Gains From Trade under Monopolistic Competition: A Simple Example with Translog Expenditure Functions and Pareto Distributions of Firm-Level Productivity*

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Abstract

In this note we provide closed-form solutions for bilateral trade flows and gains from trade in a model with monopolistic competition, translog expenditure functions, and Pareto distributions of firm-level productivity. In spite of variable mark-ups, gains from trade can be evaluated in the same way as in the quantitative trade models discussed in Arkolakis, Costinot, and Rodríguez-Clare (2010).

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1 Introduction

In this note we provide closed-form solutions for bilateral trade flows and gains from trade in a model with monopolistic competition, symmetric translog expenditure functions, and Pareto distributions of firm-level productivity. In spite of variable mark-ups, gains from trade can be evaluated in the same way as in the quantitative trade models discussed in Arkolakis, Costinot, and Rodríguez-Clare (2010).

2 Basic Environment

Our basic environment is a simple multi-country extension of the model developed by Rodríguez-López (2010) with symmetric translog expenditure functions as proposed by Bergin and Feenstra (2000, 2001) and Feenstra (2003). We consider a world economy with $i = 1, ..., n$ countries, a continuum of goods $\omega \in \Omega$, and labor as the only factor of production. Labor is inelastically supplied and immobile across countries. L_i and w_i denote the total endowment of labor and the wage in country i, respectively.

Consumers. In each country i there is a representative consumer with a symmetric translog expenditure function

$$\begin{aligned} \ln e\left(\boldsymbol{p}_i, u_i\right) &= \ln u_i + \frac{1}{2\zeta M_i} + \frac{1}{M_i} \int_{\omega \in \Omega_i} \ln p_i\left(\omega\right) d\omega \\ &+ \frac{\zeta}{2M_i} \int\int_{\omega,\omega' \in \Omega_i} \ln p_i\left(\omega\right) \left[\ln p_i\left(\omega'\right) - \ln p_i\left(\omega\right)\right] d\omega' d\omega, \end{aligned} \tag{1}$$

where $\boldsymbol{p}_i \equiv \{p_i\left(\omega\right)\}_{\omega \in \Omega_j}$ denotes the schedule of good prices in country i; Ω_i denotes the set of goods available in country i; and M_i denotes the measure of this set.

Firms. In each country i there is a large pool of monopolistically competitive firms. In order to start producing firms must hire $F_i > 0$ units of labor. After fixed entry costs have been paid, firms receive a productivity level z drawn randomly from a Pareto distribution with density:

$$g_i\left(z\right) = \theta \frac{b_i^\theta}{z^{\theta+1}} \text{ for all } z > \underline{z}. \tag{2}$$

For a firm with productivity z, the cost of producing one unit of a good in country i and selling in country j is given by $c_{ij}\left(z\right) = w_i \tau_{ij}/z$, where $\tau_{ij} \geq 1$ reflect bilateral (iceberg) trade costs between country i and country j.

3 Characterization of the Equilibrium

3.1 Firm-level variables

Consider an importing country j. By Equation (1) and Shepard's Lemma, the share of expenditure on good ω in country j is given by $\frac{1}{M_j} + \zeta\left(\frac{1}{M_j}\int_{\omega'\in\Omega_j} \ln p_j(\omega')\, d\omega' - \ln p_j(\omega)\right)$, so the sales of the producer of good ω in country j are

$$p_j(\omega)\, q_j(\omega) = \left[\frac{1}{M_j} + \zeta\left(\frac{1}{M_j}\int_{\omega'\in\Omega_j} \ln p_j(\omega')\, d\omega' - \ln p_j(\omega)\right)\right] Y_j, \tag{3}$$

where $q_j(\omega)$ are the units of good ω sold in country j and $Y_j \equiv \sum_{j=1}^{n} X_{ij}$ is the total expenditure of country j's representative consumer.

