

# Uninsured Idiosyncratic Investment Risk and Aggregate Saving\*

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## Abstract

This paper augments the neoclassical growth model to study the macroeconomic effects of uninsured idiosyncratic investment, or capital-income, risk. Under standard assumptions for preferences and technologies, individual policy rules are linear in individual wealth, ensuring that the equilibrium dynamics for aggregate quantities and prices are independent of the wealth distribution. The analysis thus remains highly tractable despite the incompleteness of markets. As compared to complete markets, the steady state is characterized by both a lower interest rate and a lower capital stock when the elasticity of intertemporal substitution is higher than the fraction of private equity in total wealth. For empirically plausible parametrizations, this condition is easily satisfied, and the reduction in aggregate saving and income is quantitatively significant. These findings contrast with Bewley models (e.g., Aiyagari, 1994), where idiosyncratic labor-income risk leads to higher aggregate saving and income.

*JEL codes:* D52, E13, G11, O16, O41.

*Keywords:* incomplete markets, idiosyncratic risk, private equity, precautionary saving.

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# 1 Introduction

The macroeconomic effects of uninsured idiosyncratic labor-income risk and precautionary saving have been the subject of a voluminous literature. Although disagreement remains in the profession about quantitative significance, no ambiguity exists about qualitative effects: as shown in Aiyagari’s 1994 seminal paper, labor-income risk necessarily increases aggregate saving in the steady state of a neoclassical economy.

In contrast, the macroeconomic effects of uninsured idiosyncratic investment, or capital-income, risk are far less well understood. Clearly, such risks are paramount in less developed economies, where the bulk of production takes place in privately-owned businesses and where risk-sharing arrangements are severely limited. But such risks are relevant even for the most developed economies. In the United States, private businesses account for almost one half of aggregate capital and employment. Moreover, the “representative” investor has a very poorly diversified portfolio. For example, consider the richest 20 percent of the US population, which accounts for almost 95 percent of the entire US non-housing wealth; the median of this group holds more than one half of non-housing wealth in private equity.<sup>1</sup> It thus seems difficult to assess the macroeconomic effects of incomplete markets without understanding the macroeconomic effects of idiosyncratic investment risk.

The goal of this paper is to contribute towards filling this gap. Like Aiyagari (1994), this paper studies the impact of incomplete markets on aggregate savings and income in the steady state of a neoclassical economy. But whereas Aiyagari focused on idiosyncratic *labor-income* risk, this paper focuses on idiosyncratic *capital-income* risk. The minimal necessary deviation from the standard paradigm is taken, providing a clear theoretical benchmark for the question of interest.

As in the standard, complete-markets, neoclassical growth model, households in my model supply labor in a competitive labor market, face no uninsurable labor-income risk, and can freely borrow and lend in a riskless bond. But unlike the standard model, households own private businesses, which are subject to undiversified idiosyncratic productivity shocks. These shocks introduce uninsurable idiosyncratic capital-income risk in the model, while assuming away borrowing constraints isolates the role of this risk from that of other forms of market incompleteness.

A common difficulty with incomplete-market models is that the wealth distribution—an infinite-dimensional object—is a relevant state variable for aggregate dynamics. This is not the case in my model. As long as private firms operate under a neoclassical CRS technology (e.g., a Cobb-Douglas technology), the individual agent faces risky, but linear, returns to his own investment. Together with the homotheticity of preferences (CRRA/CEIS), this ensures that, for any given sequence of prices, optimal decision rules are linear in individual wealth. By implication, aggregate quantities and prices in equilibrium are invariant to the wealth distribution. This avoids the “curse of dimensionality” and permits characterizing the equilibrium with a low-dimensional, closed-form dynamic system. The analysis of the steady state is then particularly tractable.

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<sup>1</sup>See Quadrini (2000), Gentry and Hubbard (2000), Carroll (2001), and Moskowitz and Vissing-Jørgensen (2002).

As in Aiyagari’s economy, the steady-state interest rate on bonds is necessarily lower than the agents’ discount rate, that is, lower than its complete-market value. But unlike Aiyagari’s economy, this result does not necessarily imply a higher capital stock. This is because investment risk introduces a wedge between the interest rate and the marginal product of capital—this wedge is the risk premium on private equity. In essence, there are two opposing effects. On the one hand, a precautionary motive stimulates the supply of savings. On the other hand, risk aversion reduces the demand for investment. As a result, the overall effect on capital accumulation is ambiguous.

This theoretical ambiguity raises a serious caveat for any attempt to quantify the steady-state effect of idiosyncratic investment risk. How could one hope for a reliable quantitative exercise without first understanding the forces that determine whether the effect is positive or negative? That’s where the tractability of my framework pays off.

I show that the capital stock is *lower* under incomplete markets if and only if the elasticity of intertemporal substitution (EIS) is higher than a certain threshold  $\underline{\theta}$ . This threshold, in turn, can be approximated by the ratio  $\phi/(2 - \phi)$ , where  $\phi$  is the fraction of private equity to total effective wealth (i.e., financial wealth plus present value of future labor income). Since  $\phi/(2 - \phi) > \phi$ , it follows that an EIS higher than the fraction of private equity to total wealth suffices for idiosyncratic investment risk to lead to both a lower interest rate and a lower capital stock.

This result extends to a variant of the model where there are two groups of households: “investors”, who hold all capital in the economy; and “hand-to-mouth workers”, who consume all their labor income and hold no financial assets. In this extension, the result stated above remains true, provided one reinterprets  $\phi$  as the fraction of private equity to the total effective wealth of the investor group, rather than the that of the entire population.

In the United States, private equity is about one half of financial net worth. Translating this fact into the model implies that  $\phi$  is no less than 0.5 (equivalently, the threshold  $\underline{\theta}$  for the EIS is no more than 0.33). Plausible parametrizations of the model predict a  $\phi$  around 0.3 (equivalently, a  $\underline{\theta}$  around 0.2). The empirical evidence on the EIS, on the other hand, suggest a value for  $\theta$  around or above 1 (see Section 4.3 for a more detailed discussion). Put together, this evidence suggests that the empirically relevant case is, indeed, that idiosyncratic investment risk reduces aggregate savings and income—the opposite of what is true for labor-income risk in Aiyagari (1994). What is more, plausible parametrizations suggest quantitatively significant effects. For my baseline specification, which assumes standard values for the preference and technology parameters and only a modest amount of idiosyncratic risk, the saving rate in the steady state is 3.8 percentage points lower than under complete markets, and aggregate income is 9% lower.

