aligned}$$

Noting that $\partial_a V_a dc = (\partial_b c) dV_b$, we can rewrite it as :

$$\begin{aligned}(\rho - r_t + \partial_a c)\, dV_a = {} & \partial_a dV_a s(a) + \mathcal{A}_t dV_a + \partial_t dV_a \\ & + dr_t V_a + \partial_a V_a\left(dr_t a + dy_t\right)\end{aligned} \tag{42}$$

An application of Feynman-Kac then directly gives (41). To verify that this constitutes a solution of the agent problem, we need to verify that the constraint is satisfied and it's enough to verify this when $\underline{a}$ is close to be hit.

Near $\underline{a}$, given that the constraint is hit in finite time, we have :

$$dV_a(a,\theta_t,t) \simeq -\frac{\partial_a c(a-\underline{a})}{s(a)} u''\left(dr_t \underline{a} + dy_t\right) + o\left(\sqrt{a-\underline{a}}\right)$$

Lasry provides the Taylor expansion of $\partial_a c$ and $s(a)$ near the constraint. We then have :

$$dc(a,\theta,t) \longrightarrow \left(dr_t \underline{a} + dy_t\right)$$

So that the constraint is satisfied. $\square$

We now have all the ingredients to rederive the results of Proposition 10 :

Proposition 15. *Consider a perturbation of the path of interest rate and income $\{dr_s\}_{t\leq s\leq T}$ and $\{dy_s\}_{t\leq s\leq T}$. The response of consumption at the time of the change is given by:*

$$dc(a_t,\theta_t,t) = -\epsilon_t c_t\, \mathbb{E}_t^{Q}\left(\int_0^{T\wedge\tau} e^{-\int_t^s \partial_a c\,dt'} dr_s ds\right) + \partial_a c_t\, \mathbb{E}_t^{Q^I}\left(\int_0^{T\wedge\tau} e^{-\int_t^s r_{t'}dt'}\,(dr_s a + dy_s)\,ds\right)$$

with $\epsilon_t = \frac{u'}{u''c}$ the elasticity of intertemporal substitution.

Proof. Consider the 2 auxiliary random variables :

$$M_t^s\, V_a(a_s,\theta_s,s)$$
$$N_t^s\, V_{aa}(a_s,\theta_s,s)$$

Where M_t^s and N_t^s are predictable processes defined by :

$$M_t^s = e^{-\int_t^s(\rho-r_{t'})dt'}\int_t^s e^{-\int_t^{s'}\partial_a c\,dt'} dr_{s'}ds'$$
$$N_t^s = e^{-\int_t^s(\rho-2r_{t'}+\partial_a c)dt'}\int_t^s e^{-\int_t^{s'} r_{t'}dt'}\,(dr_{s'}a + dy_{s'})\,ds'$$

Noting that $M_t^t = 0$ and applying Dynkin's formula between t and $T\wedge\tau$ for an arbitrary T, we have :

$$\begin{aligned}
&\mathbb{E}_t\left(M_t^{T\wedge\tau}\, V_a\right)\\
=&\,\mathbb{E}_t\left(e^{-\int_t^{T\wedge\tau}(\rho-r_{t'})dt'}V_a\int_t^{T\wedge\tau} e^{-\int_t^s \partial_a c\,dt'}dr_s ds\right)\\
=&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau} e^{-\int_t^s(\rho-r_{t'}+\partial_a c)dt'}dr_s V_a\right)\\
+&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau} M_t^s\left\{\partial_a V_a s(a) + \mathcal{A}_s V_a + \partial_t V_a - (\rho-r_s)V_a\right\}ds\right)\\
=&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau} e^{-\int_t^s(\rho-r_{t'}+\partial_a c)dt'}dr_s V_a ds\right)
\end{aligned}$$

We canceled the third line making use of (39) again.

Similarly :

$$
\begin{aligned}
&\mathbb{E}_t\left(e^{-\int_t^{T\wedge\tau}\left(\rho-2r_{t'}+\partial_a c\right)dt'}V_{bb}\int_t^{T\wedge\tau}e^{-\int_t^s r_{t'}dt'}\left(dr_s a+dy_s\right)ds\right)\\
=&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau}e^{-\int_t^s\left(\rho-r_{t'}+\partial_a c\right)dt'}\left\{\partial_a V_a\left(dr_s a+dy_s\right)\right\}ds\right)\\
+&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau}N_t^s\left\{\partial_a V_{aa}s(a)+\mathcal{A}_s V_{aa}+\partial_t V_{aa}-(\rho-2r_t+\partial_a c)V_{aa}\right\}ds\right)\\
=&\,\mathbb{E}_t\left(\int_t^{T\wedge\tau}e^{-\int_t^s\left(\rho-r_{t'}+\partial_a c\right)dt'}\left\{\partial_a V_a\left(dr_s a+dy_s\right)\right\}ds\right)
\end{aligned}
$$

