Speculative Growth: Hints from the U.S. Economy

By Ricardo J. Caballero, Emmanuel Farhi, and Mohamad L. Hammour*

We propose a framework for understanding episodes of vigorous economic expansion and extreme asset valuations. We interpret this phenomenon as a high-valuation equilibrium with a low cost of capital based on optimism about future funding. The key ingredient for such equilibrium is feedback from increased growth to a decline in the long-run cost of capital. This feedback arises when an expansion comes with technological progress in the capital sector, when fiscal rules generate procyclical fiscal surpluses, when the rest of the world has lower expansion potential or high saving needs, and when financial constraints are relaxed by the expansion itself. (*JEL* E22, O33, O41)

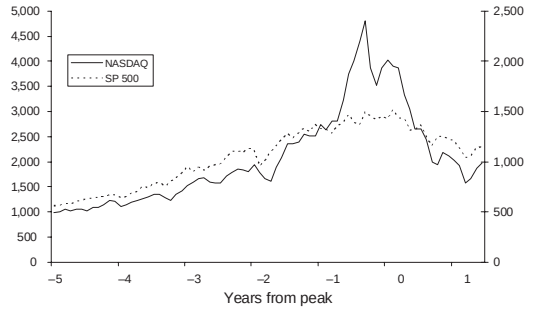
Economic history has witnessed many stark "speculative growth" episodes of extreme stock market valuations accompanied by brisk economic growth. The most notable recent experience was that of the United States in the 1990s. Figure 1A illustrates the sharp rise in the NASDAQ in the 1990s, followed by the collapse in 2000–2001. The extremes reached by valuations shown in Figure 1B, and their collapse, in the absence of obvious changes in fundamentals, are suggestive of widespread speculation. Figures 1C and 1D illustrate the growth and investment boom and bust that accompanied the market's gyrations.

The nature and policy dilemmas of speculative expansions have attracted much attention (e.g., International Monetary Fund (IMF), 2000; Robert J. Shiller, 2000; Stephen Cecchetti et al., 2000), but our formal understanding of the macroeconomic mechanisms that underlie the relation between stock market speculation and real economic activity remains quite limited. It is always possible to attribute such episodes to irrational exuberance, and indeed it is highly likely that some of that is invariably present. But what are the environments that facilitate these bouts of irrationality or, put differently, that facilitate the confusion of intelligent economic agents by having a rational path that is not too distant from observation?

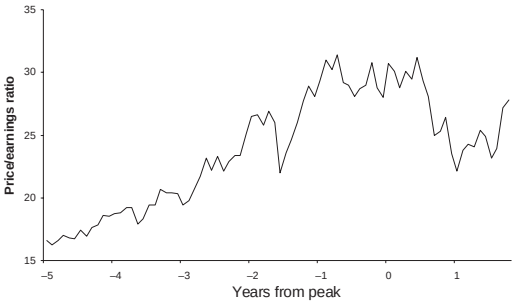
In this paper we take an extreme approach to answering this question and look for rational expectations equilibria that can mimic speculative growth episodes. We take a hint from the recent U.S. episode to formulate our theory. We start from the observation that long-run interest rates, and in particular the cost of capital faced by growing companies, declined throughout the 1990s. The dotted line in Figure 2 illustrates the path of the ten-year U.S. Treasury bond rate, from which we subtract the University of Michigan Inflation Expectations Index to obtain the corresponding real rate (solid line). The latter's decline during the 1990s is clear: the difference between the average real rates for the 1980s and 1990s (dashed lines) is about 200 basis points.

Such a decline can account for a significant share of the observed rise in asset prices: the decline in the real rates implicit in U.S. Treasury bonds alone can explain over 50 percent of the rise in broad markets' valuations during the second half of the 1990s, and if one is to consider the further decline in risk premia due to increased participation in stock markets (admittedly a

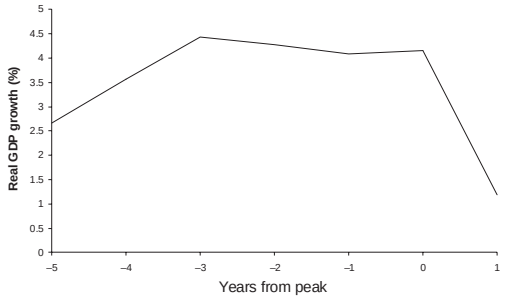
* Caballero: Department of Economics, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge, MA 02142, and National Bureau of Economic Research (e-mail: caball@mit.edu); Farhi: MIT, 50 Memorial Drive, Cambridge, MA 02142 (e-mail: efarhi@mit.edu); Hammour: Capital Guidance, 44 rue Washington, 75008 Paris, France. Part of this paper was written when Hammour was a research director at DELTA (a joint CNRS-EHESS-ENS research unit) (e-mail: hammour@delta.ens.fr). We are grateful to Björn Brügemann for outstanding research assistance and to the European Central Bank Research Department and the International Institute for Economic Studies for their hospitality. We thank Franklin Allen, Marios Angeletos, Gadi Barlevy, Olivier Blanchard, Thomas Chaney, Peter Diamond, Thomas Philippon, Frank Portier, Pietro Reichlin, Richard Rogerson, Jean Tirole, Harald Uhlig, Jaume Ventura, two anonymous referees, and seminar participants at Boston University, the CREI-CEPR Conference on Asset Price Bubbles, CORE, DELTA, ESSIM, Harvard University, IIES, MIT, the NBER EFG meetings, Northwestern University, the University of Toulouse, and the Wharton School for useful comments. Caballero thanks the National Science Foundation for financial support.



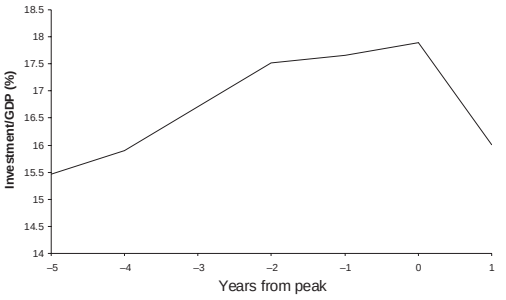
Panel A. Stock market indices



Panel B. Price/earnings ratio



Panel C. Real GDP growth (%)



Panel D. Investment/GDP (%)

FIGURE 1. SPECULATIVE GROWTH IN THE UNITED STATES

Notes: Panel B: the numerator is the real (inflation-adjusted) Datastream Total Market Index; denominator is moving average over preceding ten years of real earnings corresponding to the index.

Sources: Panel A: NASDAQ Composite Index from the NASDAQ Stock Market, Inc., SP 500 from Datastream; panel B: Datastream Total Market Index for US; panel C: Bureau of Economic Analysis (BEA); panel D: BEA. Annual Report on National Accounts (CD-ROM) 1998.

somewhat circular argument), then that share rises to 80 percent or more.[1, 2] Once such a rise has taken place, the investment and output growth boom that follow can be rationalized along standard *q-theory* arguments.

The key general equilibrium question in the above story is, however: how are expectations of a low long-term cost of capital consistent with the high *demand* for funding needed to finance a high-investment equilibrium? Limited to rational explanations, this necessarily means that along the expansionary path, agents expect an increase in the availability of funding to more than offset the demand for such funding. The source of this shift could be exogenous or endogenous. In our model and account of the U.S. episode, we include both, but highlight the latter since the crash of 2000–2001 cannot be matched by a commensurate exogenous shock. Thus the central ingredient of our model is a *growth-funding feedback* by which the *future supply* of *effective* funding increases as a result of the conditions created by a speculative expansion, and ends up lowering the cost of capital (the word "effective" is added to encompass the important decline in the price of new capital relative to consumption goods during the 1990s).

We also show that speculative growth episodes facilitate the emergence of (rational) bubbles. Of independent interest is the fact that, in

[1] See, e.g., Annette Vissing-Jorgensen (1999) for a model and evidence on the relation between participation and the equity premium.

[2] This is a fact that was not missed by economic commentators, and indeed was extrapolated to extreme limits by a sequel of bestsellers starting with "Dow 36,000: The New Strategy for Profiting from the Coming Rise in the Stock Market," by Jeremy Glassman and Kevin A. Hassett (1999).

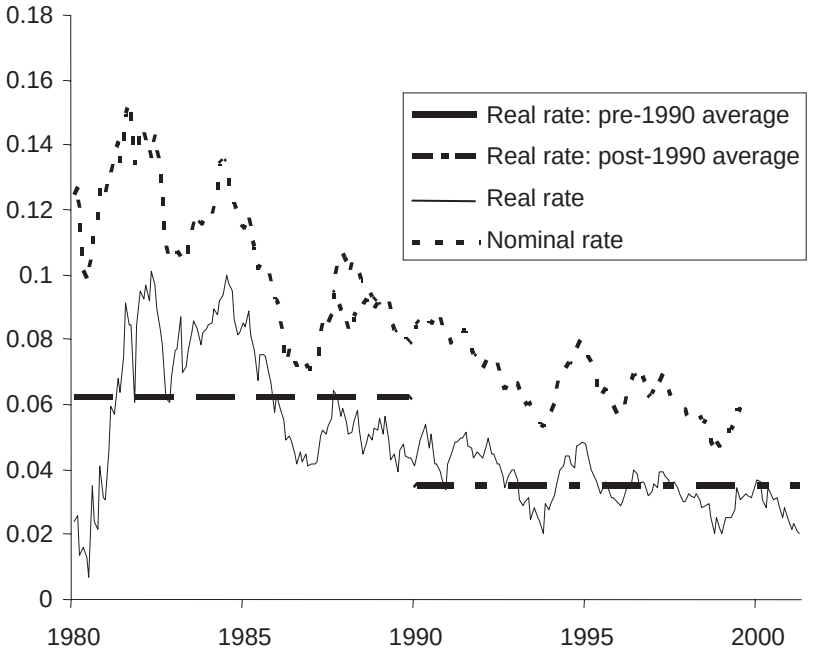


FIGURE 2. LONG REAL AND NOMINAL RATES (TEN-YEAR TREASURY BONDS RATE)

our environment, rational bubbles exhibit more realistic features than in the standard setup. In the latter, bubbles typically appear in economies that exhibit overaccumulation of capital and low interest rates (below the rate of growth of the economy) at the outset. Their emergence improves welfare by reducing investment and thereby alleviating the overaccumulation problem. The notion that bubbles and investment move in opposite directions is contrary, however, to the patterns depicted in Figure 1. Moreover, empirical testing of "dynamic inefficiency," which in steady state is equivalent to testing whether the return on capital is below the rate of growth of the economy, has been negative. Andrew Abel et al. (1989) tested an implication of dynamic inefficiency—whether the aggregate market for assets acts as a long-term "cash sink" for investors—but found no evidence of it in Organisation for Economic Co-operation and Development (OECD) economies.

In contrast, since in our model a speculative growth episode anticipates increased funding, bubbles may emerge even when the current interest rate exceeds the rate of growth of the economy and the corporate sector generates a surplus. Moreover, while it is still the case that the emergence of a bubble crowds out resources for investment, the speculative growth dynamics guarantees that investment still booms along the path. By the same token, if the speculative growth path crashes, then the conditions no longer exist for a rational bubble to survive (for the same reason that a bubble cannot exist in the dynamically efficient region of the standard model), and it must crash as well.

We show that a growth-funding feedback arises naturally when an expansion comes with technological progress, especially in the capital-producing sector; when fiscal rules generate procyclical fiscal surpluses; when the rest of the world has lower expansion potential; and when financial constraints are relaxed by the expansion itself. Arguably, these ingredients were all simultaneously present in the United States during the 1990s.

How does technological progress, especially in the capital producing sector, generate feedback from growth to funding? On one end, technological progress raises future incomes, and with it saving available for investment. On the other, relatively fast technological progress in capital-producing sectors reduces the saving needed to finance a higher steady-state capital-output ratio. In this context, a fundamental expansion in productivity and stock market speculation are not necessarily competing explanations of a speculative growth episode, as

most observers had it during the 1990s. A technological revolution might form an integral part—both as cause and consequence—of a speculative growth equilibrium.[3]

What about fiscal policy? We note that a significant share of increased available funding in the U.S. speculative expansion of the 1990s was attributable to public sector saving. To the extent that procyclical government revenues increase public saving, they reinforce the feedback from growth to saving. In the short run, fiscal surpluses can arise as a consequence of the boom and can facilitate the initial rise in investment. More importantly, fiscal surpluses can play a central role in making the speculative equilibrium feasible, by providing the funding necessary to sustain high investment in the long run. This consideration has a particularly stark implication for fiscal policy during a speculative growth episode. The possibility of using the fiscal surpluses that result from such expansions to cut taxes and raise spending could be illusory, as these surpluses might be necessary to sustain the speculative growth equilibrium that generated them.

A similar role is played by external saving, the other significant source of increase in aggregate saving during the 1990s. In the short run, capital flows alleviate the pressure on short-run interest rates brought about by the investment boom. In the long run, if the potential productivity in the high investment equilibrium is higher at home than abroad, then the speculative growth path yields a reallocation of global investment toward domestic assets. Moreover, the speculative growth path itself may be made feasible by a decline in opportunities abroad.

Finally, if the expansion relaxes financial constraints faced by productive entrepreneurs, then aggregate saving is reallocated toward them. The expansion of the 1990s clearly achieved this goal by facilitating the reallocation of capital toward the mostly small and emerging companies of the New Economy sectors. Some of these companies were undoubtedly bubbles and crowded out capital from good firms, but many others were the pillars of the information technology revolution that was so central to the episode.

The speculative growth episodes to which our theory relates are a recurrent phenomenon. Earlier episodes of vigorous economic expansion under speculative asset valuations have been documented by economic historians. In the case of the United States, this phenomenon also can be observed in the expansions of the turn of the twentieth century, the 1920s, and the 1960s (see, e.g., Shiller, 2000). Equally important, our theory can be brought to bear on sustained low cost-of-capital equilibria, such as the prolonged expansions exhibited in a number of East Asian economies in the postwar period. In fact, the key feedback from growth to funding in these economies has been documented. Examining the aggregate relationship between income growth and saving in a cross-country panel of 64 countries over the 1958–1987 period, Christopher Caroll and David N. Weil (1994) find that growth Granger-causes saving with a positive sign, but that savings does not Granger-cause growth. The pattern of an acceleration in growth followed by strong increases in saving rates is particularly clear in the high-growth, high-saving East Asian economies of Japan, South Korea, Singapore, and Hong Kong. Michael Gavin et al. (1997) elaborate on this evidence and show that the estimated impact of growth on saving is not only statistically significant but also very large in economic terms.