In my benchmark model, the entire capital stock is invested in private firms. In an extension, I give households the option to also invest in “public equity,” a sector where all risk is diversified. Because the low risk-free rate stimulates investment in public equity, the negative impact of incomplete markets on aggregate savings is significantly mitigated. Nevertheless, incomplete markets now also reduce aggregate total factor productivity by shifting resources away from the more risky

but also more productive private equity. As a result, the negative impact of incomplete markets on output can remain large even if the impact on savings is greatly mitigated (or even reversed). Indeed, in my baseline specification the reduction in the saving rate relative to complete markets is only 2 percentage points—almost half of what in the benchmark model—but the reduction in aggregate income remains as high as 8%.

Like any other numerical results, these findings are only suggestive. But they do illustrate that significant general-equilibrium effects on savings and income are both empirically plausible and consistent with low private-equity premia, like those reported in Moskowitz and Vissing-Jørgensen (2002). For example, in parametrizations where the private premium is as low as 0.9%, the reduction in aggregate income remains as high as 5%.

The contribution of this paper is thus to highlight that the macroeconomic effects of idiosyncratic investment risk can be both qualitatively distinct from those of idiosyncratic labor-income risk and quantitatively significant. The focus here is on analyzing steady-state effects. However, as I discuss in Section 6, the framework could also be used to study transitional dynamics, thereby providing insights into the potential business-cycle implications of idiosyncratic investment risk. Other tractable extensions include the introduction of a government and an open-economy version.

The paper also makes a methodological contribution by developing an incomplete-markets economy that admits exact aggregation. This result is instrumental for the tractability of the framework and the characterization of the steady state. But it is also interesting on its own, for it clarifies that non-linearity of individual policy rules—equivalently, the dependence of aggregates on wealth inequality—is not essential for incomplete markets to have significant macroeconomic effects. In my model, idiosyncratic risk has significant impact on both the steady state and the transitional dynamics, even though the wealth distribution is not a relevant state variable.

The rest of this introduction discusses related literature. Section 2 introduces the basic model. Sections 3 and 4 analyze the general equilibrium and the steady state. Section 5 revisits the steady-state effects after adding public equity. Section 6 discusses possible extensions. Section 7 concludes. All proofs appear in the Appendix.

**Related literature.** The paper is most closely related to Bewley-type models such as Aiyagari (1994), Huggett (1997) and Krusell and Smith (1998, 1999). These models also introduce uninsurable idiosyncratic risk in the neoclassical growth model, but focus on labor-income rather than capital-income risk. Given the very different general-equilibrium effects of these two types of risk, and the potential importance of capital-income risk for the wealthy, exercises like the one conducted here seem necessary if one wishes for a more complete understanding of the macroeconomic effects of incomplete markets.

In complementary earlier work, Angeletos and Calvet (2000, 2005, 2006) examine an economy with idiosyncratic production risks, as in this paper. But, unlike this paper, they assume constant *absolute* risk aversion (CARA), thus killing altogether the effect of wealth on precautionary saving, risk taking, and investment. They also do not allow for either a labor market or public equity.

The framework of this paper, instead, is much closer to the standard neoclassical paradigm. This improves, not only the qualitative, but also the quantitative content of the analysis. On the other hand, Angeletos and Calvet's CARA-normal framework also accommodates additive endowment risk, which is not the case with the CRRA framework of this paper.

Like Aiyagari (1994), this paper exogenously imposes the financial incompleteness. Complete lack of risk sharing can be the constrained efficient outcome when both income and bond trades are private information (Cole and Kocherlakota, 2001). For an alternative approach that allows for partial risk sharing in entrepreneurial activity, see Meh and Quadrini (2005).

Various authors have introduced borrowing constraints and occupational choice in the neoclassical growth paradigm to examine the impact of wealth inequality on entrepreneurial activity and capital accumulation. Some focus on the intensive margin as this paper (e.g., Bernanke and Gertler, 1989; Aghion and Bolton, 1997; Kiyotaki and Moore, 1997; Kocherlakota, 2000), others on the extensive (e.g., Banerjee and Newman, 1993; Quadrini, 2000; Cagetti and De Nardi, 2005; Buera, 2005). In these models, borrowing constraints imply a wedge between the interest rate and the marginal product of capital. A similar wedge is featured in the model of this paper: the private risk premium on investment. However, the origins and the implications of this wedge are quite different. First, the wedge here reflects a reduction in willingness, not ability, to invest; its sensitivity to wealth is due to diminishing absolute risk aversion, not collateral constraints. Second, because risk taking increases with wealth, the impact of this wedge on savings and investment, unlike that of borrowing constraints, need not vanish as agents get wealthier.

Partly motivated by this paper, Covas (2006) considers a variant in which households face borrowing constraints and have no income other than their entrepreneurial earnings. In numerical simulations, he often finds the capital stock to be higher than under complete markets. The main reason for this finding appears to be that he imposes a low EIS and no safe asset.<sup>2</sup> Covas suggests that another reason is that borrowing constraints raise precautionary saving. This intuition is correct, but only partly. Borrowing constraints also discourage risk-taking, reducing the demand for investment. There are thus two opposing effects, and it is unclear which one dominates when.

Finally, the paper is related to the literature that studies the role of rate-of-return risk in  $AK$  models (e.g., Levhari and Srinivasan, 1969; Sandmo, 1970; Obstfeld, 1994; Devereux and Smith, 1994; Krebs, 2003; Jones, Manuelli, Siu, and Stacchetti, 2005). The  $AK$  framework shares with the neoclassical framework of this paper the prediction that the impact of rate-of-return risk on savings critically depends on the EIS. However, by ignoring other sources of income beyond capital, it fails to identify the fraction of private equity as a key parameter, and it underestimates the potential for reduction in savings. Moreover, the  $AK$  framework obtains tractability only by eliminating transitional dynamics, which is not the case here.

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<sup>2</sup>The closest analogue of his exercise in my model would be to set  $\theta = 1/2$  and  $\phi = 1$  (and hence  $\bar{\theta} = 1$ ).

## 2 The Model

Time is discrete, indexed by  $t \in \{0, 1, \dots, \infty\}$ . There is a continuum of infinitely-lived households, indexed by  $i$  and distributed uniformly over  $[0, 1]$ . Each household owns a single private firm, so that firm  $i$  is identified as the firm owned by household  $i$ . Firms employ labor in a competitive labor market but use the capital stock accumulated by their respective household-owner. Households, on the other hand, are each endowed with one unit of labor, which they supply inelastically in a competitive labor market; they can invest capital in the firm they own, but in no other firm; and they can freely trade a riskless bond, but not other financial asset.