Under monopolistic competition, a firm from country i with productivity $z(\omega)$ chooses the price $p_j(\omega)$ at which it sells good ω in country j in order to maximize its profits, $p_j(\omega)\, q_j(\omega) - \frac{w_i \tau_{ij}}{z(\omega)} q_j(\omega)$. The first-order condition associated with this maximization program leads to the following pricing equation:

$$p_j(\omega) = \left[1 + \frac{1}{\zeta M_j} + \frac{1}{M_j}\int_{\omega'\in\Omega_j} \ln p_j(\omega')\, d\omega' - \ln p_j(\omega)\right] \frac{w_i \tau_{ij}}{z(\omega)}.$$

This can be rearranged as

$$p_j(\omega) = \mathcal{W}\left[\frac{z(\omega)}{z_{ij}^*} e\right] \cdot \frac{w_i \tau_{ij}}{z(\omega)}, \tag{4}$$

where $z_{ij}^* \equiv w_i \tau_{ij} \exp\left[-\left(\frac{1}{\zeta M_j} + \frac{1}{M_j}\int_{\omega'\in\Omega_j} \ln p_j(\omega')\, d\omega'\right)\right]$ is a variable that summarizes market conditions in j for firms from i, and where $\mathcal{W}(\cdot)$ is the Lambert function.[1] The term $\mathcal{W}\left[\frac{z(\omega)}{z_{ij}^*} e\right]$ is the mark-up charged by firms from i selling good ω in country j. Compare to models of monopolistic competition with CES preferences, firms' mark-ups now vary with firm-level productivity, $z(\omega)$, and market conditions, z_{ij}^*.

Combining the previous expression with Equation (3), we obtain the following expression for the sales, $x_{ij}(z)$, of firms from country i with productivity z in country j:

$$x_{ij}(z) = \left[\mathcal{W}\left(\frac{z}{z_{ij}^*} e\right) - 1\right] \zeta Y_j. \tag{5}$$

[1] For all $z > 0$, the value of the Lambert function, $\mathcal{W}(z)$, is implicity defined as the unique solution of $xe^x = z$. The Lambert function satisfies $\mathcal{W}_z > 0$, $\mathcal{W}_{zz} < 0$, $\mathcal{W}(0) = 0$ and $\mathcal{W}(e) = 1$; see **?** for detail.

Since $\mathcal{W}_z > 0$ and $\mathcal{W}(e) = 1$, firms from country i export in country j if and only if $z > z_{ij}^*$. Similar algebra implies that the profits $\pi_{ij}(z)$ of firms from country i with productivity z in country j are given by

$$\pi_{ij}(z) = \left[\mathcal{W}\left(\frac{z}{z_{ij}^*}e\right) - 1\right]\left[1 - \mathcal{W}^{-1}\left(\frac{z}{z_{ij}^*}e\right)\right]\zeta Y_j. \tag{6}$$

3.2 Aggregate variables

Bilateral Trade Flows. Let X_{ij} denote the total value of country j's imports from country i. By definition, bilateral imports can be expressed as

$$X_{ij} = \int_{\omega\in\Omega_{ij}} x_{ij}(\omega)\, d\omega, \tag{7}$$

where $\Omega_{ij} \equiv \left\{\omega \in \Omega_j | z(\omega) > z_{ij}^*\right\}$ is the set of goods exported by firms from country i in country j. Equations (2), (5), and (7) imply

$$X_{ij} = N_i \int_{z_{ij}^*}^{+\infty} \left[\mathcal{W}\left(\frac{z}{z_{ij}^*}e\right) - 1\right]\zeta Y_j \theta \frac{b_i^\theta}{z^{\theta+1}} dz,$$

where N_i is the measure of potential firms in country i, i.e. those firms which have paid the free entry costs $w_i F_i$. This measure will be endogenously determined using a free entry condition below. Using a simple change of variable, $v = z/z_{ij}^*$, and the definition of z_{ij}^*, we can rearrange the previous expression as