In its economic theme, this paper is part of a long tradition of studies of speculative growth episodes (e.g., Charles Kindleberger, 1989). In terms of the recent U.S. experience, this literature is divided between those who see a case of speculative behavior (e.g., Shiller, 2000), and those who, based on the significant acceleration in underlying U.S. productivity growth, conclude that the expansion was driven by a technological revolution that affected real fundamentals (e.g., Jeremy Greenwood and Boyan Jovanovic, 1999; Bart Hobijn and Jovanovic, 2000). From this approach, our contribution is to provide a unified perspective under which these two views need not be mutually exclusive. In fact, an expansion of technological opportunities and stock market speculation may go hand in hand.

This perspective finds support in the evidence of Paul Beaudry and Franck Portier (2004), who devise a novel semistructural VAR procedure to conclude that the U.S. business cycle is largely

[3] The link between technological progress and speculation has been noted in the literature. Jacques Olivier (2000), for example, emphasizes the feedback from bubbles to technological progress. Our setup also contains the opposite feedback, from technological progress to speculation.

driven by a shock that does not affect productivity in the short run, but that is strongly related to future productivity growth. They argue that such a shock could well represent a coordination device. Similarly, Lawrence Christiano and Jonas D. M. Fisher (2003) conclude that the explanatory power of real business cycle (RBC) models is greatly enhanced by the introduction of procyclical investment shocks. In our model, the comovements emphasized by Christiano and Fisher arise from a single shock, and this shock may be mostly of the sort of coordination hinted by Beaudry and Portier's findings.

On the methodological side, our paper belongs to the literature on multiple equilibria. This literature is too extensive to review here (see, e.g., Jess Benhabib and Roger Farmer, 1999), but it is interesting to point out some of the similarities and differences with Paul Krugman (1991), who develops a trade model with external economies and adjustment costs. In that model, if adjustment costs are small, multiple equilibria arise and expectations dominate. On the other hand, when adjustment costs are large enough, the multiple equilibria region disappears and only history matters. In contrast, in our model multiple equilibria arise only for intermediate adjustment costs. As in Krugman, we need adjustment costs to be small in order to be able to afford a transition to a high capital equilibrium. Unlike Krugman, we also need adjustment costs to be large enough to generate sufficient capital gains to justify the investment boom along the transition. We refer to this mechanism as the *capital gains mechanism,* which also illustrates that high valuations are not a side show but an integral ingredient of a speculative growth episode.

We also relate on the methodological and substantive sides to the literature on bubbles in general equilibrium. The seminal work by Jean Tirole (1985) sets up the foundation for the analysis of bubbles in general equilibrium, but when embedded in the basic unique-equilibrium overlapping generations (OLG) model it generates implications that are at odds with speculative growth episodes. In particular, it implies that investment and bubbles experience negative comovement, and that the latter can arise only in economies with return on capital below the rate of growth of the economy. Neither of these elements is observed during these episodes. Partly for this reason, a large literature has developed to modify the basic structure. Several papers have demonstrated that, in the presence of externalities that create a wedge between private and social returns on investment, bubbles can arise even if the bubbleless economy is dynamically efficient (e.g., Gilles Saint-Paul, 1992; Gene Grossman and Noriyuki Yanagawa, 1993; Ian King and Don Ferguson, 1993). Jaume Ventura (2003) takes this logic a step further and shows that with segmented financial markets, bubbles may emerge when only the marginal savers face interest rates below the rate of growth of the economy. Olivier (2000) addresses the negative-comovement problem and shows how bubbles on firm creation can lead to more rather than less investment. Aside from the difference in the specific mechanism we emphasize, in our model conventional rational bubbles exhibit positive comovement with investment and arise even when all investors and savers face high (and private) interest rates.

The rest of this paper is organized as follows. In Section I, we present a prototypical model of speculative growth and discuss the feasibility and properties of rational bubbles in this environment. Section II presents specific mechanisms that may have facilitated a speculative growth episode in the United States during the 1990s, namely, fiscal surpluses and capital flows. Section III provides microfoundations to the growth-funding feedback mechanism. For this, we develop a model of technological progress in the equipment sector and another of endogenous relaxation of financial constraints. Section IV concludes and is followed by an extensive Appendix.[4]

I. Speculative Growth

In this section we present a prototypical model of speculative growth. Our analysis is based on a linearized version of Peter Diamond's (1965) OLG model. The reason for using an OLG structure is our interest in studying the behavior of rational bubbles. The shortcoming of such a structure is the unappealing two-period assumption for phenomena that occur at higher than generational frequency and the hidden incomplete markets characteristic of OLG. We view the former as a useful simplification in a

[4] More extensive proofs and other results are available in the on-line Appendix (www.e-aer.org/data/sept06/20040445_app.pdf).

model that has no quantitative ambition, and we ensure that the latter plays no central role in our main results. Our main speculative growth conclusions survive in more cumbersome infinite horizon models.

The main substantive ingredient of the model is the *growth-funding feedback mechanism.* We capture it through a generic funding function that relaxes Diamond's stability condition.[5] We postpone any discussion of what is behind such relaxation until later in the paper, when we add context based on the U.S. experience during the 1990s.

Complementing the growth-funding feedback is a *capital gains mechanism,* which we capture with a simple adjustment cost in capital accumulation, but it could be replaced by a variety of other frictions that make capital gains possible, as long as these feed back into agents' investment decisions.

We also show in this section that bubbles can emerge in speculative growth paths, even if at the outset interest rates exceed the rate of growth of the economy, and that these bubbles exhibit positive rather than negative comovement with investment.

A. *The Growth-Funding Feedback*

Consider a standard Diamond (1965) overlapping-generations structure with no population growth and a unit mass of young and old agents who coexist at any date t. Each generation is born with a unit of labor, $\bar{L} = 1$, to be used when young, for which it receives a total wage W_t determined in a competitive, full-employment labor market. The economy's single consumption good is used as a numéraire.

Consumption goods are produced with capital, K_t, and labor, L_t. The production function at any time t is determined by the level of technology, A_t, which grows at an exogenous rate γ:

$$A_{t+1} = (1 + \gamma)A_t.$$

Production is given by a constant returns technology, $F(K_t, A_tL)$. Letting $k_t \equiv K_t/A_tL_t$ denote capital per effective worker, we write the marginal product of capital as

$$f_k(k_t), \quad f_k' < 0.$$

[5] See Diamond (1965), p. 1134.

The labor market is competitive, and the wage $w_t \equiv W_t/A_t\bar{L}$ per unit of effective labor can be written as:

$$w_t \equiv w(k_t) \quad w' > 0. \tag{1}$$

In Diamond as well as in most OLG models, each member of generation t chooses the level of saving when young, S_t, that maximizes lifetime utility. The relevant features of preferences are summarized in the level of saving per effective unit of labor, $s_t \equiv S_t/A_t\bar{L}$, which is a function of current wages and interest rates. We generalize this saving function to a *funding* function which, as we will see later on, encompasses a wide variety of channels beyond domestic private savings. In particular, we will let the funding function capture, with minor modifications, public and foreign savings, collateral constraints, and increasing returns in a capital-producing sector. Let us write this funding function as

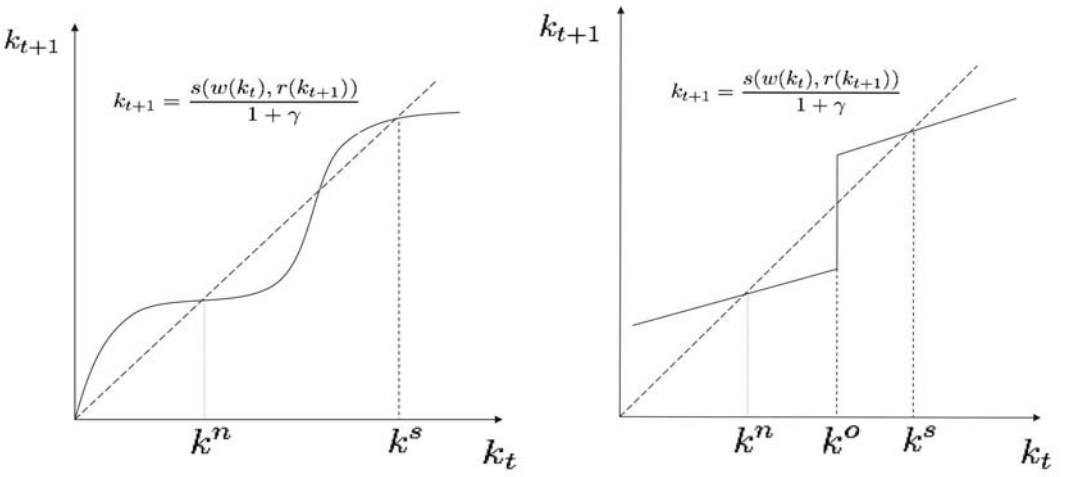
$$s_t = s(k_t, r_t), \tag{2}$$

where r_t denotes the interest rate between periods t and $t + 1$, $s_k > 0$, and $0 < s_r < \infty$. In the basic saving-function interpretation of standard OLG models, the first argument in this funding function, k_t, is the result of replacing the wage function into the saving function. In ours, there will be other channels as well.

The Long-Run Feedback.—In the Diamond (1965) model, there are no adjustment costs to capital, so that $r_t = f_k(k_{t+1})$ and the stock of capital next period is equal to today's funding. Equilibrium then is characterized by a nonlinear difference equation for capital accumulation:

$$k_{t+1} = \frac{s(k_t, r(k_{t+1}))}{1 + \gamma}. \tag{3}$$

If the stability condition in Diamond (1965) is relaxed, then multiple steady states as in panel A of Figure 3 can arise. The essential feature of this saving function is that there is a region where saving increases rapidly as capital rises. For now, we do not explain the source of this feature, but simply assume it. A particularly simple formulation of such a situation is to start with a saving function that is linear in k and r, with a step at some level of capital k^o:



Panel A. Multiple steady states in the Diamond model

Panel B. Multiple steady states in the linear model

FIGURE 3. MULTIPLE STEADY STATES

$$s(k_t, r_t) = \begin{cases} s_0 + s_k k_t + s_r r_t, & k < k^o; \\ s_0 + \delta + s_k k_t + s_r r_t, & k \geq k^o, \end{cases} \tag{4}$$

with s_0, δ, s_k, and s_r strictly positive. Now let the marginal product of capital be linear in capital, $\pi_0 - \pi_1 k$, with π_0 and π_1 strictly positive and $k \leq \pi_0/\pi_1$ (henceforth, we will focus on equilibria that satisfy this constraint), so that:

$$r_t = r(k_{t+1}) = \pi_0 - \pi_1 k_{t+1}.$$

Replacing this interest rate expression into the funding function (4), and the result into the capital accumulation expression (3), solves the model. The following assumptions ensure that we capture the scenario that concerns us.

ASSUMPTION 0: $\Delta \equiv s_r \pi_1 + (1 + \gamma - s_k) > 0$ *and* $s_0 + s_r \pi_0 < k^0 \Delta$.

ASSUMPTION 1 (Minimum growth-saving feedback): $\delta > \underline{\delta} \equiv k^0 \Delta - (s_0 + s_r \pi_0)$.

Assumption 1 states that the jump in the funding function at k^0 must be high enough for a steady state to exist to the right of k^0.

PROPOSITION 1 (Multiple steady states): *If Assumptions* 0 *and* 1 *are satisfied, the economy has two steady states,* k^n *and* k^s, *with:*

$$k^n = \frac{s_0 + s_r \pi_0}{\Delta} < k^o < \frac{s_0 + \delta + s_r \pi_0}{\Delta} = k^s, \tag{5}$$

where the superscripts "*n*" *and* "*s*" *stand for* normal *and* speculative, *respectively.*

PROOF:

See Appendix.

Note that *normal* and *speculative* are simply labels for low and high capital equilibria, respectively. The reason for these labels is that the former has low equity valuations while the latter has high equity valuations (see below).

The proposition simply states that if the growth-saving feedback, captured here by the parameter δ, is sufficiently strong, the economy exhibits multiple steady states. This is illustrated in panel B of Figure 3.

Importantly, since the marginal product of capital is decreasing in the stock of capital, the speculative steady state exhibits a lower cost of capital than the normal steady state:

$$r^s = r^n - \pi_1 \frac{\delta}{\Delta} < r^n.$$

This is the reason at times we refer to the speculative equilibrium as the "low cost of capital" equilibrium. Similarly, since the valuation

of an asset is simply its price divided by earnings, and in steady state the former is just earnings divided by cost of capital, we have that valuation is just one over the cost of capital. Thus we also refer to the speculative equilibrium as the "high-valuation equilibrium."

How can the economy have a low cost of capital in the speculative equilibrium when it requires more funding to be sustained? Precisely because higher capital raises funding enough to more than offset the reduction in funding due to the decline in interest rates resulting from lower marginal product of capital (supply of funding shifts more than demand for funding). This is at the core of the growth-funding mechanisms we discuss in this paper.

The Capital Gains Mechanism.—A central aspect of the phenomenon we wish to characterize is the possibility of a stock market boom. Moreover, the presence of a crash at the end of the U.S. speculative episode, in the absence of a clear exogenous shock, also suggests the presence of multiple equilibria. Neither of these features, stock market booms or multiple equilibria crashes, is present in the simple economy we have described up to now.

On one hand, the value of installed capital is equal to one at all times. On the other, while the model above has multiple steady states, it has a *unique equilibrium.* That is, for any given level of k_t the economy converges to a specific steady state. To see this, suppose the economy is at k^n. Then moving toward k^s would require increasing funding, which can happen only if the interest rate rises (since k_t is given). But this cannot be an equilibrium since a decreasing marginal product of capital implies that an increase in the interest rate reduces investment. Conversely, suppose that the economy is at k^s. Then moving toward k^n would require lowering funding, which requires the interest rate to fall. But this cannot be an equilibrium either, since a fall in the interest rate leads to more rather than less investment. More generally, the same logic applies to any other level of equilibrium capital: if a level of investment constitutes an equilibrium in the capital market for a given level of capital, then no other level of investment (and hence of tomorrow's capital) can be an equilibrium. The reason is simply that for any given k_t, investment and funding are, respectively, decreasing and increasing with respect to r_t.

Let us modify slightly the previous setup and introduce adjustment costs. This not only creates the possibility of a stock market boom but also breaks the short-run connection between the return on investment and the marginal product of capital (since there are capital gains), which is at the heart of the uniqueness result above.