**Preferences.** I assume an Epstein-Zin specification with constant elasticity of intertemporal substitution (CEIS) and constant relative risk aversion (CRRA). A stochastic consumption stream  $\{c_t^i\}_{t=0}^\infty$  generates a stochastic utility stream  $\{u_t^i\}_{t=0}^\infty$  according to the following recursion:

$$u_t^i = U(c_t^i) + \beta \cdot U(\mathbb{C}\mathbb{E}_t[U^{-1}(u_{t+1}^i)]), \quad (1)$$

where  $\mathbb{C}\mathbb{E}_t(u_{t+1}^i) \equiv \Upsilon^{-1}[\mathbb{E}_t\Upsilon(u_{t+1}^i)]$  represents the certainty equivalent of  $u_{t+1}^i$  conditional on period- $t$  information. The utility functions  $U$  and  $\Upsilon$  aggregate consumption across dates and states, respectively; they are given by

$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta} \quad \text{and} \quad \Upsilon(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad (2)$$

where  $\theta > 0$  is the elasticity of intertemporal substitution and  $\gamma > 0$  is the coefficient of relative risk aversion.

None of the results of the paper relies on the Epstein-Zin preference specification. Standard expected utility is nested by letting  $\theta = 1/\gamma$ , in which case (1) reduces to  $u_t^i = \mathbb{E}_t \sum_{\varsigma=0}^\infty \beta^\varsigma U(c_{t+\varsigma}^i)$ . It is nevertheless important to allow  $\theta \neq 1/\gamma$  so as to clarify that the *sign* of the steady-state effects depends most critically on the elasticity of intertemporal substitution, not the coefficient of relative risk aversion. This also guides the numerical simulations conducted later.

**Budgets.** Let  $\omega_t$  denote the wage rate in period  $t$  and  $R_t$  the gross risk-free rate between periods  $t-1$  and  $t$ . The budget constraint of household  $i$  in period  $t$  is given by

$$c_t^i + k_{t+1}^i + b_{t+1}^i = \pi_t^i + R_t b_t^i + \omega_t, \quad (3)$$

where  $c_t^i$  denotes consumption,  $k_{t+1}^i$  investment in physical capital,  $b_{t+1}^i$  savings in the risk-free bond, and  $\pi_t^i$  capital income (to be specified below).<sup>3</sup> Naturally, consumption and physical capital cannot be negative:  $c_t^i \geq 0$  and  $k_{t+1}^i \geq 0$ . Finally, households can freely borrow in the riskless bond up to the “natural” solvency constraint that debt be low enough to be paid out even under the worst realization of idiosyncratic uncertainty.

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<sup>3</sup>The budget constraint in condition (3) is expressed in terms of stock variables:  $R_t$  equals 1 plus the net risk free rate and  $\pi_t^i$  includes the value of the beginning-of-period, non-depreciated capital stock installed in firm  $i$ .

**Technology and idiosyncratic risk.** The capital income of household  $i$  is given by the earnings of firm  $i$  net of labor costs:

$$\pi_t^i = y_t^i - \omega_t n_t^i, \quad (4)$$

where  $n_t^i$  denotes the amount of labor firm  $i$  hires in period  $t$  and  $y_t^i$  the gross output it produces in the same period. Output in turn is given by

$$y_t^i = F(k_t^i, n_t^i, A_t^i),$$

where  $A_t^i$  is a shock specific to firm  $i$  and  $F: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  a neoclassical production technology.<sup>4</sup>

The shock  $A_t^i$  is realized in the beginning of period  $t$ , after capital  $k_t^i$  is installed but before employment  $n_t^i$  is chosen. It is independently and identically distributed across  $i$  and  $t$ , with continuous p.d.f.  $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . In order to interpret a higher  $A_t^i$  as higher productivity (or higher profitability), I impose  $F_A > 0$ ,  $F_{KA} > 0$ , and  $F_{LA} > 0$ . I finally let  $F(K, L, 0) = 0$ , meaning that the worst idiosyncratic event leads to zero output, and normalize  $\bar{A} \equiv \int A \psi(A) dA = 1$ .

**Equilibrium.** The initial position of the economy is given by the distribution of  $(k_0^i, b_0^i)$  across households. Households choose plans  $\{c_t^i, n_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^\infty$ , contingent on the history of their idiosyncratic shocks, so as to maximize their life-time utility. Idiosyncratic uncertainty, however, washes out at the aggregate. I thus define an equilibrium as a deterministic sequence of prices  $\{\omega_t, R_t\}_{t=0}^\infty$ , a deterministic macroeconomic path  $\{C_t, K_t, Y_t\}_{t=0}^\infty$ , and a collection of contingent plans  $\{c_t^i, n_t^i, y_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^\infty$ ,  $i \in [0, 1]$ , such that the following conditions hold:<sup>5</sup>

- (i) *Optimality:*  $\{c_t^i, n_t^i, y_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^\infty$  maximizes  $u_0^i$  for every  $i$ .
- (ii) *Labor-market clearing:*  $\int_i n_t^i = 1$  in all  $t$ .
- (iii) *Bond-market clearing:*  $\int_i b_t^i = 0$  in all  $t$ .
- (iv) *Aggregation:*  $C_t = \int_i c_t^i$ ,  $Y_t = \int_i y_t^i$ ,  $K_t = \int_i k_t^i$  in all  $t$ .

### 3 Equilibrium Characterization

#### 3.1 Individual behavior

The idiosyncratic state of agent  $i$  in period  $t$  is summarized by  $(k_t^i, b_t^i, A_t^i)$ . For a given price sequence, the value function in period  $t$  can therefore be denoted by  $V(k, b, A; t)$ . Since, by the assumption  $F(K, L, 0) = 0$ , the worst possible realization of capital income is zero, the natural solvency constraint reduces to  $b_{t+1}^i \geq -h_t$ , where

$$h_t \equiv \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{R_{t+1} \dots R_{t+j}} \quad (5)$$

<sup>4</sup>That is,  $F$  exhibits constant returns to scale with respect to  $K$  and  $L$ , has positive and strictly diminishing marginal products, and satisfies the familiar Inada conditions.

<sup>5</sup>With some abuse of notation, whenever I write  $\int_i x_t^i$  for some variable  $x \in \{c, n, y, k, b\}$ , I mean the cross-sectional expectation of  $x$  in period  $t$ . I also suppress the dependence of individual variables on the history of shocks.

denotes the present value of future labor income (a.k.a. “human wealth”). The household’s problem can thus be represented by the following program:

$$\begin{aligned}
V(k, b, A; t) &= \max_{c, n, k, b} U(c) + \beta \cdot U\Upsilon^{-1} \left( \int \Upsilon U^{-1} V(k', b', A'; t+1) \psi(A') dA' \right) \\
\text{s.t.} \quad &c + k' + b' = \pi + Rb + \omega \\
&\pi = F(k, n, A) - \omega n \\
&c \geq 0 \quad k' \geq 0 \quad b' \geq -h_t
\end{aligned}$$

I solve this problem in two steps. First, I characterize the optimal labor demand and the earnings of firm  $i$ ; the latter gives me the capital income of household  $i$ . Then, I characterize the optimal consumption, saving and investment choices of household  $i$ .