$$X_{ij} = N_i b_i^\theta (w_i \tau_{ij})^{-\theta} \exp\left(\frac{\theta}{\zeta M_j} + \frac{\theta}{M_j}\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega\right) Y_j \kappa_1, \tag{8}$$

where $\kappa_1 \equiv \zeta\theta \int_1^{+\infty} [\mathcal{W}(ve) - 1]\, v^{-\theta-1} dv$. Summing across all exporters $i = 1, ..., n$ and using the definition of Y_j, we further get

$$\exp\left(\frac{\theta}{\zeta M_j} + \frac{\theta}{M_j}\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega\right)\kappa_1 = \frac{1}{\sum_{i=1}^n N_i b_i^\theta (w_i\tau_{ij})^{-\theta}}.$$

Combining the two previous expressions, we obtain

$$X_{ij} = \frac{N_i b_i^\theta (w_i \tau_{ij})^{-\theta} Y_j}{\sum_{i'=1}^n N_{i'} b_{i'}^\theta (w_{i'}\tau_{i'j})^{-\theta}}. \tag{9}$$

This is what Arkolakis, Costinot, and Rodríguez-Clare (2010) refer to as the strong version of a "CES import demand system" with θ being the (absolute value of the) trade elasticity of that system. A similar macro-level restriction holds in Krugman (1980), Eaton and Kortum (2002), and Eaton, Kortum, and Kramarz (2010). In the rest of this note we denote by $\lambda_{ij} \equiv X_{ij}/Y_j$ the share of country j's total expenditure that is devoted to goods from country i.

Measures of Potential Firms and Consumed Varieties. To solve for the measure N_i of potential firms in country i, we now use the free entry and labor market clearing conditions. On the one hand, free entry requires that total expected profits are equal to entry costs. By Equation (2), we can express this condition as

$$\sum_{j=1}^{n} \int_{z_{ij}^*}^{+\infty} \pi_{ij}(z) \frac{\theta b_i^\theta}{z^{\theta+1}} dz = w_i F_i.$$

Using Equation (6) and the same change of variables as before, $v = z/z_{ij}^*$, we can rearrange the previous expression as

$$\sum_{j=1}^{n} \left(\frac{b_i}{z_{ij}^*}\right)^\theta Y_j = \frac{w_i F_i}{\kappa_2}, \tag{10}$$

where $\kappa_2 \equiv \zeta\theta \int_1^{+\infty} [\mathcal{W}(ve) - 1]\left[1 - \mathcal{W}^{-1}(ve)\right] v^{-\theta-1} dv$. On the other hand, labor market clearing requires that total costs are equal to the total wage bill. By Equation (2), we can express this condition as:

$$\sum_{j=1}^{n} N_i \int_{z_{ij}^*}^{+\infty} [x_{ij}(z) - \pi_{ij}(z)] \frac{\theta b_i^\theta}{z^{\theta+1}} dz + N_i w_i F_i = w_i L_i.$$

Using Equation (5), Equation (6), and the same change of variables as before, $v = z/z_{ij}^*$, we can rearrange the previous expression as

$$N_i \left[\kappa_3 \sum_{j=1}^{n} \left(\frac{b_i}{z_{ij}^*}\right)^\theta Y_j + w_i F_i\right] = w_i L_i, \tag{11}$$

where $\kappa_3 \equiv \zeta\theta \int_1^{+\infty} \left[1 - \mathcal{W}^{-1}(ve)\right] v^{-\theta-1} dv$. Combining Equations (10) and (11) we obtain

$$N_i = \frac{L_i}{F_i \left(1 + \frac{\kappa_3}{\kappa_2}\right)}. \tag{12}$$

Like in a simple Krugman (1980) model, N_i is increasing in the total endowment L_i of labor in country i and decreasing in the magnitude of free entry costs F_i.