Let investment to maintain the effective capital stock be frictionless, but all deviations from this level of investment be subject to a convex adjustment cost:[6]

$$c(I_t, K_t) = \frac{1}{2}\,\theta^{-1}K_t\left(\frac{I_t}{K_t} - \gamma\right)^2, \tag{6}$$

where I is investment. The first-order condition for investment is

$$c_I - c_K = q_t - 1,$$

where q_t is the price of a unit of *installed* capital, with

$$c_I = \theta^{-1}(x_t - \gamma)$$

$$c_K = -\theta^{-1}(x_t - \gamma)\,\frac{x_t + \gamma}{2},$$

where $x_t \equiv I_t/K_t$. The investment function follows from:

$$(x(q_t) - \gamma)[1 + \gamma + \tfrac{1}{2}(x(q_t) - \gamma)] = \theta(q_t - 1).$$

Note that a first-order approximation of $x(q_t)$ around $q_t = 1$ takes the usual simple q-theory form:

$$x(q_t) = \gamma + \frac{\theta}{1 + \gamma}\,(q_t - 1). \tag{7}$$

When adjustment costs are large (θ low), investment is relatively unresponsive to deviations of the value of installed capital from that of uninstalled capital. When they are low (θ large), investment is very responsive to such deviations. The simple model in the previous section

[6] Subtracting effective-maintenance investment from the adjustment cost function facilitates our analysis later, without any substantive cost.

obtains when θ goes to infinity. The capital accumulation equation can now be written as:

$$k_{t+1} = \frac{1 + x(q_t)}{1 + \gamma} k_t. \tag{8}$$

Importantly, it is no longer the case that the interest rate is equal to the marginal product of capital. A unit of installed capital costs q_t and saves adjustment costs for $c_K(I_t, K_t)$ today. This unit of capital yields q_{t+1} and $f_k(k_{t+1})$ tomorrow. Thus, the return from investing in capital is

$$(1 + r_t) = \frac{q_{t+1} + f_k(k_{t+1})}{q_t + c_K(I_t, K_t)}.$$

After replacing f_k in it, this expression can be rearranged as

$$q_{t+1} = (1 + r_t)(q_t + c_K(I_t, K_t)) - (\pi_0 - \pi_1 k_{t+1}), \tag{9}$$

with

$$\begin{aligned} c_K(I_t, K_t) &= \frac{1}{2}\theta^{-1}\left(\frac{I_t}{K_t} - \gamma\right)^2 - \theta^{-1}\left(\frac{I_t}{K_t} - \gamma\right)\frac{I_t}{K_t} \\ &= \frac{1}{2}\theta^{-1}(x_t - \gamma)^2 - \theta^{-1}(x_t - \gamma)x_t \\ &= -\left(\gamma + \frac{\theta}{2}(q_t - 1)\right)(q_t - 1). \end{aligned}$$

The last step to specify fully the dynamic system is to solve the interest rate as a function of q_t and k_t. For this, let us find the capital market equilibrium condition. Following production in period t, the old sell their capital to the young at price q_t. Funding is allocated to the purchase of the existing stock of capital, to invest in new capital, and to pay for the adjustment costs of installing this new capital:

$$s(k_t, r_t) = (q_t + x_t(q_t))k_t + c(x(q_t), k_t). \tag{10}$$

Replacing (4), (6), and (7) into (10), and solving out for the interest rate, yields

$$\begin{aligned} r(q_t, k_t) &= \frac{1}{s_r}\left\{\left[q_t + x_t(q_t) + \frac{\theta^{-1}}{2}(x(q_t) - 1)^2 - s_k\right]\right. \\ &\qquad \left. \times k_t - s_0 - 1\{k_t \geq k^o\}\delta\right\}, \end{aligned} \tag{11}$$

with $1\{k_t \geq k^o\}$ an indicator function that takes value one when $k_t \geq k^o$, and zero otherwise.

Putting things together, the system governing equilibrium dynamics can be written as a two-dimensional system in (k_t, q_t)-space:

$$\begin{aligned} k_{t+1} &= \frac{1 + x(q_t)}{1 + \gamma} k_t, \\ q_{t+1} &= (1 + r(q_t, k_t))(1 + \theta^{-1}(x(q_t) - \gamma)) \\ &\quad - \left(\pi_0 - \pi_1 \frac{1 + x(q_t)}{1 + \gamma} k_t\right). \end{aligned}$$

It is straightforward to verify that this dynamic system has the same steady states described in Proposition 1. For this, simply note that in steady state $q = 1$, in which case the steady state equations of this system collapse to those of the model without adjustment costs.

But the question that concerns us here is under which circumstances the presence of a capital gains mechanism can break the uniqueness result of the simpler model, in particular, whether it is possible for an economy near its "natural" steady state to initiate a path toward the "speculative" one. That is, whether the economy can embark on a "speculative growth" path.

Let us assume that δ is small so that we can make a first-order approximation around $q = 1$ and $k = k^n$ for the transition from k^n to k^s, if such transition is possible. (See the Appendix for a precise statement about this approximation, which also requires the distance between k^o and k^n to be small.) Let $\widehat{q_t} \equiv (q_t - 1)$ and $\widehat{k_t} \equiv (k_t - k^n)$; then the linearized dynamic system can be written as:

$$\begin{aligned} \hat{r}(q_t, k_t) &= \frac{1}{s_r}\left(1 + \frac{\theta}{1 + \gamma}\right)k^n \widehat{q_t} \\ &\quad + \frac{1}{s_r}[1 + \gamma - s_k]\widehat{k_t} - 1\{k_t \geq k^o\}\frac{\delta}{s_r}, \end{aligned}$$

$$(12)\quad \widehat{q_{t+1}} = \left[\frac{1+r^n}{1+\gamma} + \left(\frac{\pi_1}{1+\gamma}\frac{\theta}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r}\right)k^n\right] \times \widehat{q_t} + \frac{\Delta}{s_r}\widehat{k_t} - 1\{\widehat{k_t} \geq \widehat{k^o}\}\frac{\delta}{s_r},$$

$$(13)\quad \widehat{k_{t+1}} = \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t}.$$

In the Appendix, we provide the exact statements that make this linear system the limit of the nonlinear one, as δ and $(k^n - k^o)$ go to zero. In the main text, our analysis refers to the dynamic system described by (12) and (13). We shall add a technical condition that ensures that the interest rate and marginal product, f_k, rather than the adjustment cost saving feature of investment, c_K, dominates local dynamics for all values of θ.[7]

ASSUMPTION 0′ (Technical condition): $(k^n/s_r) + (1 + r^n)/(1 + \gamma) - 1 > 0$.

Let us define a function $\Lambda(\theta)$ which corresponds to the ratio of the slope of the unstable path stemming from the normal steady state to (minus) the slope of the saddle path stemming from the speculative steady state (see the Appendix). It can be shown that Λ is strictly positive, nonmonotonic on $[0, +\infty)$, decreasing on $[0, \tilde{\theta}]$, and increasing on $[\tilde{\theta}, +\infty)$, with $\lim_{\theta\to 0}\Lambda(\theta) = \lim_{\theta\to+\infty}\Lambda(\theta) = +\infty$, where

$$\tilde{\theta} = \frac{\frac{k^n}{s_r} + \frac{1+r^n}{1+\gamma} - 1}{\left(\frac{\pi_1}{1+\gamma} + \frac{1}{s_r}\right)\frac{1}{1+\gamma}k^n}.$$

ASSUMPTION 1′ (Minimum growth saving feedback for a transition): $\delta > \underline{\delta} \equiv (\Lambda(\tilde{\theta}) + 1)\Delta(k^0 - k^n)$.

Note that since $\Lambda(\tilde{\theta}) > 0$, Assumption 1′ implies Assumption 1.

ASSUMPTION 2 (Speculative adjustment costs region): $\underline{\theta}(\delta) < \theta < \bar{\theta}(\delta)$, *with* $\underline{\theta}(\delta) = \inf\{\theta > 0, \Lambda(\theta) = -1 + \delta/[\Delta(k^0 - k^n)]\}$ *and* $\bar{\theta}(\delta) = \sup\{\theta > 0, \Lambda(\theta) = -1 + \delta/[\Delta(k^0 - k^n)]\}$.

PROPOSITION 2 (Multiple equilibria and speculative growth): *If Assumptions* 0, 0′, 1′, *and* 2 *hold, there is a speculative growth path that takes the economy from* k^n *to* k^s. *Along this path,* $\widehat{q_t} > 0$.

PROOF:
See Appendix.

Let us discuss the structure of the proof with the help of the (heuristic) phase diagrams in Figure 4. The central ingredients in each of these panels are the unstable arm of the normal steady state, the vertical line at k^o, and the stable arm (saddle path) of the speculative steady state. Naturally, for a speculative growth path to exist, it must be the case that the unstable path of the low capital equilibrium intersects the k^o line *below* the intersection of the latter and the saddle path of the high capital equilibrium. This is precisely the scenario depicted in panel A. By continuity, in this case there is a path starting at (k^n, q^+) with $q^+ > 1$, such that it hits k^o at the saddle path of the speculative equilibrium.

Before further characterizing the speculative growth path, it is instructive to discuss scenarios where the conditions for such a path do not exist. Panels B and C in Figure 4 provide these examples. The former is one in which adjustment costs are too low (hence $\theta > \bar{\theta}$) while the latter represents the other extreme (hence $\theta < \underline{\theta}$).[8]

On one end, when adjustment costs are too low, the slope of the saddle path for the speculative steady state is very flat (small deviations of q from one lead to large investment re-

[7] This condition is automatically verified if the normal equilibrium is dynamically efficient, which we view as the most interesting case and implicitly assume throughout.

[8] The only reason Proposition 2 is not stated as an *if and only if,* is that since our linear system is an approximation to the nonlinear one, we cannot state whether a speculative growth path is feasible or not when θ is *exactly* at the boundaries of the range in Assumption 2. If the inequalities in Assumption 2 are reversed, there is no speculative growth path (see the Appendix). The same consideration applies to most of the Propositions in the main text.

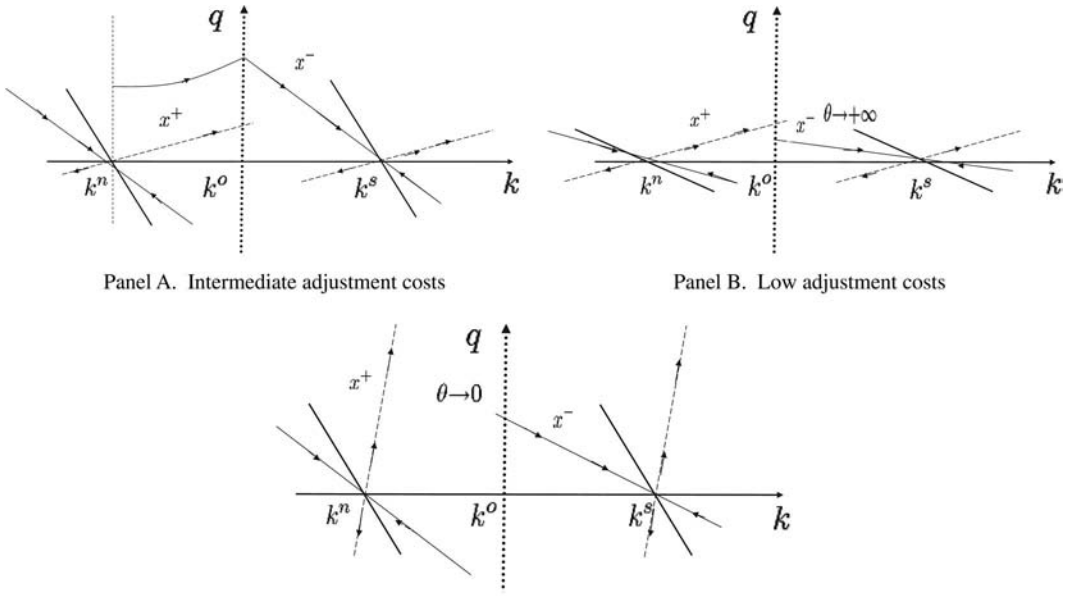


FIGURE 4. MULTIPLE EQUILIBRIA

sponses). On the other, when adjustment costs are too large, the slope of the unstable path for the normal steady state is very steep. (It takes enormous capital gains to justify costly investment beyond maintenance investment.) That is, for a speculative growth path to exist, adjustment costs must be large enough to generate sufficient capital gains to decouple the return on investment from the marginal product of capital in the short run, but not so large that it is simply too costly to finance an investment boom.

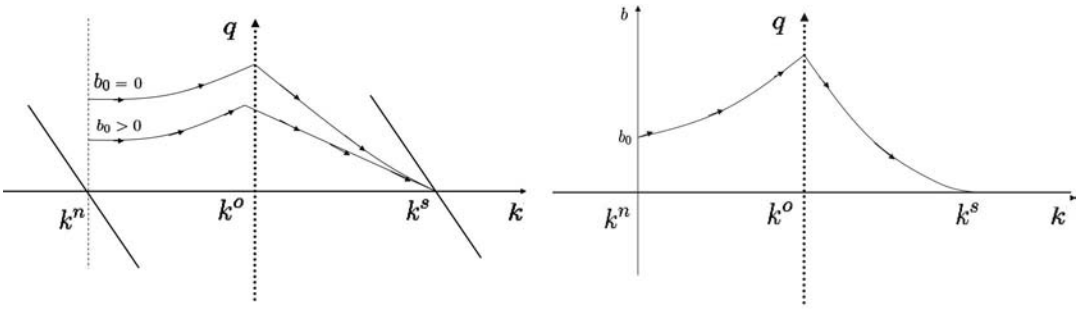
Returning to panel A, we can see that *along the speculative growth path the value of installed capital is booming and with it are investment and growth.* And since marginal product is declining, the *price-earnings ratios* (*valuations*) *are also rising.* These are precisely the features we highlighted in the introduction for the U.S. episode during the 1990s.

B. *Bubbles*

Up to now, we have shown that along a speculative growth path, the valuation of installed capital's fundamentals rises. In this section we show that speculative growth environments also are conducive to the emergence of pure bubbles. This is important not only because it captures some of the most extreme aspects of the NASDAQ during the 1990s, but also because it refines the theory of equilibrium rational bubbles (see Tirole, 1985) in empirically sound dimensions.

The fact that most evidence points in the direction of dynamic efficiency in the United States (Abel et al., 1989), and that episodes of speculative growth as illustrated in Figure 1 exhibit strong *positive* comovement between valuations and investment, would seem to be conclusive evidence against the presence of rational bubbles during these episodes. It turns out that neither property needs to hold in our speculative growth environment. Along a speculative growth path, pure bubbles may emerge even in the region where the interest rate exceeds the rate of growth of the economy, and they exhibit positive rather than negative comovement with physical investment.[9]

[9] In the standard theory, bubbles can emerge when the steady-state interest rate is below the rate of growth of the economy (Tirole, 1985). Essentially, since a bubble must grow at the interest rate, and the latter is below the rate of growth of the economy, a bubble in this environment never outgrows the economy. Thus, if it is feasible at the outset, it remains so at later dates. Moreover, rational bubbles in this context solve an "excessive" investment problem driven by a high demand for a store of value rather than by productivity considerations. It follows that the emergence of a



Panel A. Paths of q and k with and without bubbles

Panel B. Path of the bubble

FIGURE 5. BUBBLES

We augment the previous model with an additional asset: a pure bubble with value B_t. We define the normalized (by productivity) bubble as $b_t \equiv B_t/A_t$.