Since employment  $n_t^i$  affects only earnings  $\pi_t^i$  in period  $t$ , and since it is chosen after the capital stock  $k_t^i$  has been installed and the contemporaneous shock  $A_t^i$  has been observed, the optimal  $n_t^i$  maximizes  $\pi_t^i$  state by state. By CRS, then, the optimal  $n_t^i$  and the maximal  $\pi_t^i$  are linear in  $k_t^i$ : the individual firm can always adjust its employment in proportion to its capital stock, implying that the individual household faces linear returns to private investment.

**Lemma 1** *Individual labor demand and capital income are linear in  $k_t^i$ , decreasing in  $\omega_t$ , and increasing in  $A_t^i$ :*

$$n_t^i = n(A_t^i, \omega_t) k_t^i \quad \text{and} \quad \pi_t^i = r(A_t^i, \omega_t) k_t^i, \quad (6)$$

where  $r(A, \omega) \equiv \max_L [F(1, L, A) - \omega L]$  and  $n(A, \omega) \equiv \arg \max_L [F(1, L, A) - \omega L]$ .

Let  $w_t^i \equiv \pi_t^i + R_t b_t^i + \omega_t$  denote the financial wealth of household  $i$  in period  $t$ . The budget constraint reduces to  $c_t^i + k_{t+1}^i + b_{t+1}^i = w_t^i$  and, by Lemma 1,

$$w_t^i = r(A_t^i, \omega_t) k_t^i + R_t b_t^i + \omega_t. \quad (7)$$

Note then that conditioning on  $(k_t^i, b_t^i, A_t^i)$  is useful only for evaluating the optimal employment  $n_t^i$  and earnings in condition (6), and the resulting wealth  $w_t^i$  in condition (7). The household’s consumption-saving problem thus reduces to the following:<sup>6</sup>

$$\begin{aligned}
V(w; t) &= \max_{(c, k', b') \in \mathbb{R}_+^2 \times [-h_t, \infty)} U(c) + \beta \cdot U\Upsilon^{-1} \left( \int \Upsilon U^{-1} V(w'; t+1) \psi(A') dA' \right) \\
\text{s.t.} \quad &c + k' + b' = w, \quad w' = r(A', \omega_{t+1}) k' + R_{t+1} b' + \omega_{t+1}.
\end{aligned} \quad (8)$$

This problem is formally similar to the classic portfolio problem studied by Samuelson (1969) and Merton (1969): preferences are homothetic (by assumption) and wealth is linear in assets (by Lemma 1). That the risky asset is a privately-held business rather than a publicly-traded financial

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<sup>6</sup>With obvious abuse of notation,  $V$  henceforth denotes the value function in terms of  $w$  rather than  $(k, b, A)$ .

security, that the payoff of this asset depends on the wage rate and thereby on the aggregate capital stock, or that the risk is idiosyncratic, are important for the general-equilibrium properties of the economy, but do not affect the mathematical structure of the individual decision problem.

**Lemma 2** *Given prices, optimal consumption, investment and bond holdings are linear in wealth:*

$$c_t^i = (1 - \varsigma_t)(w_t^i + h_t) \quad (9)$$

$$k_{t+1}^i = \varsigma_t \phi_t (w_t^i + h_t) \quad (10)$$

$$b_{t+1}^i = \varsigma_t (1 - \phi_t)(w_t^i + h_t) - h_t \quad (11)$$

where  $w_t^i$  and  $h_t$  are given by conditions (7) and (5), and where

$$\varsigma_t = \left\{ 1 + \left[ \sum_{\tau=t}^{\infty} \prod_{j=\tau}^{\tau} \beta^\theta \rho_j^{\theta-1} \right]^{-1} \right\}^{-1} \quad (12)$$

$$\rho_t = \rho(\omega_{t+1}, R_{t+1}) \equiv \max_{\varphi \in [0,1]} \mathbb{C}\mathbb{E}_t [\varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi)R_{t+1}] \quad (13)$$

$$\phi_t = \phi(\omega_{t+1}, R_{t+1}) \equiv \arg \max_{\varphi \in [0,1]} \mathbb{C}\mathbb{E}_t [\varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi)R_{t+1}] \quad (14)$$

To interpret the above conditions, note that the sum  $w_t^i + h_t$  represents the “effective” wealth of household  $i$ ,  $\varsigma_t$  the fraction of effective wealth that is saved,<sup>7</sup>  $\phi_t$  the fraction of savings that is allocated to capital, and  $\rho_t$  the risk-adjusted return to savings (a.k.a. the certainty equivalent of the overall portfolio return). Condition (12) follows from the Euler condition and gives the saving rate as a function of current and future risk-adjusted returns.

Conditions (13) and (14), on the other hand, mean that the allocation of savings between private equity and bonds maximizes the risk-adjusted return to savings. Note that

$$\phi_t \approx (\ln \bar{r}_{t+1} - \ln R_{t+1}) / (\gamma \sigma_{t+1}^2) \quad \text{and} \quad \rho_t \approx R_{t+1} \exp \left\{ (\ln \bar{r}_{t+1} - \ln R_{t+1})^2 / (2\gamma \sigma_{t+1}^2) \right\}, \quad (15)$$

where  $\bar{r}_{t+1} \equiv \mathbb{E}_t [r(A_{t+1}, \omega_{t+1})]$  in the mean and  $\sigma_{t+1} \equiv \text{Var}_t [\ln r(A_{t+1}, \omega_{t+1})]$  the volatility of private returns.<sup>8</sup> Assuming that  $\sigma_{t+1}$  and  $\omega_{t+1}$  are unrelated,  $\phi_t$  and  $\rho_t$  are decreasing in both  $\sigma_{t+1}$  and  $\omega_{t+1}$ . The first effect is due to risk aversion; the second is because a higher wage reduces capital income for every realization of the productivity shock.

### 3.2 General equilibrium

By Lemma 1, the fact that there is a continuum of agents, and the assumption that the shocks are i.i.d. across them, aggregate employment and capital income are given by  $N_t = \bar{n}(\omega_t)K_t$  and  $\Pi_t = \bar{r}(\omega_t)K_t$ , where  $\bar{n}(\omega) \equiv \int n(A, \omega)\psi(A)dA$  and  $\bar{r}(\omega) \equiv \int r(A, \omega)\psi(A)dA$ . It follows that the labor market clears in period  $t$  if and only if  $\omega_t = \omega(K_t)$ , where  $\omega(K) \equiv \bar{n}^{-1}(1/K)$ . Similarly,

<sup>7</sup>Note that  $\varsigma_t$  is ratio of the stock of savings to the stock of wealth; this is different from the ratio of flow-savings to flow-income, which is the standard definition of the “saving rate”.