Measure of available goods. By definition, the measure M_j of goods available in country j is given by $M_j = \sum_{i=1}^{n} N_i \left(b_i/z_{ij}^*\right)^{\theta}$. By Equation (8) and the definition of z_{ij}^*, we also know that $X_{ij} = N_i \left(b_i/z_{ij}^*\right)^{\theta} Y_j \kappa_1$. Combining the two previous expressions with trade balance, $\sum_{i=1}^{n} X_{ij} = \sum_{i=1}^{n} X_{ji}$, we therefore have $Y_j M_j = \sum_{i=1}^{n} N_j \left(b_j/z_{ji}^*\right)^{\theta} Y_i$. Together with Equations (10) and (12), this implies $Y_j M_j = \left[L_j/F_j\left(1+\frac{\kappa_3}{\kappa_2}\right)\right] \cdot \left[w_j F_j/\kappa_2\right]$. Noting that $Y_j = w_j L_j$ by labor market clearing and simplifying we obtain

$$M_j = \frac{1}{\kappa_2 + \kappa_3}.$$

This implies that the measure of goods available in each country is independent of labor endowments and trade costs, and hence is the same in all countries.

4 Welfare Evaluation

Let $P_j \equiv e(\mathbf{p}_j, 1)$ denote the consumer price index in country i. We are interested in evaluating the changes in real income, $W_j \equiv Y_j/P_j$, associated with foreign shocks in country j. To do so, we first introduce the following formal definition.

Definition 1 *A foreign shock in country j is a change from $(\boldsymbol{L}, \boldsymbol{F}, \boldsymbol{\tau})$ to $(\boldsymbol{L}', \boldsymbol{F}', \boldsymbol{\tau}')$ such that $L_j = L_j'$, $F_j = F_j'$, $\tau_{jj} = \tau_{jj}'$, with $\boldsymbol{L} \equiv \{L_i\}$, $\boldsymbol{F} \equiv \{F_i\}$ and $\boldsymbol{\tau} \equiv \{\tau_{ij}\}$.*

Put simply, foreign shocks correspond to any changes in labor endowments, fixed entry costs, and trade costs that do not affect either country j's endowment or its ability to serve its own market.

Ex post welfare evaluation. We are now ready to state our first welfare prediction.

Proposition 1 *The change in real income associated with any foreign shock in country j can be computed as $\widehat{W}_j = \widehat{\lambda}_{jj}^{-1/\theta}$, where $\widehat{v} \equiv v'/v$ denotes the change in any variable v between the initial and the new equilibrium.*

Proof. See Appendix. ■

Since θ corresponds to (the absolute value of the) trade elasticity in this environment, this is the exact same welfare formula as in Arkolakis, Costinot, and Rodríguez-Clare (2010). As emphasized in Arkolakis, Costinot, and Rodríguez-Clare (2010), Proposition 1 is an *ex* *post* result in the sense that the percentage change in real income is expressed as a function of the change in the share of domestic expenditure, $\widehat{\lambda}_{jj}$. Thus, it is only useful to the extent that $\widehat{\lambda}_{jj}$ is observed. For instance, looking at historical trade data, Proposition 1 can be used to infer the welfare consequences of past episodes of trade liberalization.

Ex ante welfare evaluation. We now turn to a very particular, but important type of shock: *moving to autarky.* Formally, we assume that trade costs in the new equilibrium are such that $\tau'_{ij} = +\infty$ for any pair of countries $i \neq j$. All other technological parameters and endowments are the same as in the initial equilibrium. For this particular counterfactual exercise, we know that $\hat{\lambda}_{jj} = 1/\lambda_{jj}$ since $\lambda'_{jj} = 1$ under autarky. Combining this observation with Proposition 1, we immediately get:

Corollary 1 *The change in real income associated with moving to autarky in country j can be computed as $\widehat{W}_j^A = \lambda_{jj}^{1/\theta}$.*

Unlike Proposition 1, Corollary 1 is an *ex ante* result in the sense that it does not require any information on trade flows in the new equilibrium. In spite of variable mark-ups, we see that gains from trade can be evaluated in the exact same way as in the quantitative trade models discussed in Arkolakis, Costinot, and Rodríguez-Clare (2010). Gains from trade, defined as the absolute value of the percentage change in real income associated with moving from the initial equilibrium to autarky, only depends on two sufficient statistics: (i) the share of domestic expenditure; and (ii) the trade elasticity.