ASSUMPTION 3 (Bubble region): $\pi_0 - \pi_1 k^n > \gamma > \pi_0 - \pi_1 k^0$.

Assumption 3 guarantees that the normal steady state is dynamically efficient, while the speculative steady state is dynamically inefficient. The latter condition is necessary for the existence of a bubble. The former is sufficient to ensure that bubbles can appear even if initial interest rates exceed the rate of growth of the economy.

PROPOSITION 3 (Bubbles in high-interest region): *Let Assumptions* 0, 0′, 1′, 2, *and* 3 *hold, and initial capital be equal to* k^n. *Then, if the economy initiates a speculative growth path, there is a range of feasible rational bubbles* (*normalized by* $A(t)$), b_t, *such that* $b_0 > 0$ *and*

$$b_{t+1} = \frac{1 + r_t}{1 + \gamma} b_t.$$

PROOF:

See Appendix.

This result is intuitive. While at the outset of the speculative growth path the economy is in a region where the interest rate exceeds the rate of growth of the economy, it is headed toward a dynamically inefficient region where bubbles can be sustained. But then, by backward induction, bubbles can start before the economy reaches that region. All that is required is that the initial bubble be sufficiently small so that, despite its fast early growth, it reaches the dynamically inefficient region with a size consistent with the bounds imposed by the size of the economy, and that it allows capital to grow and reach that region.

Figure 5 shows one such example. Panel A illustrates the paths of q and k for two economies along a speculative path, one with and the other without a rational bubble. Panel B illustrates the path of the bubble (normalized by productivity).

In order to highlight the novel investment-bubble comovement aspect of bubbles in a speculative environment, let us momentarily introduce a sunspot as an equilibrium-selection device. Suppose that in the multiple equilibria region the sunspot variable is a Markov chain with two states, $\{u, d\}$, and initial value $e_0 = u$. When the state is u, all agents coordinate their expectations on the speculative growth path. When the state is d, expectations coordinate on the path toward the low capital steady state. Define the crash time, τ_d,

$$\tau_d = \inf\{t > 0, e_t = d\},$$

rational bubble in this context lowers aggregate investment, and that a crash of the bubble boosts aggregate investment.

Now assume that the interest rate in steady state exceeds the rate of growth of the economy. It is also well known from Tirole (1985) that rational bubbles cannot exist in such an environment. If the interest rate exceeds the rate of growth of the economy, a bubble grows faster than the economy until it eventually becomes inconsistent with the aggregate endowment constraint. Backward induction then shows that a rational bubble can simply not emerge in this region.

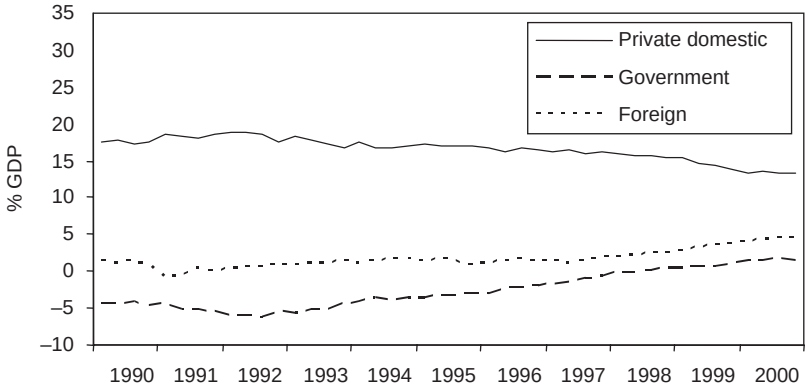


FIGURE 6. FUNDING SOURCES IN THE UNITED STATES

and let the transition probabilities be such that once a crash takes place, the economy converges to the low capital equilibrium with probability one. Finally, suppose that preferences are such that only the expected return of a project matters to agents (we can use Epstein-Zin preferences for this purpose). With this structure, sunspots do not alter the expression defining the equilibrium interest rate. All that is modified is the expression for expected future price of capital while in the speculative growth path, which is now $(\mu q^u_{t+1} + (1 - \mu)q^d_{t+1})$ instead of simply q_{t+1}.

PROPOSITION 4 (Positive investment-bubbles comovement): *In the bubble region of a speculative path, investment and the bubble grow in tandem. Moreover, if the speculative path crashes, so must the bubble.*

Note that a bubble still crowds out investment. This is reflected in the fact that the larger the initial bubble, the lower q for a given k along the transition path (see Figure 5, panel A), and hence the lower investment. But more importantly, the bubble now exhibits positive rather than negative comovement with aggregate investment. In particular, the bubble can emerge only when the investment is booming along a speculative path. And if the latter crashes, so will the bubble.[10]

[10] A point that needs to be emphasized is that along all the paths considered here, there is no bubble on q. The bubbles we consider are distinct assets. Along every path we consider, and at every point in time, q, the price of a unit of installed capital, is equal to the net present value of the dividends this unit of capital produces in the future and the reduction in adjustment costs permitted by this extra unit of installed capital in future periods:

$$q_u = \sum_{t=u}^{+\infty} r_t \Pi_{s=u}^{t} (1 + r_s)^{-1} - \sum_{t=u}^{+\infty} c_K(I_t, K_t) \Pi_{s=u}^{t-1} (1 + r_s)^{-1}.$$

Although it may not be immediately obvious, it turns out that our speculative growth model does satisfy Manuel Santos and Michael Woodford's (1997) general conditions for the existence of bubbles. We return to this issue after we have developed specific mechanisms for the growth-funding feedback in Section III.

II. Fiscal Surpluses and Capital Flows

Section I describes an environment conducive to speculative growth episodes that resembles the paths portrayed in Figure 1. At the core of this result is optimism about future effective funding of large capital accumulation and the short-run response of funding to expected capital gains. In this and the next section we discuss a few prominent features of the U.S. economy during the 1990s that supported this optimism.

Figure 6 shows that much of the funding for the onset of the speculative growth path in the United States during the 1990s came from fiscal surpluses and capital flows. In this section we discuss these two channels and their potential role in facilitating speculative growth episodes.

A. *Fiscal Surpluses*

The fiscal surpluses generated during the U.S. speculative growth experience were the combined result of fiscal consolidation measures and the automatic effect of procyclical tax revenues in a booming environment. The reduction in public debt had a moderating effect on interest rates in the short run and substituted for increased private saving to fund the investment boom. It is in this sense that a tight policy of paying down the national debt supported the boom in the short term. Turning to the more novel issue of longer-term sustainability, we argue that a continuing policy rule of generating fiscal surpluses provides critical support to the speculative equilibrium.

To develop our argument, we add a government to the model. We assume the government taxes wage income at a rate τ_t and spends $g_t A_t L$ on goods that do not enter agents' utility function. Thus the government's budget constraint is

$$(14)\quad (1+\gamma)d_{t+1} = (1+r_t)[d_t - (\tau_t w_t - g_t)],$$

where $d_t \equiv D_t/A_t L$ denotes public debt per unit of effective labor.

Adding the public sector alters two equations in the model. If we now interpret s more narrowly as the saving of the young, it becomes a function of after-tax wages:

$$(15)\qquad s_t = s((1-\tau_t)w(k_t),\, r_t),$$

while the capital-market equilibrium condition is now given by

$$(16)\quad (\tau_t w_t - g_t) + s_t = d_t + (q_t + x(q_t))k_t + c(x(q_t), k_t).$$

The saving of the government and of the young must fund the purchase of the existing public debt and capital stock from the old, and new, investment. Combined, equations (15) and (16) implicitly define a new interest rate function $r_t = r(k_t, q_t, d_t, \tau_t, g_t)$. The economy's dynamics are described by system (8) and (9), with the new interest rate function together with the government budget constraint, (14), and a fiscal policy rule.

Let us consider a benchmark *fixed-parameters* policy rule under which detrended government spending and the tax rate are fixed at $\bar{g} > 0$ and $\bar{\tau} > 0$. To facilitate comparisons, we assume that this and other policy rules result in a balanced budget in the low-valuation steady state—which requires $\bar{g} = \bar{\tau} w^n$—and, therefore, result in the same low valuation steady state. The important point to notice is that this fixed-parameters policy creates primary surpluses during expansions beyond k^n and primary deficits during contractions. To see this, note that the fiscal rule implies that the primary government surplus, $(\bar{\tau} w_t - \bar{g})$, increases with w_t in an expansion. Thus the response of the combined "gross" saving of government and the young to an increase in wages is given by $(\bar{\tau} + s_w(1-\bar{\tau}))dw_t$.

The increase in saving generated by the fixed-parameters rule not only facilitates the funding of investment in the short run but also, more importantly, plays a central role in facilitating the emergence of a speculative growth scenario. This feasibility point is made most clearly by focusing on the speculative steady state, $(k^s, 1)$, rather than on the entire path. One can show that if, instead of being fixed, government spending is raised with the endogenous increase in the wage, then aggregate saving falls and so does the level of k^s. In fact, this fiscal expansion experiment is similar to *reducing* δ in Section I. For a high enough indexation of fiscal spending to wages, the speculative equilibrium is no longer feasible. In other words, fiscal surpluses are not only a symptom of the speculative growth episode, but also can be a central element of the factors that support it.

The notion that the fiscal surpluses generated by the speculative equilibrium can be partly spent or rebated to the taxpayer may be an illusion. The surpluses could be a pillar of the speculative equilibrium, and might swiftly disappear if this equilibrium unravels—giving rise to what one might describe as a *surplus illusion.* The following proposition summarizes this discussion.[11]

PROPOSITION 5 (Surplus illusion):[12] *Let Assumptions* 0, 0′, 1′, 2, *and* 4 *hold. Let* $g_t = \bar{g} + \alpha\bar{\tau}(w_t - w^n)$. *If the economy has two saddle-*

[11] In the proposition, we also add the condition that the speculative equilibrium be dynamically inefficient in order to avoid explosive assets or debt. This is necessary only because we are analyzing simple linear fiscal rules.

[12] Proof of this proposition can be found in the on-line Appendix.

path steady-state equilibria for $\alpha = 0$, *there exists an* $\alpha^m > 0$ *such that for any* $\alpha \geq \alpha^m$, *the speculative equilibrium disappears. An upper bound for* α^m *is*

$$\max\left\{\frac{\Delta}{\pi_1 k^n}\frac{\delta/\Delta}{k^0 - k^n}\left(1 - \frac{k^0 - k^n}{\delta/\Delta}\right), \bar{\tau}\right\}.$$

Importantly, the parameter α^m is less than one for a wide configuration of parameters.

B. *Capital Flows*

The other major source of funding for the investment boom in the United States was the current account. As a short-term funding mechanism, international capital flows can moderate the rise in interest rates needed to fund the investment boom. Over time, the whole world may be dragged into a speculative path.[13]

We capture the key role played by capital flows in facilitating a speculative growth episode with a stark contrast, one in which no such episode will be feasible without the presence of external funding. For this purpose, let us assume that the closed economy funding function does not depend on the interest rate, $s_r = 0$. In this case, if the home country were in autarky, the transition from the normal steady state to the speculative one would be impossible. This is because funding, and therefore investment, are fully determined by k^n, with no role for interest rates in facilitating an investment boom.

Let us now introduce a simple foreign country. In it, the young receive an aggregate endowment $\tilde{e}_t = \tilde{e}(1 + \gamma)^t$ in each period, which they save in full. The economy also has a technology that uses only capital to produce consumption goods:

$$\tilde{F}(Z_t) = A_t \tilde{f}\left(\frac{Z_t}{A_t}\right) = A_t \tilde{f}(z_t); \quad \tilde{f}' > 0, \tilde{f}'' < 0,$$

where (foreign) capital Z_t is accumulated without adjustment costs. Because this production function exhibits decreasing returns to scale, there are quasi-rents which accrue to an unmodeled factor of production, which does not save (this is not important). Let the marginal product of capital from this technology be linear in capital, $\tilde{\pi}_0 - \tilde{\pi}_1 z$, with $\tilde{\pi}_0$ and $\tilde{\pi}_1$ strictly positive and $z \leq \tilde{\pi}_0/\tilde{\pi}_1$ (henceforth, we focus on equilibria that satisfy this constraint). Since capital markets are integrated, it must be that

$$r_t = r(z_{t+1}) = \tilde{\pi}_0 - \tilde{\pi}_1 z_{t+1}. \tag{17}$$

Equilibrium in *global* capital markets is now

$$s(k_t) + \tilde{e} = (q_t + x_t(q_t))k_t + c(x(q_t), k_t) + (1 + \gamma)z_{t+1}, \tag{18}$$

while the arbitrage and capital accumulation equations for k are still given by (9) and (8).

We can now solve z_{t+1} out from (17):

$$z_{t+1} = \frac{\tilde{\pi}_0 - r_t}{\tilde{\pi}_1},$$

and replace it in (18) to obtain an effective funding function (by which we mean saving available for investment in k):

$$s^e(k_t, r_t) \equiv s(k_t) + \tilde{e} - \frac{\tilde{\pi}_0(1 + \gamma)}{\tilde{\pi}_1} + \frac{1 + \gamma}{\tilde{\pi}_1} r_t.$$

It is now straightforward to see that with $s_0^e = s_0 + \tilde{e} - \tilde{\pi}_0(1 + \gamma)/\tilde{\pi}_1$, $s_k^e = s_k$ and $s_r^e = (1 + \gamma)/\tilde{\pi}_1$, we can fully reproduce the analysis of Section I with $s^e(k_t, r_t)$ replacing $s(k_t, r_t)$ in Section I's formulae. In particular, the domestic economy now can initiate a speculative path which would have been infeasible in autarky.

Early on in this path, the interest rate rises, and foreign saving flows into the home economy while foreign output declines. Over time, as the growth-funding feedback gains in strength, the domestic economy pulls the foreign economy by reversing the capital flows.

Interestingly, as it may have played an important role during the 1990s, the mechanism above also implies that if there is an exogenous decline in foreign opportunities, this may

[13] Ventura (2001) emphasizes an alternative portfolio channel connecting the current account and a domestic bubble. In his model, the main effect of the bubble is to raise domestic wealth. As domestic agents attempt to rebalance their portfolios by investing in non-bubbly domestic equity, external borrowing rises to finance the investment that is required to build domestic equity.

facilitate a transition into a speculative episode in the home economy. Furthermore, if the transition comes together with the decline in opportunities abroad, interest rates need not rise in the short run either.