<sup>8</sup>See Appendix for the derivation of 15; this condition is exact in the continuous-time variant of the model.

aggregate gross output (including the value of non-depreciated capital) is given by  $Y_t = \Pi_t + \omega_t = f(K_t)$ , where  $f(K) \equiv \bar{r}(\omega(K))K + \omega(K)$ . By Lemma 2, in turn, consumption, bond holdings, and private investment are linear in individual wealth and therefore the corresponding aggregates are not affected by wealth inequality. Using these properties, and aggregating across agents, we arrive at the following closed-form recursive characterization of the general equilibrium.

**Proposition 1 (General Equilibrium)** *In equilibrium, the aggregate dynamics satisfy*

$$C_t + K_{t+1} = Y_t = f(K_t) \quad (16)$$

$$C_t = (1 - \varsigma_t) [f(K_t) + H_t] \quad (17)$$

$$(1 - \varsigma_t)^{-1} = 1 + \beta^\theta \rho_{t+1}^{\theta-1} (1 - \varsigma_{t+1})^{-1} \quad (18)$$

$$K_{t+1} = \phi_t \varsigma_t [f(K_t) + H_t] \quad (19)$$

$$\bar{n}(\omega_t) K_t = 1 \quad (20)$$

$$H_t = \frac{\omega_{t+1} + H_{t+1}}{R_{t+1}} \quad (21)$$

where  $\phi_t = \phi(\omega_{t+1}, R_{t+1})$  and  $\rho_t = \rho(\omega_{t+1}, R_{t+1})$ .

The interpretation of this result is straightforward. Condition (16) is the resource constraint. Condition (17) gives aggregate consumption, and condition (18) the associated Euler condition. Condition (19) gives the aggregate capital stock, and condition (20) the clearing condition for the labor market. Finally, condition (21) is the present value of aggregate labor income.

Note that the same system characterizes the complete-markets equilibrium, with only one modification: since investment is risk-free under complete markets, the optimal portfolio condition  $\phi_t = \phi(\omega_{t+1}, R_{t+1})$  reduces to the arbitrage condition  $R_{t+1} = f'(K_{t+1})$ ; and similarly the return to savings reduces to  $\rho_t = R_{t+1} = f'(K_{t+1})$ .

Also note that the equilibrium system is recursive in  $(K_t, H_t, \varsigma_t)$ . To see this, use conditions (16), (20) and (21) to eliminate  $C_t$ ,  $\omega_t$ , and  $R_{t+1}$ . The equilibrium dynamics then reduce to a three-dimensional, first-order, difference-equation system in  $(K_t, H_t, \varsigma_t)$ . If  $\theta = 1$  (logarithmic utility), then  $\varsigma_t = \beta$  for all  $t$  and the dimensionality further reduces to two. This means a dramatic gain in tractability as compared to other incomplete-market models (e.g., Aiyagari, 1994, Krusell and Smith, 1998), where the entire wealth distribution—an infinitely-dimensional object—is a relevant state variable for the aggregate dynamics.

The exact-aggregation result presented above is also interesting for the following reason. With diminishing absolute risk aversion, the precautionary motive associated with labor-income risk leads to strict concavity in the consumption function; the same is true for borrowing constraints. Such non-linearity in individual policy rules is important because it breaks the representative-agent framework and makes wealth inequality matter for aggregates. How pronounced these effects are is a debatable quantitative issue (see, for example, the discussions in Krusell and Smith (1997) and

Carroll (2000)). However, whether an economy admits exact aggregation—as in the model of this paper—or approximate aggregation—as in Krusell and Smith (1997)—is not necessarily related to how significant the macroeconomic impact of idiosyncratic risk is. As the numerical results in the next section illustrate, in my model exact aggregation is consistent with significant quantitative effects on aggregate savings and income.

## 4 Steady State

### 4.1 Characterization

A steady state is a fixed point of the dynamic system (16)-(21).<sup>9</sup> Since the general equilibrium is characterized in closed form for any kind of idiosyncratic risk, so is the steady state as well. For expositional purposes, however, it is useful to assume that the shock is lognormally distributed and augmented to capital.<sup>10</sup> I thus impose the following for the rest of the analysis.

**Assumption A1.**  $F(K, L, A) = F(AK, L, 1)$  and  $\ln A \sim \mathcal{N}(-\sigma^2/2, \sigma^2)$ .

The first part implies that  $f(K) = F(K, 1, 1)$ ,  $\bar{r}(\omega(K)) = f'(K)$ , and  $r(A, \omega(K)) = Af'(K)$ , for every  $A$  and  $K$ ; a mean preserving spread in  $A$  is thus equivalent to a mean-preserving spread in individual returns. The second part of Assumption A1 then parsimoniously parameterizes the amount of uninsured idiosyncratic risk with  $\sigma$ .

**Proposition 2 (Steady State)** *In steady state, the capital stock  $K$  and the interest rate  $R$  solve*

$$\beta^\theta \rho^{\theta-1} [\phi f'(K) + (1 - \phi)R] = 1 \quad (22)$$

$$\frac{f(K) - f'(K)K}{(R - 1)K} = \frac{1 - \phi}{\phi} \quad (23)$$

where  $\phi = \phi(\omega(K), R)$  and  $\rho = \rho(\omega(K), R)$ .

Condition (22) follows from combining the resource constraint with the Euler condition and has a simple interpretation. The first term in the left-hand side of condition (22),  $\beta^\theta \rho^{\theta-1} = \varsigma$ , is the steady-state value of the savings-to-wealth ratio; this is increasing (respectively, decreasing) in the risk-adjusted return  $\rho$  if and only if  $\theta > 1$  (respectively,  $\theta < 1$ ) and reduces to  $\varsigma = \beta$  when  $\theta = 1$ . The second term,  $\phi f'(K) + (1 - \phi)R$ , represents the aggregate return to savings; this is a weighted average of the marginal product of capital and the risk-free rate. The product of these two terms gives the growth rate of aggregate effective wealth. In steady state, aggregate wealth must be constant, which gives condition (22).

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<sup>9</sup>Note that, although there is a steady state for the aggregates, there is no steady state for the cross section: at steady-state prices, the log of individual wealth follows a random walk. See Section 6 for a modification of the model that admits a stationary distribution of wealth.

<sup>10</sup>One can then interpret the shock also as random capital depreciation.