Finally, like in Arkolakis, Costinot, and Rodríguez-Clare (2010), ex-ante results can be derived for *any* change in trade costs using the following proposition.

Proposition 2 *The percentage change in real income associated with any change in trade costs in country j can be computed using $\widehat{W}_j = \widehat{\lambda}_{jj}^{-1/\theta}$ combined with*

$$\widehat{\lambda}_{jj} = \left[\sum_{i=1}^{n} \lambda_{ij} \left(\hat{w}_i \hat{\tau}_{ij}\right)^{-\theta}\right]^{-1}, \tag{13}$$

where $\{\hat{w}_i\}_{i=1,..,n}$ are such that

$$\widehat{w}_i = \sum_{j=1}^{n} \frac{\lambda_{ij} \hat{w}_j Y_j \left(\hat{w}_i \hat{\tau}_{ij}\right)^{-\theta}}{Y_i \sum_{i'=1}^{n} \lambda_{i'j} \left(\hat{w}_{i'} \hat{\tau}_{i'j}\right)^{-\theta}}. \tag{14}$$

Proof. The proof is the same as the proof of Proposition 2 in Arkolakis, Costinot, and Rodríguez-Clare (2010) and omitted. ■

Since Equations (13) and (14) only depend on trade data and the trade elasticity, Proposition 2 implies that the welfare consequences of any change in trade costs, not just moving to autarky, must be the same in the present model as in the quantitative trade models discussed in Arkolakis, Costinot, and Rodríguez-Clare (2010) for which Proposition 2 applies.

5 Concluding Remarks

In this note we have characterized bilateral trade flows, welfare, and most importantly, the relationship between the two in a model with monopolistic competition, translog expenditure functions, and Pareto distributions of firm-level productivity, as in Rodríguez-López (2010). In spite of variable mark-ups, we have shown that gains from trade can be evaluated in the same way as in the quantitative trade models discussed in Arkolakis, Costinot, and Rodríguez-Clare (2010). This simple model is, of course, very special. We do not expect the welfare equivalence between models with CES and translog expenditure functions to be true for all distributions of firm-level productivity. We also do not claim that this simple model is empirically realistic. Unlike in the case of CES expenditure functions, Pareto distributions of firm-level productivity combined with translog expenditure functions do not lead to Pareto distribution of sales for which there is considerable empirical support. The empirical work of Feenstra and Weinstein (2010) on translog expenditure functions also provides direct evidence against this distributional assumption. Still, the previous theoretical results provide a nice illustration that new sources of gains from trade, here variable mark-ups, do not necessarily lead to larger gains from trade.

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A Proof of Proposition 1

Throughout this proof we use labor in country j as our numeraire, $w_j = 1$. Under this normalization, we have $W_j = Y_j/P_j = L_j/P_j$. Thus changes in real income are equal to the inverse of changes in the consumer price index:

$$\widehat{W}_j = 1/\widehat{P}_j. \tag{15}$$

We now characterize changes in the consumer price index. By Equation (1), we know that

$$\ln P_j = \frac{1}{2\zeta M_j} + \frac{1}{M_j}\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega + \frac{\zeta}{2M_j}\int\int_{\omega,\omega'\in\Omega_j} \ln p_j(\omega)\left[\ln p_j(\omega') - \ln p_j(\omega)\right] d\omega' d\omega,$$

which can be rearranged as

$$\ln P_j = \frac{1}{2\zeta M_j} + \frac{1}{M_j}\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega + \frac{\zeta}{2M_j}\left[\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega\right]^2 - \frac{\zeta}{2}\int_{\omega\in\Omega_j}\left[\ln p_j(\omega)\right]^2 d\omega. \tag{16}$$