Let us suppose that with $s^e(k_t, r_t)$ replacing $s(k_t, r_t)$, Assumptions 0, 0′, and 1 hold but Assumption 2 does not hold, so that no transition from the normal steady state to the speculative steady state is possible. Consider a situation where the economy is initially in the low steady state. Let us now imagine that $\tilde{e}$ and $\tilde{\pi}_0$ unexpectedly shift to $\tilde{e}' \geq \tilde{e}$ and $\tilde{\pi}_0' \leq \tilde{\pi}_0$, so that s_0^e shifts to $s_0^{e\prime} = s_0^e + (\tilde{e}' - \tilde{e}) - (\tilde{\pi}_0' - \tilde{\pi}_0)[(1 + \gamma)/\tilde{\pi}_1] \geq s_0^e$.

ASSUMPTION 4: $\Delta(k^0 - k^n) > s_0^{e\prime} - s_0^e > \Delta(k^0 - k^n) - \delta/(\Lambda(\theta) + 1) > 0$.

This assumption implies that there are two steady states under the new parameters. Denote them by $k_n' = k_n + (s_0^{e\prime} - s_0^e)/\Delta$ and $k_s' = k_s + (s_0^{e\prime} - s_0^e)/\Delta$.

PROPOSITION 6 (Change in conditions in the rest of the world): *If Assumption* 4 *holds, there is no transition from k^n to k^s under the parameters $\tilde{e}$ and $\tilde{\pi}_0$. By contrast, there are multiple equilibria under the parameters $\tilde{e}'$ and $\tilde{\pi}_0'$: starting with capital stock k^n, there is a saddle path that takes the economy from k^n to $k^{n'}$ and a speculative growth path that takes the economy from k^n to $k^{s'}$. Along these paths, $\widehat{q_t} > 0$.*

The result is intuitive. If saving in the foreign country increases or if the rate of return in the foreign country decreases, causing a reallocation of saving to the domestic economy, a speculative growth scenario can emerge where it would have been impossible before the shift.

III. Growth-Funding Feedback Mechanisms

In this section we go beyond amplification and discuss mechanisms that can be directly responsible for the long-run growth-funding mechanism (that is, the equivalents of the jump in the funding function): endogenous technological progress in the equipment sector and relaxation of financial constraints.

A. *Technological Progress in the Equipment Sector*

The productivity growth that came with the U.S. speculative growth episode was exceptional. After two disappointing decades beginning with the oil price shocks of the 1970s, the economy recovered more than half of its lost productivity growth. Most prominently, productivity growth in industrial and electronic machinery accelerated from an annual rate of 2 percent for 1973–1990 to more than 6 percent for 1995–2000. This acceleration reduced the price of machinery and electronic devices (see Figure 7), which in turn contributed to the investment boom. This mechanism fits within the speculative growth perspective we have highlighted up to now, once the concept of funding is understood more broadly than saving. That is, for any given level of saving, a decline in the price of new capital raises the power of that saving to build the economy's stock of capital.

While some of the technological progress in equipment-producing sectors may have been exogenous to the capital-deepening process itself, we continue here with our endogenous perspective (which in any event would act as a multiplier to the exogenous shocks). We capture this endogeneity through a positive spillover from the capital accumulation process to the production of equipment goods.[14] This simple channel allows us to preserve much of our previous formulae. Moreover, here we set $\delta = 0$, as the technological spillover ends up playing a similar role in the current model.

We assume that the technology in the equipment goods sectors can transform one unit of consumption goods into $\lambda(k)$ units of equipment goods (with $\lambda' > 0$). Perfect competition in this sector ensures that the price of equipment goods is

$$p(k_t) = \frac{1}{\lambda(k_t)}.$$

Since q_t corresponds to Tobin's q—that is, as the value of a unit of installed capital

[14] See Kevin Murphy et al. (1989) for a model of this sort based on increasing returns in the equipment sector, and Nair Jaimovich (2004) for a countercyclical markups version.

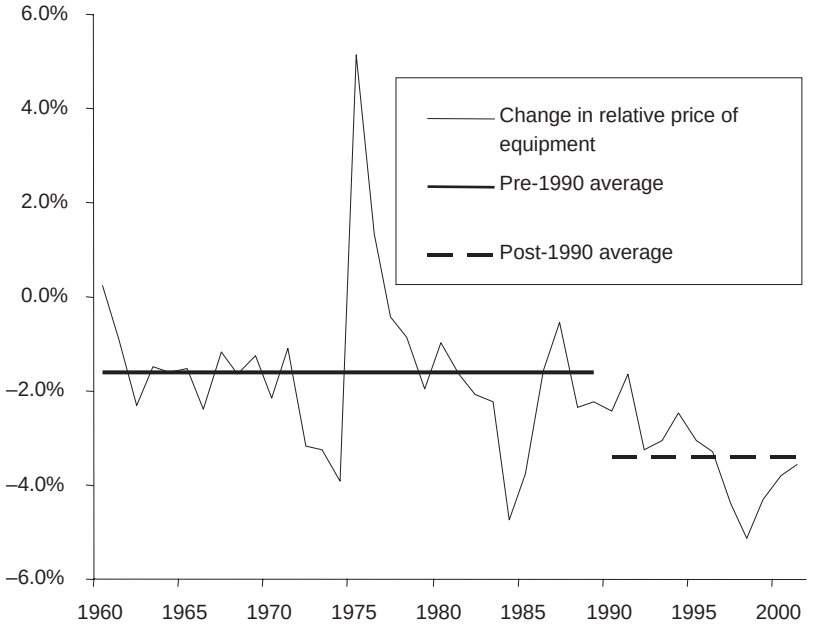


FIGURE 7. CHANGE IN THE RELATIVE PRICE OF EQUIPMENT

over the price on a new (uninstalled) unit of capital—we preserve the formulae of Section I, with two exceptions: the capital market equilibrium condition and the arbitrage equation.

The former is now:

$$s(k_t, r_t) = p(k_t)(q_t + x_t(q_t))k_t + p(k_t)c(x(q_t), k_t). \tag{19}$$

But it is apparent that by dividing both sides of this condition by $p(k_t)$ and defining the funding function as

$$s^p(k_t, r_t) \equiv \frac{s(k_t, r_t)}{p(k_t)},$$

we can rewrite (19) as

$$s^p(k_t, r_t) = (q_t + x_t(q_t))k_t + c(x(q_t), k_t), \tag{20}$$

which is entirely analogous to the equilibrium capital market condition in Section I.

Moreover, suppose that we model the aggregate increasing returns aspect of the equipment sector as a simple step function:

$$p(k_t) = \begin{cases} 1, & k_t < k^o \\ p < 1, & k \geq k^o \end{cases}. \tag{21}$$

Then $(1 - p) > 0$ is isomorphic to δ in terms of the effective funding function.

The only difference with the model in Section I is that $(1 - p)$ affects not only effective funding, but also the arbitrage equation, since now

$$q_{t+1} = (1 + r_t)(q_t + c_K(I_t, K_t))\frac{p(k_t)}{p(k_{t+1})} - \frac{(\pi_0 - \pi_1 k_{t+1})}{p(k_{t+1})}. \tag{22}$$

However, nothing of our fundamental message is changed by this modification. In particular, the preconditions and features of the speculative path remain valid.[15, 16]

[15] In Saint-Paul (1992), private saving is too low because his technology exhibits external returns to capital. Public debt reduces growth and hence cannot be Pareto-improving. In our model, there is also an externality on capital accumulation, pointing to a potential cost of public debt (see our discussion of surplus illusion in the previous section).

[16] We prove these claims in the on-line Appendix.

B. *Financial Constraints and Growth-Funding Feedback*

Another key ingredient in the U.S. episode is the reallocation of funding toward small growth firms in new technology sectors. We develop this model here with two purposes. First, it shows that a growth-funding mechanism need not operate through aggregate saving; the reallocation of funds can play a similar role. Second, and more importantly, it shows another explicit model of δ; here the growth-funding feedback stems from the relaxation of financial constraints brought about by the expansion.[17]

Let us simplify interest rate determination by assuming that there is a constant returns to capital, h, sector with (low) productivity, r, which is always active so that the "riskless" interest rate is pinned down at r.

More substantively, assume that only a fraction $\mu < 1$ of the young can invest and work in the conventional sector described in Section I, while the rest of the young (lenders) work in a sector that has no capital. Thus, aggregate output is

$$Y_t = A_t(\mu L f(k_t) + (1 - \mu)L + rh_t).$$

Young lenders have undiscounted log-utility preferences, so they save half their wages, which in aggregate amounts to $(1 - \mu)/2$. We assume that $(1 - \mu)$ is large enough so their saving is never binding in equilibrium. Entrepreneurs, on the other hand, satisfy a minimum (detrended) consumption $\bar{c}$, after which they save and invest as much as they can:

$$\mu(w(k_t) - \bar{c}).$$

In equilibrium, entrepreneurs borrow from lenders, d_t:

$$d_t = (x_t + q_t + \tfrac{1}{2}\theta^{-1}(x_t - \gamma)^2)k_t - \mu(w(k_t) - \bar{c}). \tag{23}$$

Against this loan, entrepreneurs post as collateral the capital they acquire:

$$q_{t+1}k_t/(1 + r).$$

Neither new investment nor output can be pledged (this is our financial constraint). Collateralized loans are made at the "riskless" interest rate r. Uncollateralized loans consume (per unit) δ/s_r resources in monitoring, and therefore are made at interest rate $r + \delta/s_r$.

For a given interest rate, neither the accumulation equation nor the arbitrage condition described in Section I changes in this model. The main change is in the criterion separating the two regions of the phase diagram. Rather than k^o, now there is a condition that splits the region according to whether the marginal loan is collateralized (and hence at rate r) or not (and hence at rate $r + \delta/s_r$). For this purpose, let us define

$$n_t \equiv d_t - \frac{q_{t+1}k_t}{1 + r}.$$

By replacing (23), the arbitrage condition, and the accumulation equation into this expression, we obtain n_t as a function n of q_t and k_t. When $n(q_t, k_t) \leq 0$, the interest rate is r, while when $n_t > 0$, the interest rate is $r + \delta/s_r$. This places us in the setting of Section I with only a slightly more complicated function separating the two regions of the phase diagram. Note that if $n_t = 0$, then $r_t = r$. Using this and simplifying, the separating region is defined by the following equation:

$$0 = \left(\gamma + (x(q_t) - \gamma)(1 - \theta^{-1}) + (q_t - 1) + \frac{1}{2}\theta^{-1}(x(q_t) - \gamma)^2\right)k_t - \mu(w(k_t) - \bar{c}) + \frac{f'\left(\frac{1 + \gamma + \theta(q_t - 1)}{1 + \gamma}k_t\right)}{1 + r}k_t.$$

[17] Note that the financial constraint being relaxed in this section is *within* generations. In contrast, OLG models also have intergenerational financial constraints, which have been present throughout our analysis, but do not have a central role in our results. In particular, without the within-generation financial constraint, the economy we describe here has a unique "bubbleless" equilibrium.

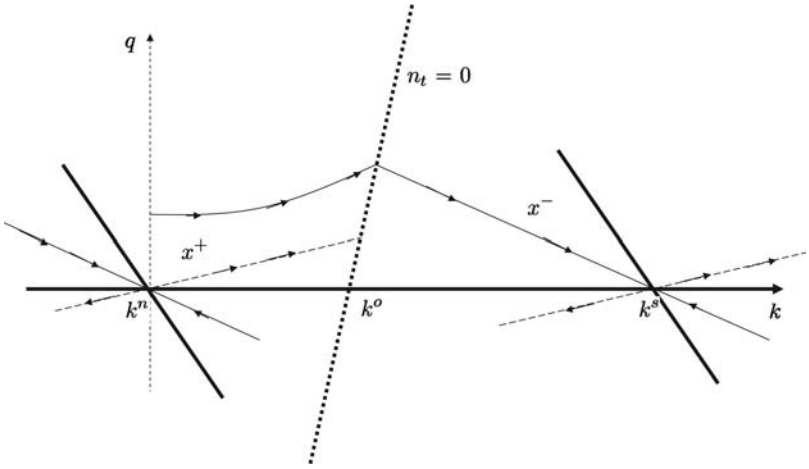


FIGURE 8. FINANCIAL CONSTRAINTS

Figure 8 illustrates the analogue of panel A in Figure 4. As before, there are two steady states and the possibility of a speculative growth path. The key growth-funding feedback now arises from the fact that the financial constraint is no longer binding when k is sufficiently large, which significantly drops the interest rate on loans. The proofs to the main claims in this section are in the Appendix.

C. *Bubbles in a Speculative Growth Environment and the Santos-Woodford Condition*

Now that we have developed specific mechanisms behind the growth-funding feedback, we can connect our results on bubbles with the general conditions in Santos and Woodford (1997). They consider an endowment economy with dynamic asset markets, and show that a necessary condition for bubbles on positive net-supply assets to occur is that the net present value of the endowment is infinite. Fixing the production decisions in our model, we can analyze a particular path of the economy in their general setup, the endowment being the part of the production which is consumed every period.

Let us first make assumptions such that bubbles are possible in our environment. In the technological progress setup, Assumption 3 has to be replaced by the following assumption.

ASSUMPTION 3A (Bubble region in the technological progress setup): $\pi_0 - \pi_1 k^n > \gamma > (\pi_0 - \pi_1 k^0)/p$.

In the financial constraints setup, Assumption 4 has to be replaced by the following assumption.

ASSUMPTION 3B (Bubble region in the financial constraints setup): $r + \delta/s_r > \gamma > r$.

In both the technological progress setup and the financial constraints setup, it is trivially verified that the net present value of the endowment (the part of the production which is consumed) is infinite along a speculative growth path with a bubble. This follows directly from the fact that close to the speculative steady state, the interest rate is lower than the growth rate of the economy. We formulate these results in two propositions.

PROPOSITION 7 (Possibility of bubbles in the technological progress setup): *Let Assumptions* 0, 0′, 3A, 5, *and* 6 *hold in the technological progress setup. Consider a path with a positive initial bubble, initial capital* k^n, *and final capital* k^s. *Then along this path, the net present value of the endowment is infinite:*

$$\sum_{t=0}^{\infty} \frac{F(K_t, L_t) - p_t I_t - p_t C_t}{\prod_{s=0}^{t} (1 + r_s)} = +\infty.$$

PROPOSITION 8 (Possibility of bubbles in the financial constraint setup):[18] *Let Assumptions* 3B, A1, A2, A3, *and* A4 *hold in the financial constraint setup. Consider a path with a positive initial bubble, initial capital* k^n, *and final capital* k^s. *Then along this path, the net present value of the endowment is infinite:*

$$\sum_{t=0}^{\infty} \frac{F(K_t, \mu L_t) - I_t - C_t + (1-\mu) L_t}{(1+r)^t} = +\infty.$$

IV. Final Remarks

This paper builds a theoretical framework for thinking about episodes of speculative growth. We characterize this phenomenon as a low effective cost-of-capital equilibrium based on optimism about the *future* availability of funds for investment. Our framework highlights the key short-term and long-term funding mechanisms necessary to sustain a speculative growth equilibrium.