Condition (23), on the other hand, follows from clearing the bond market. In particular, it follows from combining  $B = 0$  with  $K = \phi\zeta(W + H)$  and  $B + H = (1 - \phi)\zeta(W + H)$ —the aggregate versions of conditions (10) and (11)—to get  $(1 - \phi)/\phi = H/K$ . This condition thus requires that the individuals' optimal  $\phi$  choices be consistent with the equilibrium ratio of the present value of labor income to the capital stock. (As shown in Proposition 3, this condition allows me to directly calibrate  $\phi$  to the data.)

When markets are complete, the optimality condition for  $\phi$  reduces to the familiar arbitrage condition  $f'(K) = R$ , while condition (22) reduces to  $R = 1/\beta$  and then condition (23) pins down  $\phi$ . When instead markets are incomplete, condition (22) together with  $\phi = \phi(\omega(K), R)$  gives the steady-state capital stock  $K$  as a function of  $R$ , and condition (23) can then be solved for the steady-state interest rate  $R$ .

Clearly, it must be that  $R < 1/\beta$ ; if the interest rate were higher or equal to the discount rate, aggregate consumption would diverge to infinity (contradicting the existence of steady state). This is true for the same reason as in Aiyagari (1994). However, whereas in Aiyagari (1994) the absence of investment risk imposed  $f'(K) = R$ , here it must be that  $f'(K) > R$ , or otherwise agents would hold no capital in equilibrium (and a steady state would again fail to exist). In other words, there is a tension between the precautionary motive for savings and the risk premium on investment: the one effect pushes the interest rate below the discount rate, but the other pushes the marginal product of capital above the interest rate. As a result, whereas in Aiyagari (1994) it was necessarily true that  $f'(K) < 1/\beta$ , so that the capital stock was necessarily higher than under complete markets, here it is possible that  $f'(K) > 1/\beta$  even though  $R < 1/\beta$ , so that the impact of incomplete markets on the capital stock is ambiguous in general.

Can we find a condition that resolves this ambiguity? The following result provides the answer: the steady-state effect on savings and income is negative if and only if the elasticity of intertemporal substitution (EIS) is sufficiently high relative to the fraction of wealth invested in private equity.

**Proposition 3** (i) *For any economy  $\mathcal{E} = (\sigma, \beta, \gamma, \theta, F)$  with  $\sigma > 0$ , there exists  $\underline{\theta} = \underline{\theta}(\mathcal{E}) < 1$  such that the steady-state levels of capital, output, and consumption are lower under incomplete markets if and only if  $\theta > \underline{\theta}$ .*

(ii) *In the continuous-time version of the model,*

$$\underline{\theta} = \frac{\phi}{2 - \phi} < \phi. \quad (24)$$

(iii) *Let  $\alpha$  denote the income share of capital and  $s$  the saving rate, evaluated in steady state.<sup>11</sup> The steady-state value of  $\phi$  satisfies*

$$\phi \leq \frac{\alpha - s}{1 - s}, \quad (25)$$

*with equality only when  $\sigma = 0$ .*

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<sup>11</sup>That is,  $\alpha \equiv \hat{f}'(K) K / \hat{f}(K)$  and  $s = I / \hat{f}(K)$ , where  $\hat{f}(K) = f(K) - (1 - \delta)K$  is GDP and  $I = \hat{f}(K) - C = \delta K$  is gross investment, both evaluated at steady state. Note that  $s \neq \zeta$  and that  $s > (=) 0$  if  $\delta > (=) 0$ .

Part (i) states that incomplete markets lead to lower capital if and only if the EIS is higher than a threshold  $\underline{\theta}$ . Part (ii) then characterizes this threshold as an increasing function of  $\phi$ , the fraction of private equity in total wealth. The formula (24) is exact only in the continuous-time limit of the model.<sup>12</sup> However, simulations suggest that this is an extremely good approximation for the discrete-time model as well (at least for plausible parameters). Finally, part (iii) gives an upper bound for the equilibrium value of  $\phi$ . This bound becomes binding only when  $\sigma = 0$ ; but it is also nearly binding for all plausible parameter values. Therefore, for all practical reasons, we can think of  $\underline{\theta} \approx \phi/(2 - \phi)$  and  $\phi \approx (\alpha - s)/(1 - s)$  as very good approximations.

The key insight here is that the critical value for the EIS increases with  $\phi$ . This is easy to explain. An increase in risk  $\sigma$  implies a reduction in the risk-adjusted rate of return  $\rho$ . As in the case of a deterministic savings problem, this has opposing income and substitution effects: the substitution effect contributes to lower savings, while the income effect to higher. The strength of the substitution effect is governed by  $\theta$ : the higher the EIS, the stronger the substitution effect. The strength of the income effect, on the other hand, depends on  $\phi$ : the smaller the fraction of an asset to total wealth, the weaker the income effect of an increase in the return of that asset. The overall effect of risk on aggregate saving thus depends on the relation between  $\theta$  and  $\phi$ .

This result is closely related to the one about the role of rate-of-return risk in simple decision-theoretic or *AK* models (e.g., Levhari and Srinivasan, 1969; Sandmo, 1970; Obstfeld, 1994; Devreux and Smith, 1994). An important difference, however, emerges in the equilibrium determination of  $\phi$ , which is reflected in part (iii) of the proposition. In *AK* models, risky capital accounts for the entire wealth of a household, so that  $\phi = 1$ . In the neoclassical economy of this paper, instead, risky capital is only a fraction of effective wealth, so that  $\phi < 1$ . In particular,  $\phi \rightarrow 1$  and  $\underline{\theta} \rightarrow 1$  as  $\alpha \rightarrow 1$ , but  $\underline{\theta} < \phi < \alpha$  for any  $\alpha < 1$ . In other words, it is the existence of labor income—or other income beyond private equity, such as income from land properties and public equity—that explains why the critical threshold for the EIS is lower here than in *AK* models.

Another important difference is that here there is a novel multiplier effect, which affects the magnitude (although not the sign) of the steady-state effects. This multiplier effect originates in the general-equilibrium interaction of wealth and risk taking. As an increase in idiosyncratic risk makes all agents cut back on their investments, wages fall in equilibrium. Because of diminishing absolute risk aversion, the reduction in the present value of wages discourages risk taking and triggers a further reduction in investment. This in turn causes a further reduction in wages, and so on. This interaction between wealth and risk taking introduces a macroeconomic complementarity that tends to increase the overall impact of market incompleteness. This complementarity is totally absent in *AK* models, and may also have interesting implications for business cycles.<sup>13</sup>

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<sup>12</sup>A previous version of the paper had misstated this formula as  $\underline{\theta} = \phi$  instead of  $\underline{\theta} = \phi/(2 - \phi)$ . I thank Vasia Panousi for spotting the mistake, and for working out together the continuous-time version of the model.