Let us now compute $\int_{\omega\in\Omega_j} \ln p_j(\omega)$ and $\int_{\omega\in\Omega_j}\left[\ln p_j(\omega)\right]^2 d\omega$. We first want to show that

$$\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega = M_j\left\{\left[1 - \ln\left(\frac{z^*_{jj}}{\tau_{jj}}\right)\right] - \kappa_4\right\}, \tag{17}$$

where $\kappa_4 \equiv \theta\int_1^{+\infty}\left[\mathcal{W}(ve)\right] v^{-\theta-1} dv$. To do so, we first note that Equation (4) and the definition of the Lambert function imply

$$\ln p_j(\omega) = 1 - \ln\left(\frac{z^*_{ij}}{w_i\tau_{ij}}\right) - \mathcal{W}\left[\frac{z(\omega)}{z^*_{ij}}e\right]. \tag{18}$$

Combining this expression with Equation (2), we get

$$\int_{\omega\in\Omega_j} \ln p_j(\omega)\, d\omega = \textstyle\sum_{i=1}^n N_i \displaystyle\int_{z^*_{ij}}^{+\infty}\left[1 - \ln\left(\frac{z^*_{ij}}{w_i\tau_{ij}}\right)\right]\theta\frac{b_i^\theta}{z^{\theta+1}}dz - \textstyle\sum_{i=1}^n N_i \displaystyle\int_{z^*_{ij}}^{+\infty}\mathcal{W}\left(\frac{z}{z^*_{ij}}e\right)\theta\frac{b_i^\theta}{z^{\theta+1}}dz. \tag{19}$$

By definition of z^*_{ij}, we know that $\frac{z^*_{ij}}{w_i\tau_{ij}} = \frac{z^*_{jj}}{\tau_{jj}}$. Since $M_j = \sum_{i=1}^n N_i\left(b_i/z^*_{ij}\right)^\theta$, we therefore have

$$\textstyle\sum_{i=1}^n N_i \displaystyle\int_{z^*_{ij}}^{+\infty}\left[1 - \ln\left(\frac{z^*_{ij}}{w_i\tau_{ij}}\right)\right]\theta\frac{b_i^\theta}{z^{\theta+1}}dz = M_j\left[1 - \ln\left(\frac{z^*_{jj}}{\tau_{jj}}\right)\right]. \tag{20}$$

For the second term on the right-handside of Equation (19), a simple change of variable, $v = z/z_{ij}^*$, and $M_j = \sum_{i=1}^n N_i \left(b_i/z_{ij}^*\right)^\theta$ further imply

$$\sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \mathcal{W}\left(\frac{z}{z_{ij}^*}e\right) \theta \frac{b_i^\theta}{z^{\theta+1}} dz = \theta M_j \int_1^{+\infty} \left[\mathcal{W}(ve)\right] v^{-\theta-1} dv. \tag{21}$$

Equation (17) directly derive from Equations (19)-(21).

Let us now turn to $\int_{\omega\in\Omega_j} \left[\ln p_j(\omega)\right]^2 d\omega$. We want to show that

$$\int (\ln p_\omega)^2 d\omega = M_j \left\{ \left[1 - \ln\left(\frac{z_{jj}^*}{\tau_{ij}}\right)\right]^2 + \kappa_5 - 2\kappa_4 \left[1 - \ln\left(\frac{z_{jj}^*}{\tau_{ij}}\right)\right] \right\}, \tag{22}$$

where $\kappa_5 \equiv \theta \int_1^{+\infty} \left[\mathcal{W}(ve)\right]^2 v^{-\theta-1} dv$. To do so, we first note that Equation (18) implies

$$\left[\ln p_j(\omega)\right]^2 = \left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right]^2 + \left[\mathcal{W}\left(\frac{z}{z_{ij}^*}e\right)\right]^2 - 2\left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right] \mathcal{W}\left(\frac{z}{z_{ij}^*}e\right).$$