Ours is not a framework of irrational exuberance, although it offers a natural interpretation of such episodes. These occur when a speculative growth path is not backed by a long-run funding mechanism strong enough to eventually validate the capital gains dynamics that are needed in the short run to reallocate resources toward the sectors that drive the boom. A prototypical example is when investment and speculation focus on the "wrong" sector, as may have been the case in Japan during the 1980s with its large real estate boom.

The U.S. experience during the 1990s probably had elements of both: a rational speculative growth component built on the information technology sector, sound fiscal policy, and foreigners' confidence in U.S. economic conditions, as well as some elements of irrationality, especially during the late 1990s. From this perspective, it is not at all clear whether much of the U.S. asset markets boom during the 1990s, even if bubbly, was condemned to be reversed or undesirable as a package. To paraphrase John Maynard Keynes's (1931) account of the investment boom that preceeded the Great Depression:[19]

> "While some part of the investment which was going on ... was doubtless ill judged and unfruitful, there can, I think, be no doubt that the world was enormously enriched by the constructions of the quienquennium from 1925 to 1929; its wealth expanded in those five years by as much as in any other ten or twenty years in its history ... . A few more quinquennia of equal activity might, indeed, have brought us near to the economic Eldorado where all our reasonable economic needs would be satisfied."

In this regard, an interesting question that we leave for future research is whether a central bank should attempt to raise interest rates upon the emergence of bubbles. It may well be the case that doing so crashes not only the bubble but also the underlying, and mostly welfare enhancing, speculative growth episode.[20]

[18] The proofs to the propositions in this subsection are in the on-line Appendix.

[19] We owe this quote to Andrei Shleifier. It is the concluding quote used by Bradford De Long et al. (1989) as a way of qualifying their conclusion on the negative economic impact of "noise traders."

[20] Do speculative growth episodes represent a Pareto improvement across generations involved in it? The answer to this question is not generic, as it depends on the specific channel behind the growth-funding feedback. But what is generic in our models is the welfare improvement of all generations born before t^o. It is also generic that if the economy is dynamically efficient, then all generations with capital close enough to the high capital equilibrium are also better off.

APPENDIX

Proof of Propositions in Section I

PROOF OF PROPOSITION 1:

The conditions for a steady state are $q = 1$, $r = \pi_0 - \pi_1 k$, $s_0 + s_k k + s_r r + 1\{k \geq k^o\}\delta = (1 + \gamma)k$; $k^n < k^0$ is a steady state if and only if $k^n = (s_0 + s_r \pi_0/\Delta) < k^0$. This inequality holds by Assumption 0. Similarly, $k^s \geq k^0$ is a steady state if and only if $k^s = (s_0 + s_r \pi_0 + (\delta/s_r)/\Delta) > k^0$. This last inequality holds by Assumption 1.

PROOF OF PROPOSITION 2:

Let $\widehat{q_t} \equiv (q_t - 1)$ and $\widehat{k_t} \equiv (k_t - k^n)$. Then the linearized dynamic system around the normal steady state can be written as

$$\widehat{q_{t+1}} = E\widehat{q_t} + \left[\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right]\widehat{k_t}, \qquad \widehat{k_{t+1}} = \frac{\theta k^n}{(1 + \gamma)^2}\,\widehat{q_t} + \widehat{k_t},$$

where

$$E^n = \left[\frac{1 + r^n}{1 + \gamma} + \left(\frac{\pi_1 \dfrac{\theta}{1 + \gamma}}{1 + \gamma} + \frac{\dfrac{\theta}{1 + \gamma} + 1}{s_r}\right)k^n\right].$$

This system is characterized by the following 2×2 matrix:

$$\Omega = \begin{bmatrix} E^n & \pi_1 + \dfrac{1 + \gamma - s_k}{s_r} \\ \dfrac{\theta k^n}{(1 + \gamma)^2} & 1 \end{bmatrix}.$$

The two eigenvalues λ^+ and λ^- of Ω are real since the discriminant D of $\det(\Omega - xI)$ is positive:

$$D = [E^n - 1]^2 + 4\left(\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right)\frac{\theta k^n}{(1 + \gamma)^2} > 0.$$

λ^+ and λ^- are given by

$$\lambda^+ = 1 + \frac{E^n - 1 + \sqrt{[E^n - 1]^2 + 4\left(\pi_1 + \dfrac{1 + \gamma - s_k}{s_r}\right)\dfrac{\theta k^n}{(1 + \gamma)^2}}}{2}$$

and

$$\lambda^- = 1 + \frac{E^n - 1 - \sqrt{[E^n - 1]^2 + 4\left(\pi_1 + \dfrac{1 + \gamma - s_k}{s_r}\right)\dfrac{\theta k^n}{(1 + \gamma)^2}}}{2}.$$

It is straightforward to check that

$$\lambda^- < 1 < \lambda^+.$$

Let us denote by $(x^+, 1)'$ and $(x^-, 1)'$ the corresponding eigenvectors:

$$x^+ = \frac{\lambda^+ - 1}{\frac{\theta k^n}{(1+\gamma)^2}} > 0, \qquad x^- = \frac{\lambda^- - 1}{\frac{\theta k^n}{(1+\gamma)^2}} < 0.$$

This shows that the normal steady state $(k^n, 1)$ is a saddle point.

Similarly, let $\widehat{q_t^s} \equiv (q_t - 1)$ and $\widehat{k_t^s} \equiv (k_t - k^s)$. Then the linearized dynamic system around the speculative steady state can be written as

$$\widehat{q_{t+1}^s} = E^s \widehat{q_t^s} + \left[\pi_1 + \frac{1+\gamma-s_k}{s_r}\right]\widehat{k_t^s}, \qquad \widehat{k_{t+1}^s} = \frac{\theta k^s}{(1+\gamma)^2}\,\widehat{q_t^s} + \widehat{k_t^s},$$

where

$$E^s = \left[\frac{1+r^s}{1+\gamma} + \left(\pi_1 \frac{\frac{\theta}{1+\gamma}}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r}\right)k^s\right].$$

This system is characterized by the following 2×2 matrix:

$$\Omega = \begin{bmatrix} E^s & \pi_1 + \frac{1+\gamma-s_k}{s_r} \\ \frac{\theta k^s}{(1+\gamma)^2} & 1 \end{bmatrix}.$$

The two eigenvalues λ^{s+} and λ^{s-} of Ω^s are real since the discriminant D^s of $\det(\Omega^s - xI)$ is positive:

$$D = [E^s - 1]^2 + 4\left(\pi_1 + \frac{1+\gamma-s_k}{s_r}\right)\frac{\theta k^s}{(1+\gamma)^2} > 0.$$

λ^{s+} and λ^{s-} are given by

$$\lambda^{s+} = 1 + \frac{E^s - 1 + \sqrt{[E^s-1]^2 + 4\left(\pi_1 + \frac{1+\gamma-s_k}{s_r}\right)\frac{\theta k^s}{(1+\gamma)^2}}}{2}$$

and

$$\lambda^{s-} = 1 + \frac{E^s - 1 - \sqrt{[E^s-1]^2 + 4\left(\pi_1 + \frac{1+\gamma-s_k}{s_r}\right)\frac{\theta k^s}{(1+\gamma)^2}}}{2}.$$

It is straightforward to check that

$$\lambda^{s-} < 1 < \lambda^{s+}.$$

Let us denote by $(x^{s+}, 1)'$ and $(x^{s-}, 1)'$ the corresponding eigenvectors:

$$x^{s+} = \frac{\lambda^{s+} - 1}{\frac{\theta k^s}{1+\gamma}} > 0, \qquad x^{s-} = \frac{\lambda^{s-} - 1}{\frac{\theta k^s}{1+\gamma}} < 0.$$

This shows that the speculative steady state $(k^s, 1)$ is a saddle point.

Let us now fix k^n, s_0, s_r, π_0, π_1, γ, and θ, and let us vary k^0 and δ such that Assumptions 0 and 1 are verified and $|k^0 - k^n| > \varepsilon\delta$ for some $\varepsilon > 0$.

As we do this, k^s, x^{s+}, x^{s-}, λ^{s+}, and λ^{s-} also vary. These variables are all continuous functions of δ, and do not depend on k^0 as long as Assumptions 0 and 1 are verified. Let us denote them by $k^s(\delta)$, $x^{s+}(\delta)$, $x^{s-}(\delta)$, $\lambda^{s+}(\delta)$, and $\lambda^{s-}(\delta)$. It is clear that $k^s(\delta) = k^n + O(\delta)$, $x^{s+}(\delta) = x^+ + O(\delta)$, $x^{s-}(\delta) = x^- + O(\delta)$, $\lambda^{s+}(\delta) = \lambda^+ + O(\delta)$, and $\lambda^{s-}(\delta) = \lambda^- + O(\delta)$.

We will prove the following result. If $(\lambda^+ - 1/1 - \lambda^-) < (k^s - k^0/k^0 - k^n)$, then a transition is possible for δ small enough, and that if $(\lambda^+ - 1/1 - \lambda^-) > (k^s - k^0/k^0 - k^n)$, then for δ small enough, no transition is possible.

Because the dynamic system we consider is twice continuously differentiable in the regions $k < k^0$ and $k > k^0$, the following results hold:

- There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$, there exist continuously differentiable functions $q_\delta^{s-} : [k^0, k^s] \to [1, +\infty[$ and $q^{n+} : [k^n, k^0] \to [1, +\infty[$ such that the stable manifold of $(k^s(\delta), 1)$ is parameterized by $(k, q_\delta^{s-}(k))$ in the region $k \in [k^0, k^s(\delta)]$, and the unstable manifold of $(k^n, 1)$ is parameterized by $(k, q^{n+}(k))$ in the region $k \in [k^n, k^0]$.
- There exists $\bar{\bar{\delta}} > \bar{\bar{\bar{\delta}}} > 0$ and $m > 0$ such that for all $\delta < \bar{\bar{\bar{\delta}}}$ and $k \in [k^0, k^s(\delta)]$, and $k' \in [k^n, k^0]$,

$$|q_\delta^{s-}(k) - 1 - x^{s-}(\delta)(k - k^s(\delta))| < m\delta^2$$

and

$$|q^{n+}(k') - 1 - x^+(k - k^n)| < m\delta^2.$$

Using the fact that $k^s(\delta) = k^n + O(\delta)$, $x^{s+}(\delta) = x^+ + O(\delta)$, $x^{s-}(\delta) = x^- + O(\delta)$, $\lambda^{s+}(\delta) = \lambda^+ + O(\delta)$, and $\lambda^{s-}(\delta) = \lambda^- + O(\delta)$, we see that there exists $\bar{\bar{\delta}} > \bar{\bar{\bar{\delta}}} > 0$ and $M > 0$ such that for all $\delta < \bar{\bar{\bar{\delta}}}$ and $k \in [k^0, k^s(\delta)]$, and $k' \in [k^n, k^0]$,

$$|q_\delta^{s-}(k) - 1 - x^-(k - k^s(\delta))| < M\delta^2$$

and

$$|q^{n+}(k') - 1 - x^+(k - k^n)| < M\delta^2.$$

The condition for a transition from the normal steady state to the speculative steady state to be possible is that the intersection of the unstable arm of the normal steady state with the vertical $k = k_0$ line lies below the intersection of the stable arm (the saddle path) of the speculative steady state with the vertical $k = k_0$ line.

The $q = 1 + x^+(k - k^n)$ line intersects the $k = k^0$ line at the point $(k^0, 1 + x^+(k^0 - k^n))$. Similarly the $q = 1 + x^-(k - k^s(\delta))$ line intersects the $k = k^0$ line at the point $(k^0, 1 + x^-(k^0 - k^s(\delta)))$. Therefore, if $\delta < \bar{\bar{\delta}}$, a necessary condition for the intersection of the unstable arm of the normal steady state with the vertical $k = k_0$ line to lie below the intersection of stable arm (the saddle path) of the speculative steady state with the vertical $k = k_0$ line is

$$1 + x^-(k^0 - k^s(\delta)) + M\delta^2 > 1 + x^+(k - k^n) - M\delta^2. \tag{A1}$$

Similarly, a sufficient condition is

$$1 + x^-(k^0 - k^s(\delta)) - M\delta^2 > 1 + x^+(k - k^n) + M\delta^2. \tag{A2}$$

We can rewrite (A1) and (A2) as

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2M\delta^2}{k^0 - k^n}$$

and

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2M\delta^2}{k^0 - k^n}.$$

Using the fact that we have kept $k^0 - k^n > \varepsilon\delta$, we see that a necessary condition for a transition when $\delta < \bar{\bar{\delta}}$ is

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2M\delta}{\varepsilon}$$

and that a sufficient condition for a transition is

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2M\delta}{\varepsilon}.$$

Notice that $(2M\delta/\varepsilon) \to 0$ when $\delta \to 0$. This shows that if $(\lambda^+ - 1/1 - \lambda^-) < (k^s - k^0/k^0 - k^n)$, then a transition is possible for δ small enough, and that if $(\lambda^+ - 1/1 - \lambda^-) > (k^s - k^0/k^0 - k^n)$, then for δ small enough, no transition is possible. Let us now reexamine this condition.

We have that

$$\frac{\lambda^+ - 1}{1 - \lambda^-} = -1 + 2\frac{1}{1 - \frac{1}{\sqrt{1 + \Upsilon}}} = \Lambda(\theta).$$

Λ is a decreasing function of

$$\Upsilon \equiv \frac{4\left(\pi_1 + \dfrac{1 + \gamma - s_k}{s_r}\right)\dfrac{\theta k^n}{(1+\gamma)^2}}{[E^s - 1]^2}.$$

Υ is nonmonotonic in θ. It is increasing in the interval $(0, \tilde{\theta})$ and decreasing in the interval $(\tilde{\theta}, +\infty)$, where

$$\tilde{\theta} = \frac{\dfrac{k^n}{s_r} + \dfrac{1 + r^s}{1 + \gamma} - 1}{\left(\dfrac{\pi_1}{1+\gamma} + \dfrac{1}{s_r}\right)1\,\dfrac{k^n}{1+\gamma}}.$$

In addition, $\lim_{\theta \to 0}\Upsilon = 0$, and $\lim_{\theta \to +\infty}\Upsilon = 0$. As a result, Λ is nonmonotonic in θ. It is decreasing on $(0, \tilde{\theta})$ and increasing on $(\tilde{\theta}, +\infty)$, and reaches a minimum for $\theta = \tilde{\theta}$. In addition, we have $\lim_{\theta \to 0}\Lambda = +\infty$ and $\lim_{\theta \to +\infty}\Lambda = +\infty$.