<sup>13</sup>This discussion presumes  $\theta > \underline{\theta}$ ; if instead  $\theta < \underline{\theta}$ , the multiplier works in the opposite direction, amplifying the increase of savings. Also note that this multiplier effect is cleanest when  $R$  is fixed—as in the case of a small open economy—for then a reduction the steady-state wage translates one-to-one to a reduction in the steady-state  $H$ . The endogenous adjustment of  $R$  in the closed economy of this paper tends to mitigate this effect.

## 4.2 Extension: hand-to-mouth workers

At this point it is important to generalize the formula for  $\underline{\theta}$  in a way that takes the model a step closer to capturing the significant heterogeneity in wealth and investment choices across the population. One way to do this without complicating the analysis is to separate the population into two groups. The one group consists of the households that we have modeled so far. The other consists of households that supply labor but do not hold any assets—they simply work and consume their entire labor income in every period. Think of the first group as “investors”, and the second as “hand-to-mouth workers”.

Hand-to-mouth workers do not hold any assets in the steady state either by assumption (e.g., they live one period), or because they have a discount rate higher than the steady-state interest rate (e.g., the same discount rate as investors) but cannot borrow. In any case, they serve a single purpose in the model: they absorb a fraction  $\xi \in (0, 1)$  of aggregate human wealth.<sup>14</sup>

As shown in the Appendix, parts (i) and (ii) of Proposition 2 continue to hold in this extension, provided we reinterpret  $\phi$  as the ratio of private equity to the total effective wealth of the investors rather than that of the entire population. That is, while  $\phi = K/(W + H)$  in the benchmark model, here  $\phi = K/(W + (1 - \xi)H)$ . Accordingly, while the critical threshold for the EIS remains  $\underline{\theta} \approx \phi/(2 - \phi)$ , the steady-state value of  $\phi$  becomes

$$\phi \approx \left( (1 - \xi) \left( \frac{\alpha - s}{1 - s} \right)^{-1} + \xi \right)^{-1}, \quad (26)$$

where  $\alpha$  and  $s$  are again the income share of capital and the saving rate.

Not surprisingly,  $\underline{\theta}$  increases with  $\xi$ , meaning that under-accumulation becomes less likely the higher the fraction of non-proprietary income absorbed by hand-to-mouth (non-investing) agents. At the one extreme,  $\xi = 0$  nests the benchmark case (condition (24)); at the other,  $\xi = 1$  nests the  $AK$  case in the sense that the threshold for the EIS becomes 1 when investors have no source of income other than their private equity. However, as we discuss in the next section, for plausible values of  $\xi$  the threshold  $\underline{\theta}$  remains quite below 1.

## 4.3 Empirically relevant case?

Assessing the quantitative impact of labor-income risk and precautionary savings has been the subject of a long debate (e.g., Zeldes, 1989; Caballero, 1990; Aiyagari, 1994; Krusell and Smith, 1998; Carroll, 2000; Gourinchas and Parker, 2001), but the qualitative effect is unambiguous: at least in the context of the neoclassical growth model, idiosyncratic labor-income risk necessarily

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<sup>14</sup>If we relax the borrowing constraint of these agents so that they can borrow up to a fraction of their income, then this constraint will bind in steady state. This in turn reduces the fraction  $\xi$  of labor income that these agents consume in the steady state, but otherwise leaves the analysis unaffected. For example, if these agents face only their natural borrowing constraint, then  $\xi$  would be zero, and the steady state would be the same as in the economy without hand-to-mouth workers. Of course, things would not have been that simple if these agents faced labor-income risk; their savings in steady state would then be determined as in Aiyagari (1994).

increases savings as compared to complete markets. With idiosyncratic investment risk, the case is more delicate—before arguing about magnitudes, one has to settle about signs. In this section I assess whether a negative or a positive effect on capital is more plausible empirically (within the context, of course, of the model). For this purpose, I need an estimate for the EIS  $\theta$ , and an estimate for the threshold  $\underline{\theta}$ , or, equivalently, for the fraction  $\phi$  of private equity to total wealth.

Consider first  $\underline{\theta}$ . Let the income share of capital be  $\alpha = 36\%$  and the saving rate  $s = 23\%$ , as in US data.<sup>15</sup> If we consider the benchmark model where all households are (potentially) investors, then from condition (25) we get  $\phi \approx 0.17$  and hence  $\underline{\theta} \approx 0.09$ . If, instead, we allow for hand-to-mouth workers, we need to pick a value for  $\xi$ . My approach then is to calibrate  $\xi$  so that the fraction of aggregate consumption that is accounted by hand-to-mouth workers in my model matches the corresponding estimate in Campbell and Mankiw (1989), namely 50%. Doing so gives  $\phi \approx 0.33$ , and hence  $\underline{\theta} \approx 0.2$ . Even if I allow  $\xi$  to be high enough that hand-to-mouth consumers account for 75% of aggregate consumption,  $\underline{\theta}$  does not exceed 0.5.

Some information about the contribution of private equity to the wealth of the typical US investor, and hence about the threshold  $\underline{\theta}$ , is also contained in the data. Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002) document that private equity accounts for about one half of financial wealth, whether one looks at the entire US economy, or at the portfolio of the rich 2%-20% (who account for the bulk of US wealth). Adjusting this for human wealth implies that 0.5 is an upper bound for  $\phi$  and therefore for  $\underline{\theta}$  as well.<sup>16</sup>

Consider next  $\theta$ . The empirical estimates of the EIS vary a lot. Hall (1988) and Campbell and Mankiw (1989) obtain very low estimates using US macro data on aggregate consumption growth and T-bill rates. However, estimates of this kind suffer from aggregation bias and also confound heterogeneity in the population. For example, using British data, Attanasio and Weber (1993) show that correcting for aggregation bias raises the estimated EIS from about 0.3 to about 0.7. Moreover, the estimated EIS tends to increase with wealth and/or asset holdings (Blundell, Browning and Meghir, 1994; Attanasio and Browning, 1995; Vissing-Jørgensen, 2002; Attanasio, Banks and Tanner, 2002). The exact estimates vary a lot across studies and different specifications, but in most cases they are around 1, especially if one concentrates on the top layers of wealth or asset holdings. For example, using data from the US Consumer Expenditure Survey (CEX) and an Epstein-Zin specification as in this paper, Vissing-Jørgensen and Attanasio (2003) report baseline estimates between 1 and 1.4 for stockholders.

What seems most relevant for the purposes of this paper is the estimated EIS for rich households, for they hold almost the entire capital stock in the US economy and they are the most natural counterpart in the data of what “investors” are in the model. Indeed, if poor households are well

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<sup>15</sup>The value  $s = 23\%$  is also the value predicted by the complete-markets steady state of the model under the benchmark parametrization, which is the same as that of Aiyagari (1994); see the next subsection for details.