Combining the previous expression with Equation (2), we get

$$\begin{aligned}
\int_{\omega\in\Omega_j} \left[\ln p_j(\omega)\right]^2 d\omega \;=\; & \sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right]^2 \theta \frac{b_i^\theta}{z^{\theta+1}} dz \\
& + \sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[\mathcal{W}\left(\frac{z}{z_{ij}^*}e\right)\right]^2 \theta \frac{b_i^\theta}{z^{\theta+1}} dz \\
& - 2\sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right] \mathcal{W}\left(\frac{z}{z_{ij}^*}e\right) \theta \frac{b_i^\theta}{z^{\theta+1}} dz.
\end{aligned} \tag{23}$$

Using the exact same logic as above, it is easy to check that

$$\sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right]^2 \theta \frac{b_i^\theta}{z^{\theta+1}} dz = M_j \left[1 - \ln\left(\frac{z_{jj}^*}{\tau_{jj}}\right)\right]^2, \tag{24}$$

$$\sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[\mathcal{W}\left(\frac{z}{z_{ij}^*}e\right)\right]^2 \theta \frac{b_i^\theta}{z^{\theta+1}} dz = M_j \int_1^{+\infty} \left[\mathcal{W}(ve)\right]^2 v^{-\theta-1} dv, \tag{25}$$

$$\sum_{i=1}^n N_i \int_{z_{ij}^*}^{+\infty} \left[1 - \ln\left(\frac{z_{ij}^*}{w_i\tau_{ij}}\right)\right] \mathcal{W}\left(\frac{z}{z_{ij}^*}e\right) \theta \frac{b_i^\theta}{z^{\theta+1}} dz = M_j \int_1^{+\infty} \left[1 - \ln\left(\frac{z_{jj}^*}{\tau_{jj}}\right)\right] \mathcal{W}(ve)\, v^{-\theta-1}dv. \tag{26}$$

Equation (22) directly derives from Equations (23)-(26).

Combining Equations (16), (17), and (22), we obtain

$$\ln P_j = -\ln\left(\frac{z^*_{jj}}{\tau_{jj}}\right) + 1 - \kappa_4 \frac{1}{2\zeta M_j} + \frac{\zeta M_j}{2}\left(\kappa_4^2 - \kappa_5\right).$$

By Equation (8) and the definition of z^*_{jj}, we also know that

$$\lambda_{jj} = N_j \left(\frac{b_j}{z^*_{jj}}\right)^{\theta} \kappa_1.$$

The two previous expressions imply

$$\ln P_j = \frac{1}{\theta}\ln \lambda_{jj} + 1 - \kappa_4 \frac{1}{2\zeta M_j} + \frac{\zeta M_j}{2}\left(\kappa_4^2 - \kappa_5\right) - \frac{1}{\theta}\ln\left(N_j \kappa_1\right) - \ln b_j + \ln \tau_{jj}.$$

Given our analysis in the main text we know that $M_j = M'_j$ and $N_j = N'_j$ if $L_j = L'_j$ and $F_j = F'_j$. Since we also have $\tau_{jj} = \tau'_{jj}$ under a foreign shock, we immediately get

$$\ln P'_j - \ln P_j = \frac{1}{\theta}\ln \lambda'_{jj} - \frac{1}{\theta}\ln \lambda_{jj}.$$

This implies $\widehat{P}_j = \widehat{\lambda}^{1/\theta}$, and by Equation (15), $\widehat{W}_j = \widehat{\lambda}^{-1/\theta}$. **QED**