This shows that if $(\lambda^+ - 1/1 - \lambda^-) < (k^s - k^0/k^0 - k^n)$, then a transition is possible for δ small enough, and that if $(\lambda^+ - 1/1 - \lambda^-) > (k^s - k^0/k^0 - k^n)$, then for δ small enough, no transition is possible. In the rest of the Appendix, we will often have to go through a similar argument. We will not go through the same details and instead state directly the equivalent of the condition $(\lambda^+ - 1/1 - \lambda^-) > (k^s - k^0/k^0 - k^n)$. The only exception concerns the discussion of bubbles where more complex issues arise.

Financial Constraints

The curve $n_t = 0$ intersects the $q = 1$ line at a point $(1, k)$ if and only if

$$\gamma = \mu\,\frac{w(k) - \bar{c}}{k} - \frac{f'(k)}{1 + r}.$$

We assume that this equation has at least one solution. If we assume, for example, that $f(k_t) = Ak_t^{\alpha}$, with $0 < \alpha < 1$ and $\mu(1 + r) > (\alpha/(1 - \alpha))$ and $\bar{c} < \max_{k>0}\{\mu w(k) - (f'(k)k/(1 + r)) - \gamma k\}$, this assumption will be verified and the equation $n_t = 0$ will have exactly two solutions.

Let $k^0 < k^1$ be the two first intersections of this curve with the $q = 1$ line ($k_1 = +\infty$ if there is no second intersection).

We assume that there are two steady states $k^n < k^s$ where $k^n < k^0$, and $k^0 < k^s < k^1$.

Because in this application it is the interest rate in the normal steady state $r + \delta$ that depends on δ, we will approximate the dynamics of the system around the speculative steady state (where the interest rate is r). Letting δ tend to 0, we are moving the normal steady state while keeping the speculative steady state fixed.

ASSUMPTION A1: $\Delta \equiv s_r\pi_1 + (1 + \gamma - s_k) > 0$, $k^s \equiv (\pi_0 - r/\pi_1) > k^0$.

ASSUMPTION A2 (Minimum growth-funding feedback): $\delta > \underline{\delta} \equiv s_r\pi_1(k^s - k^0)$.

ASSUMPTION A3: $(1 + \mu)\pi_1 k^s - \gamma - (r/1 + r) > 0$ *and* $\theta + \gamma - (\pi_1\theta k^s/1 + \gamma) > 0$.

The last assumption is made to ensure that locally the $n = 0$ curve is an upward-sloping line with the low interest rate region to its right.

Let

$$\Xi_t = \left(\gamma + (x(q_t) - \gamma)(1 - \theta^{-1}) + (q_t - 1) + \frac{1}{2}\theta^{-1}(x(q_t) - \gamma)^2\right)k_t - \mu(w(k_t) - \bar{c}) + \frac{f'\left(\frac{1 + \gamma + x(q_t)}{1 + \gamma}k_t\right)}{1 + r}k_t.$$

Assuming that δ and n^s are small, let us perform a first-order approximation around $q = q^* = 1$ and $k = k^s$.

Up to second-order terms in $\widehat{q_t} = (q_t - 1)$, $\widehat{k_t} = (k_t - k^s)$, $\widehat{k_0} = (k^0 - k^s)$, $\widehat{r_t} = (r_t - r^s)$, $\widehat{n_t} = n_t - n^s$, δ, and n^s, we have:

$$\widehat{\Xi}_t = \gamma\widehat{k_t} + \left(\frac{\theta}{1 + \gamma} + \frac{\gamma}{1 + \gamma}\right)k^s\widehat{q_t} - (1 + \mu)\pi_1 k^s\widehat{k_t} - \frac{\pi_1\theta k^{s^2}}{(1 + \gamma)^2}\widehat{q_t} + \frac{r}{1 + r}\widehat{k_t}$$

$$= \left[\frac{\theta + \gamma}{1 + \gamma}k^s - \frac{\pi_1\theta k^{s^2}}{(1 + \gamma)^2}\right]\widehat{q_t} - \left[(1 + \mu)\pi_1 k - \gamma - \frac{r}{1 + r}\right]\widehat{k_t}.$$

We have $\widehat{r_t} = 0$ if $\widehat{\Xi}_t \leq -\Xi^s$ and $\widehat{r_t} = -(\delta/s_r)$ if $\widehat{\Xi}_t > -\Xi^s$. Note that, by assumption, $\Xi^s = \gamma k^s - \mu(w^s - \bar{c}) + (r/1 + r)k^s < 0$ and $\widehat{k^n} = -(\delta/s_r\pi_1) < \widehat{k_0} < 0$.

(A3) $$\widehat{q_{t+1}} = \left[\frac{1 + r^s}{1 + \gamma} + \frac{\pi_1\theta k^s}{(1 + \gamma)^2}\right]\widehat{q_t} + \pi_1\widehat{k_t} + 1\{\widehat{\Xi}_t \leq -\Xi^s\}\frac{\delta}{s_r}.$$

(A4) $$\widehat{k_{t+1}} = \frac{\theta k^s}{(1 + \gamma)^2}\widehat{q_t} + \widehat{k_t}.$$

The analysis and proofs that follow refer to the dynamic system described by (A3) and (A4).

For a speculative growth path to exist, two conditions must be satisfied. First, it must be the case that the unstable arm of the normal steady state is less steep than the $\widehat{\Xi}_t = -\Xi^s$ line. Second, it must be the case that the unstable arm of the normal steady state intersects the $\widehat{\Xi}_t = -\Xi^s$ line *below* the intersection of the latter and the saddle path of the speculative equilibrium. We can express these conditions as

$$\Psi^f \equiv \frac{(1 + \mu)\pi_1 k^s - \gamma - \frac{r}{1 + r}}{(1 + \gamma)\left(1 + \frac{\gamma}{\theta}\right) - \pi_1 k^s} - (\lambda^{f+} - 1) \geq 0, \qquad \Lambda^f \equiv \frac{\Omega_1}{\Omega_2} < \frac{k^s - k^0}{k^0 - k^n},$$

where

$$\Omega_1 = \frac{\lambda^{f+} - 1}{\frac{(1 + \mu)\pi_1 k^s - \gamma - \frac{r}{1 + r}}{(1 + \gamma)\left(1 + \frac{\gamma}{\theta}\right) - \pi_1 k^s} - (\lambda^{f+} - 1)},$$

$$\Omega_2 = \frac{1 - \lambda^{f-}}{\dfrac{(1+\mu)\pi_1 k^s - \gamma - \dfrac{r}{1+r}}{(1+\gamma)\left(1+\dfrac{\gamma}{\theta}\right) - \pi_1 k^s} + (1 - \lambda^{f-})},$$

$$\lambda^{f+} = 1 + \frac{\hat{E} - 1 + \sqrt{[\hat{E} - 1]^2 + 4\,\dfrac{\pi_1 \theta k^s}{1+\gamma}}}{2},$$

$$\lambda^{f-} = 1 + \frac{\hat{E} - 1 - \sqrt{[\hat{E} - 1]^2 + 4\,\dfrac{\pi_1 \theta k^s}{1+\gamma}}}{2},$$

and

$$\hat{E} = \frac{1 + r^s}{1+\gamma} + \frac{\pi_1 \theta k^s}{(1+\gamma)^2}.$$

We can rewrite Ψ^f as:

$$\Psi^f = 1 + \frac{(1+\mu)\pi_1 k^s - \gamma - \dfrac{r}{1+r}}{(1+\gamma)\left(1+\dfrac{\gamma}{\theta}\right) - \pi_1 k^s} \times \frac{\lambda^-}{\lambda^+ - 1 - \dfrac{(1+\mu)\pi_1 k^s - \gamma - \dfrac{r}{1+r}}{(1+\gamma)\left(1+\dfrac{\gamma}{\theta}\right) - \pi_1 k^s}}.$$

We have

$$\lim_{\theta \to 0} \Psi^f < 0,$$

so that no transition is possible for too large adjustment costs because the slope of the unstable arm of the normal steady state goes to $+\infty$ while the slope of the $\widehat{\Xi}_t = -\Xi^s$ tends to a finite positive limit.

Similarly

$$\lim_{\theta \to +\infty} \Psi^f = -\infty,$$

so that no transition is possible because the unstable arm of the normal steady state is too steep (the slope of which tends to a finite strictly positive limit) and overshoots the $\widehat{\Xi}_t = -\Xi^s$ line (the slope of which tends to 0).

Therefore, a transition can occur only for intermediate adjustment costs.

Let us define Assumption A4 as requiring that the two conditions for a transition be satisfied.

ASSUMPTION A4: $\Psi^f(\theta) > 0$ *and* $\Lambda^f(\theta) < (k^s - k^0/k^0 - k^n)$.

We have proved the following propositions.

PROPOSITION A1 (Multiple steady states): *If Assumptions* A1, A2, *and* A3 *are satisfied, the economy has two nondegenerate steady states,* k^n *and* k^s, *with:*

$$k^n = \frac{\pi_0 - \left(r + \frac{\delta}{s_r}\right)}{\pi_1} < k^o < k^s = \frac{\pi_0 - \left(r + \frac{\delta}{s_r}\right)}{\pi_1}. \tag{A5}$$

PROPOSITION A2 (Multiple equilibria and speculative growth): *If Assumptions* A1, A2, A3, *and* A4 *hold, there is a speculative growth path that takes the economy from* k^n *to* k^s. *Along that path,* $\hat{q}_t > 0$.

Proof of Claims in Section IB

In principle, there could be a third steady state once bubbles are feasible: $(k^b, 1, b)$, where $\pi_0 - \pi_1 k^b = \gamma$ and $b > 0$. Assumption 4 guarantees that this steady state does not exist. To see why, assume the contrary. It is clear from Assumption 3 that $k^b < k^0$. But the associated bubble in this steady state would be negative, since $b = s_0 + s_r\pi_0 - \Delta k^b < s_0 + s_r\pi_0 - \Delta k^n = 0$, which is a contradiction.

In the proof of this proposition, we have to perform two linearizations, one around the normal steady state and one around the speculative steady state. This is because the characteristics of the dynamic system are different around the two steady states. The normal steady state has a two-dimensional unstable manifold and a one-dimensional stable manifold, whereas the speculative steady state has a one-dimensional unstable manifold and a two-dimensional stable manifold.

We consider the possibility of small initial bubbles, $0 < b_0 < M_b(\delta)\delta$ for some function $M_b(\delta) > 0$, such that

$$\lim_{\delta \to 0} M_b(\delta) = 0$$

and

$$\lim_{\delta \to 0} \frac{M_b(\delta)}{\delta} = +\infty$$

and show that if

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n},$$

then there exists a function $m_b(\delta) > 0$ such that a transition is possible for δ small enough and $0 < b_0 < m_b(\delta)\delta$, and that if

$$\frac{\lambda^+ - 1}{1 - \lambda^-} > \frac{k^s - k^0}{k^0 - k^n}$$

then for δ small enough, for every $\widetilde{m_b} > 0$, there exists $0 < b_0 < \widetilde{m_b}\delta$ such that no transition is possible with initial bubble b_0.

Let us first linearize the system around the normal steady state. Up to second-order terms in $\widehat{q_t} = (q_t - 1)$, $\widehat{k_t} = (k_t - k^n)$, $\widehat{r_t} = (r_t - r^n)$, $\widehat{b_t} = b_t$, and δ, we have that

$$\widehat{r_t} = \frac{\frac{\theta}{1+\gamma} + 1}{s_r} k^n \widehat{q_t} + \frac{1+\gamma-s_k}{s_r}\widehat{k_t} + \frac{1}{s_r}\widehat{b_t}, \qquad \widehat{q_{t+1}} = \tilde{E}^n \widehat{q_t} + \left[\pi_1 + \frac{1+\gamma-s_k}{s_r}\right]\widehat{k_t} + \frac{1}{s_r}\widehat{b_t},$$

$$\widehat{k_{t+1}} = \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t}, \qquad \widehat{b_{t+1}} = \frac{1+r^n}{1+\gamma}\widehat{b_t},$$

where

$$\tilde{E}^n = \frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n.$$

Similarly, we linearize the system around the speculative steady state. Up to second-order terms in $\widehat{q_t} = (q_t - 1)$, $\widehat{k_t} = (k_t - k^s)$, $\widehat{r_t} = (r_t - r^s)$, $\widehat{b_t} = b_t$, and δ, we have that

$$\widehat{r_t} = \frac{\frac{\theta}{1+\gamma} + 1}{s_r} k^s \widehat{q_t} + \frac{1+\gamma-s_k}{s_r}\widehat{k_t} + \frac{1}{s_r}\widehat{b_t}, \qquad \widehat{q_{t+1}} = \tilde{E}^s \widehat{q_t} + \left[\pi_1 + \frac{1+\gamma-s_k}{s_r}\right]\widehat{k_t} + \frac{1}{s_r}\widehat{b_t},$$

$$\widehat{k_{t+1}} = \frac{\theta k^s}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t}, \qquad \widehat{b_{t+1}} = \frac{1+r^s}{1+\gamma}\widehat{b_t},$$

where

$$\tilde{E}^s = \frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s.$$

The dynamic system around the normal steady state is characterized by the matrix

$$\Omega^n = \begin{bmatrix} \tilde{E}^n & \pi_1 + \frac{1+\gamma-s_k}{s_r} & \frac{1}{s_r} \\ \frac{\theta k^n}{(1+\gamma)^2} & 1 & 0 \\ 0 & 0 & \frac{1+r^n}{1+\gamma} \end{bmatrix}.$$

Similarly, the dynamic system around the speculative steady state is characterized by the matrix

$$\Omega^s = \begin{bmatrix} \tilde{E}^s & \pi_1 + \frac{1+\gamma-s_k}{s_r} & \frac{1}{s_r} \\ \frac{\theta k^s}{(1+\gamma)^2} & 1 & 0 \\ 0 & 0 & \frac{1+r^s}{1+\gamma} \end{bmatrix}.$$

Keeping the same notations as above, it is clear that the eigenvalues of Ω^n are $\lambda^{n+} > 1$, $\lambda^{n-} < 1$, and $(1 + r^n/1 + \gamma) > 1$, with corresponding eigenvectors $(x^{n+}, 1, 0)'$, $(x^{n-}, 1, 0)'$, and z_b^n where

$$\lambda^{n+} = 1 + \frac{\tilde{E}^n - 1 + \sqrt{[\tilde{E}^n - 1]^2 + 4\left(\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right)\frac{\theta k^n}{(1+\gamma)^2}}}{2}$$

and

$$\lambda^{n-} = 1 + \frac{\tilde{E}^n - 1 - \sqrt{[\tilde{E}^n - 1]^2 + 4\left(\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right)\frac{\theta k^n}{(1+\gamma)^2}}}{2}.$$

Therefore, the tangent plane to the unstable manifold of the normal steady state is the plane through the normal steady state with directing vectors $(x^{n+}, 1, 0)'$ and z_b^n, and the tangent line to its stable manifold is the line through the normal steady state with directing vector $(x^{n-}, 1, 0)'$.