Using our second lemma, we then have :

$$
dV_a=\mathbb{E}_t\left(\int_t^{\tau\wedge T}e^{-\int_t^s\left(\rho-r_{t'}+\partial_a c\right)dt'}\left\{dr_s V_a+\partial_a V_a\left(dr_s a+dy_s\right)\right\}ds\right)
$$

$$
=\mathbb{E}_t\left(e^{-\int_t^{\tau\wedge T}\left(\rho-r_{t'}\right)dt'}V_a\int_t^{\tau\wedge T}e^{-\int_t^s\partial_a c\,dt'}dr_s ds\right)
$$

$$
+\,\mathbb{E}_t\left(e^{-\int_t^{\tau\wedge T}\left(\rho-2r_{t'}+\partial_a c\right)dt'}V_{aa}\int_t^{\tau\wedge T}e^{-\int_t^s r_{t'}dt'}\left(dr_s a+dy_s\right)ds\right)
$$

Since $dV_a=u''dc$ using (37) and (38) gives us the first formula.

□

We now derive the welfare cost of shocks that occur before τ.

Proposition 16. *The welfare cost of shocks occurring at the planing horizon is given by :*

$$
\begin{aligned}
dV(a_t,\theta_t,t)&=\partial_a V\,\mathbb{E}_t^Q\left(\int_0^{T\wedge\tau}e^{-\int_t^s r_{t'}dt'}\left(dr_s a+dy_s\right)ds\right)\\
&=u'\,\mathbb{E}_0^Q\left(\int_0^{T\wedge\tau}e^{-\int_t^s r_{t'}dt'}\left(dr_s a+dy_s\right)ds\right)
\end{aligned}
$$

Proof. Differentiating the HJB equation at an arbitrary point in the interior of the state space :

$$\rho dV = \partial_a dV s(a) + \mathcal{A}_t dV + \partial_t dV \\ + \partial_a V \left(dr_t a + dy_t \right)$$

Applying Feynman-Kac under the assumption that shocks past τ are compensated for at τ, we have :

$$dV(a_t, \theta_t, t) = \mathbb{E}_t \left(\int_t^{T \wedge \tau} e^{-\int_t^s \rho dt'} \left(dr_s a + dy_s \right) V_a ds \right)$$

Applying Dynkin's formula between t and $T \wedge \tau$ to $\tilde{M}_t^s V_a$ with

$$\tilde{M}_t^s = e^{-\int_t^s (\rho - r_{t'}) dt'} \int_t^s e^{-\int_t^{s'} r_{t'} dt'} \left(dr_{s'} a + dy_{s'} \right) ds'$$

we have

$$\begin{aligned} & \mathbb{E}_t \left(e^{-\int_t^{T \wedge \tau} (\rho - r_{t'}) dt'} V_a \int_t^{T \wedge \tau} e^{-\int_t^s r_{t'} dt'} \left(dr_s a + dy_s \right) ds \right) \\ = & \mathbb{E}_t \left(\int_t^{T \wedge \tau} e^{-\int_t^s \rho dt'} \left(dr_s a + dy_{s,} \right) V_a ds \right) \\ = & dV(a_t, \theta_t, t) \end{aligned}$$

Using (37) then gives us the result. □

We need $W_t = \mathbb{E}_t^Q \left(\int_0^{T \wedge \tau} e^{-\int_t^s r_{t'} dt'} \left(dr_s a + dy_s \right) ds \right)$ to compensate the agent for the change in interest rate and income. We can then decompose the consumption response as:

$$dc(a_t,\theta_t,t) = \underbrace{\epsilon_t c\, \mathbb{E}_t^Q \left(\int_t^{T\wedge\tau} e^{-\int_t^s \partial_a c\, dt'} dr_s ds \right)}_{\text{substitution effect}}$$

$$+ \underbrace{\partial_a c\, \mathbb{E}_t^Q \left(\int_t^{T\wedge\tau} e^{-\int_t^s r dt'} \left(dr_s a + dy_s \right) ds \right)}_{\text{wealth effect}}$$

$$+ \underbrace{\partial_a c\, \mathrm{Cov}_t^Q \left(\frac{dQ^I_{T\wedge\tau}}{dQ_{T\wedge\tau}}, \int_t^{T\wedge\tau} e^{-\int_t^s r dt'} \left(dr_s a + dy_s \right) ds \right)}_{\text{precautionary effect}}$$

Where the precautionary effect does not depend on the precise timing of the transfers $\{t_s\}_{s\geq t}$ as long as they satisfy $1.w_t = W_t$, $2.\dot{w}_s = r_s w_s - t_s$ $3.w_\tau = 0$. That is as long as the transfer are done through the risk free bond.