Similarly, the eigenvalues of Ω^s are $\lambda^{s+} > 1$, $\lambda^{s-} < 1$, and $(1 + r^s/1 + \gamma) < 1$, with corresponding eigenvectors $(x^{s+}, 1, 0)'$, $(x^{s-}, 1, 0)'$, and z_b^s where

$$\lambda^{s+} = 1 + \frac{\tilde{E}^s - 1 + \sqrt{[\tilde{E}^s - 1]^2 + 4\left(\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right)\frac{\theta k^s}{(1+\gamma)^2}}}{2}$$

and

$$\lambda^{s-} = 1 + \frac{\tilde{E}^s - 1 - \sqrt{[\tilde{E}^s - 1]^2 + 4\left(\pi_1 + \frac{1 + \gamma - s_k}{s_r}\right)\frac{\theta k^s}{(1+\gamma)^2}}}{2}.$$

Therefore, the tangent line to the unstable manifold of the speculative steady state is the line through the speculative steady state with directing vectors $(x^{s+}, 1, 0)'$, and the tangent plane to its stable manifold is the plane through the speculative steady state with directing vectors $(x^{s-}, 1, 0)'$ and z_b^s.

Let us now vary δ, $k^0 = k^0(\delta)$, $\gamma = \gamma(\delta)$, and $s^0 = s^0(\delta)$ in such a way that $k^n = (s_0 + s_r\pi_0)/(s_r\pi_1 + (1 + \gamma - s_k))$ is fixed, Assumptions 0, 1, and 4 are verified, and $|k^0 - k^n| > \varepsilon\delta$ for some $\varepsilon > 0$. Note that $\gamma(\delta) < r^n = \pi_0 - \pi_1 k^n$ for $\delta > 0$ and $\lim_{\delta \to 0}\gamma(\delta) = r^n = \pi_0 - \pi_1 k^n$. Also $k^0(\delta) > k^n$ for $\delta > 0$ and $\lim_{\delta \to 0} k^0(\delta) = k^n$.

As we do this, k^s, x^{n+}, x^{n-}, λ^{n+}, λ^{n-}, x^{s+}, x^{s-}, λ^{s+}, and λ^{s-} also vary. These variables are all continuous functions of δ. Let us denote them by $k^s(\delta)$, $x^{n+}(\delta)$, $x^{n-}(\delta)$, $\lambda^{n+}(\delta)$, $\lambda^{n-}(\delta)$, $x^{s+}(\delta)$, $x^{s-}(\delta)$, $\lambda^{s+}(\delta)$, and $\lambda^{s-}(\delta)$. It is clear that $k^s(\delta) = k^n + O(\delta)$, $x^{n+}(\delta) = x^+ + O(\delta)$, $x^{n-}(\delta) = x^- + O(\delta)$, $\lambda^{n+}(\delta) = \lambda^+ + O(\delta)$, $\lambda^{n-}(\delta) = \lambda^- + O(\delta)$, $x^{s+}(\delta) = x^+ + O(\delta)$, $x^{s-}(\delta) = x^- + O(\delta)$, $\lambda^{s+}(\delta) = \lambda^+ + O(\delta)$, and $\lambda^{s-}(\delta) = \lambda^- + O(\delta)$, with

$$\lambda^+ = 1 + \frac{\hat{E} - 1 + \sqrt{[\hat{E} - 1]^2 + 4\left(\pi_1 + \frac{1 + r^n - s_k}{s_r}\right)\frac{\theta k^n}{(1+r^n)^2}}}{2},$$

$$\lambda^- = 1 + \frac{\hat{E} - 1 - \sqrt{[\hat{E} - 1]^2 + 4\left(\pi_1 + \frac{1 + r^n - s_k}{s_r}\right)\frac{\theta k^n}{(1 + r^n)^2}}}{2},$$

$$x^+ = \frac{\lambda^+ - 1}{\frac{\theta k^n}{(1 + r^n)^2}} > 0, \quad x^- = \frac{\lambda^- - 1}{\frac{\theta k^n}{(1 + r^n)^2}} < 0,$$

and

$$\hat{E} = \frac{1 + r^n}{1 - r^n} + \frac{\pi_1 \frac{\theta}{1 + r^n} k^n}{1 + r^n} + \frac{\frac{\theta}{1 + r^n} + 1}{s_r} k^n.$$

Because the dynamic system we consider is twice continuously differentiable in the regions $k < k^0$ and $k > k^0$, the following results are true:

- There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$, there exist continuously differentiable functions $q_\delta^{s-} : [k^0, k^s] \times [0, M_b(\delta)\delta] \to [1, +\infty)$ and $q^{n+} : [k^n, k^0] \times [0, M_b(\delta)\delta] \to [1, +\infty)$ such that the stable manifold of $(k^s(\delta), 1, 0)$ is parameterized by $(k, b, q_\delta^{s-}(k, b))$ in the region $(k, b) \in [k^0, k^s(\delta)] \times [0, M_b(\delta)\delta]$, and the unstable manifold of $(k^n, 1, 0)$ is parameterized by $(k, b, q_\delta^{n+}(k, b))$ in the region $(k, b) \in [k^n, k^0] \times [0, M_b(\delta)\delta]$.
- There exists $\bar{\bar{\delta}} > 0$ and $m^n > 0$ such that for all $\delta < \bar{\bar{\delta}}$, $(k, b) \in [k^0, k^s(\delta)] \times [0, M_b(\delta)\delta]$, and $(k', b') \in [k^n, k^0] \times [0, M_b(\delta)\delta]$,

$$\left|q_\delta^{s-}(k, b) - 1 - x^{s-}(\delta)(k - k^s(\delta))\right| < m^n M_b(\delta)\delta$$

and

$$\left|q_\delta^{n+}(k', b') - 1 - x^{n+}(\delta)(k - k^n)\right| < m^n M_b(\delta)\delta.$$

Using the fact that $k^s(\delta) = k^n + O(\delta)$, $x^{s-}(\delta) = x^- + O(\delta)$, $x^{n+}(\delta) = x^+ + O(\delta)$, $\lambda^{s-}(\delta) = \lambda^- + O(\delta)$, and $\lambda^{n+}(\delta) = \lambda^+ + O(\delta)$, we see that there exists $\bar{\bar{\delta}} > \bar{\bar{\bar{\delta}}} > 0$ and $m^s > 0$ such that for all $\delta < \bar{\bar{\bar{\delta}}}$, $(k, b) \in [k^0, k^s(\delta)] \times [0, M_b(\delta)\delta]$, and $(k', b') \in [k^n, k^0] \times [0, M_b(\delta)\delta]$,

$$\left|q_\delta^{s-}(k, b) - 1 - x^-(k - k^s(\delta))\right| < m^s M_b(\delta)\delta$$

and

$$\left|q_\delta^{n+}(k', b') - 1 - x^+(k - k^n)\right| < m^s M_b(\delta)\delta.$$

Let $M = \max\{m^n, m^s\}$. The $q = 1 + x^+(k - k^n)$ plane intersects the $k = k^0$ plane along the line $k = k^0$, $q = 1 + x^+(k^0 - k^n)$. Similarly the $q = 1 + x^-(k - k^s(\delta))$ plane intersects the $k = k^0$ plane along the line $k = k^0$, $q = 1 + x^-(k^0 - k^s(\delta))$. Therefore, if $\delta < \bar{\bar{\bar{\delta}}}$, we have that

$$1 + x^-(k^0(\delta) - k^s(\delta)) + MM_b(\delta)\delta > 1 + x^+(k^0(\delta) - k^n) - \delta - MM_b(\delta)\delta \tag{A6}$$

is a necessary condition for the intersection of the unstable manifold of the normal steady state with the vertical $k = k_0$ plane to lie below the intersection of stable manifold of the speculative steady state with the vertical $k = k_0$ plane in the region $b \in [0, M_b(\delta)]$.

Similarly, a sufficient condition is

$$1 + x^-(k^0(\delta) - k^s(\delta)) - MM_b(\delta)\delta > 1 + x^+(k^0(\delta) - k^n) - \delta + MM_b(\delta)\delta. \tag{A7}$$

We can rewrite (A6) and (A7) as

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2MM_b(\delta)\delta}{k^0 - k^n}$$

and

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2MM_b(\delta)\delta}{k^0 - k^n}.$$

Using the fact that $k^0 - k^n > \varepsilon\delta$, we see that a necessary condition for a transition when $\delta < \bar{\bar{\bar{\delta}}}$ is

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2MM_b(\delta)}{\varepsilon}$$

and that a sufficient condition for a transition is

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2MM_b(\delta)}{\varepsilon}.$$

Notice that $(2MM_b(\delta)/\varepsilon) \to 0$ when $\delta \to 0$. This shows that if

$$\frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^s - k^0}{k^0 - k^n},$$

then there exists $m_b(\delta) > 0$ such that a transition is possible for δ small enough and $0 < b_0 < m_b(\delta)\delta$, and that if

$$\frac{\lambda^+ - 1}{1 - \lambda^-} > \frac{k^s - k^0}{k^0 - k^n},$$

then for δ small enough, for every $\widetilde{m_b} > 0$, there exists $0 < b_0 < \widetilde{m_b}\delta$ such that no transition is possible with initial bubble b_0.

This condition holds if and only if

$$\Lambda = \frac{\lambda^+ - 1}{1 - \lambda^-} < \frac{k^n - k^0}{k^0 - k^s},$$

which is exactly the same condition as without bubbles. The discussion of the possibility of a transition is therefore entirely similar.

REFERENCES

Abel, Andrew B.; Mankiw, N. Gregory; Summers, Lawrence H. and Zeckhauser, Richard J. "Assessing Dynamic Efficiency: Theory and Evidence." *Review of Economic Studies*, 1989, *56*(1), pp. 1–19.

Beaudry, Paul and Portier, Franck. "Stock Prices, News and Economic Fluctuations." National Bureau of Economic Research, Inc., NBER Working Papers: No. 10548, 2004.

Benhabib, Jess and Farmer, Roger E. A. "Indeterminacy and Sunspots in Macroeconomics," in John B. Taylor and Michael Woodford, eds., *Handbook of macroeconomics*. Vol. 1A. Amsterdam: Elsevier Science, North-Holland, 1999, pp. 387–448.

Carroll, Christopher D. and Weil, David N. "Saving and Growth: A Reinterpretation." *Carnegie-Rochester Conference Series on Public Policy*, 1994, 40, pp. 133–92.

Cecchetti, Stephen G.; Genberg, Hans; Lipsky, John and Wadhwani, Sushil B. *Asset prices and central bank policy: What to do about it*? Geneva Reports on the World Economy, Vol. 2. London: Centre for Economic Policy Research, 2000.

Christiano, Lawrence J. and Fisher, Jonas D. M. "Stock Market and Investment Goods Prices: Implications for Macroeconomics." National Bureau of Economic Research, Inc., NBER Working Papers: No. 10031, 2003.

De Long, J. Bradford; Shleifer, Andrei; Summers, Lawrence H. and Waldmann, Robert J. "The Size and Incidence of the Losses from Noise Trading." *The Journal of Finance*, 1989, *44*(3), pp. 681–96.

Diamond, Peter A. "National Debt in a Neoclassical Growth Model." *American Economic Review*, 1965, *55*(5), pp. 1126–50.

Gavin, Michael; Hausmann, Ricardo and Talvi, Ernesto. "Saving Behavior in Latin America: Overview and Policy Issues." Inter-American Development Bank, Office of the Chief Economist Working Paper: No. 346, 1997.

Glassmen, James K. and Hassett, Kevin A. *Dow 36,000: The new strategy for profiting from the coming rise in the stock market*. New York: Three Rivers Press, 1999.

Greenwood, Jeremy and Jovanovic, Boyan. "The Information-Technology Revolution and the Stock Market." *American Economic Review*, 1999 (*Papers and Proceedings*), *89*(2), pp. 116–22.

Grossman, Gene M. and Yanagawa, Noriyuki. "Asset Bubbles and Endogenous Growth." *Journal of Monetary Economics*, 1993, *31*(1), pp. 3–19.

Hobijn, Bart and Jovanovic, Boyan. "The Information-Technology Revolution and the Stock Market: Evidence." *American Economic Review*, 2001, *91*(5), pp. 1203–20.

International Monetary Fund. *World economic outlook: Asset prices and the business cycle*. Washington, DC: International Monetary Fund, 2000.

Jaimovich, Nir. "Firm Dynamics, Markup Variations, and the Business Cycle." Unpublished Paper, 2004.

Keynes, John M. "An Economic Analysis of Unemployment," in Donald M. Moggridge, ed., *The collected writings of John Maynard Keynes*. Vol. 13. London: Macmillan, 1931.

Kindleberger, Charles P. *Manias, panics, and crashes: A history of financial crises*. New York: Basic Books, 1989.

King, Ian and Ferguson, Don. "Dynamic Inefficiency, Endogenous Growth, and Ponzi Games." *Journal of Monetary Economics*, 1993, *32*(1), pp. 79–104.

Krugman, Paul. "History versus Expectations." *Quarterly Journal of Economics*, 1991, *106*(2), pp. 651–67.

Murphy, Kevin M.; Shleifer, Andrei and Vishny, Robert W. "Building Blocks of Market Clearing Business Cycle Models," in Olivier J. Blanchard and Stanley Fischer, eds., *NBER macroeconomics annual: 1989*. Cambridge, MA: MIT Press, 1989, pp. 247–87.

Olivier, Jacques. "Growth-Enhancing Bubbles." *International Economic Review*, 2000, *41*(1), pp. 133–51.

Saint-Paul, Gilles. "Fiscal Policy in an Endogenous Growth Model." *Quarterly Journal of Economics*, 1992, *107*(4), pp. 1243–59.

Santos, Manuel and Woodford, Michael. "Rational Asset Pricing Bubbles." *Econometrica*, 1997, *65*(1), pp. 19–58.

Shiller, Robert J. *Irrational exuberance*. Princeton: Princeton University Press, 2000.

Tirole, Jean. "Asset Bubbles and Overlapping Generations." *Econometrica*, 1985, *53*(6), pp. 1499–1528.

Ventura, Jaume. "A Portfolio View of the U.S. Current Account Deficit." *Brookings Papers on Economic Activity*, 2001, 1, pp. 241–53.

Ventura, Jaume. "Economic Growth with Bubbles." Universitat Pompeu Fabra, Centre de Recerca en Economia Internacional (CREI) Economics Working Papers: No. 848, 2003.

Vissing-Jorgensen, Annette. "Limited Stock Market Participation and the Equity Premium Puzzle." Unpublished Paper, 1999.