<sup>16</sup>In the model analyzed so far, private equity accounts for the entire financial wealth. This however can be relaxed either by letting the safe asset to be in exogenously fixed positive supply, or by introduce a public-equity sector; see Section 5.

approximated by the “hand-to-mouth workers” of the model, then their EIS is utterly irrelevant. This logic together with the empirical evidence reviewed above suggests setting  $\theta$  around 1.<sup>17</sup>

The case for a high  $\theta$  is further supported by the following observations. The aforementioned empirical studies—with either macro or micro data—typically attempt to identify the EIS by instrumenting current rates of return with lagged rates of return. This identification strategy, however, is invalid if persistent macroeconomic shocks impose a correlation between interest rates and the precautionary motive (the residual in the linearized Euler equations estimated in these studies). An arguably better identification strategy is to instrument real returns with predictable variation in tax rates on capital income. This strategy, which is employed by Mulligan (2002) for macro data and by Gruber (2005) for micro data, delivers consistently high point estimates—indeed closer to 2. What is more, unlike many other studies, Gruber’s point estimates are quite robust to different specifications and different choices on whether one measures the rate of return with T-bill rates or with stock-market returns.<sup>18</sup>

To recap, the most likely scenario appears to be that  $\theta > \underline{\theta}$ , and therefore that idiosyncratic investment risk leads to lower aggregate savings and income. At the same time, a broader lesson that comes out of the analysis here is that the EIS and the fraction of private equity to total wealth are key variables for evaluating the impact of idiosyncratic investment risk, presumably not only within the simple model of this paper but also within richer quantitative models.

#### 4.4 Numerical simulation

Having discussed what is the most likely qualitative scenario for the steady-state effects of idiosyncratic investment risk, I now turn to a quantitative evaluation of these effects.<sup>19</sup>

For this purpose, I assume that the technology is Cobb-Douglas, so that  $f(k) = k^\alpha + (1 - \delta)k$ . The economy is then fully parameterized by  $(\beta, \gamma, \theta, \alpha, \delta, \sigma, \xi)$ , where  $\beta$  is the discount factor,  $\gamma$  is the coefficient of relative risk aversion,  $\theta$  is the elasticity of intertemporal substitution,  $\alpha$  is the income share of capital,  $\delta$  is the depreciation rates,  $\sigma$  is the standard deviation of idiosyncratic returns, and  $\xi$  is the fraction of labor income absorbed by hand-to-mouth workers. For any such vector of parameters, numerical solution of the steady state is trivial: substituting  $\phi = \phi(\omega(K), R)$  and

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<sup>17</sup>Consistent with this logic are Guvenen’s (2005) results. He augments a standard RBC model for heterogeneity in stock-market participation and in intertemporal substitution preferences. In particular, he assumes that the EIS is 1 for (wealthy) stockholders and 0.1 for (poor) non-stockholders. He then shows that the poor are important for the business-cycle properties of aggregate consumption but virtually irrelevant for those of aggregate investment and output: a Hall-type regression based on aggregate consumption data generated by the model gives estimated EIS around 0.25; but aggregate investment and output are better approximated by a representative-agent economy with an EIS equal to 1.

<sup>18</sup>However, the standard errors are often large enough that one cannot rule out a significantly lower EIS. For example, Gruber’s baseline point estimate is 2.032 with a standard error of 0.796 (see Table 2, “Tax IV” row, “T-bill rate” column, in his paper).

<sup>19</sup>Levine and Zame (2002) have shown that the effect of market incompleteness in Bewley economies vanishes as the discount rate converges to zero. An analogue of this limit result appears to hold here: as  $\beta \rightarrow 1$ , conditions (22) and (23) can hold only if  $R \rightarrow 1$  and  $\phi \rightarrow 0$ , which in turn implies that the risk premium vanishes. Nevertheless, the simulations presented below deliver significant quantitative effects with low discount rates and modest i.i.d. shocks.

$\rho = \rho(\omega(K), R)$  into conditions (22)-(23) gives a simple system of two equations in two unknowns, the steady-state levels of  $K$  and  $R$ .

I interpret the time period as one year and let  $\beta = 0.96$ ,  $\alpha = 0.36$ , and  $\delta = 0.08$ . These values are standard in the literature. They are indeed identical to those assumed in Aiyagari (1994). They also ensure that the steady-state values of the output-to-capital ratio, the investment-to-GDP rate, and the interest rate predicted by the model are close to their empirical counterparts in US data.<sup>20</sup>

For my baseline parametrization, I also set  $\gamma = 3$ , and then consider  $\gamma = 1$  and  $\gamma = 5$  for robustness. These values are again standard in the literature and identical to those in Aiyagari (1994). Where I deviate from the norm is that I allow the EIS to differ from the reciprocal of the coefficient of relative risk aversion: following the discussion in the previous section, I set  $\theta = 1$  for my baseline parametrization, but then consider  $\theta$  as low as  $1/3$  or as high as  $2$  for robustness. Finally, I set  $\xi = 0.6$ , which is the value that makes hand-to-mouth workers account for about 50% of aggregate consumption.

What remains is  $\sigma$ , the standard deviation of private returns.<sup>21</sup> There are various indications that idiosyncratic investment risks are significant for the typical investor in the US economy. The probability that a privately-held firm survives 5 years after entry is less than 40 percent. The variation of returns in the cross-section of private investors is very large even conditional on survival. The estimated value of private equity in the United States is about as high as the value of public equity today; and it was about twice as large in the 70's and 80's. More than 75% of aggregate private equity is owned by households for whom private equity constitutes at least half of their total net worth. The median rich household (top 5%-10%) holds almost half of his total net worth, or almost 60% of his non-housing net wealth, in private equity; and more than 70% of this is invested in a single company in which the household has an active management interest.<sup>22</sup>

Unfortunately, however, there are no precise estimates of the level of idiosyncratic investment risk. For example, Moskowitz and Vissing-Jørgensen (2002) and Bitler, Moskowitz, and Vissing-Jørgensen (2005) explore the cross-section of private-equity investors in the Survey of Consumer Finances, but are unable to provide a reliable measure for the risk faced by individual investors due to lack of enough time-series variation in the data. For their numerical exercises, they instead proxy  $\sigma$  with the standard deviation of the annual return to an individual publicly-traded stock, which is about 50% (Campbell *et al.*, 2001).

One possibility is that publicly-held firms are willing to engage in more risky projects than privately-held firms. Another possibility, however, is that privately-held firms, being on average younger and smaller, face higher bankruptcy rates and more volatile return than publicly-held firms. Based on these observations, I choose the conservative value of  $\sigma = 20\%$  for my baseline

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<sup>20</sup>With complete markets, the output-to-capital ratio is 4, the investment-to-GDP rate is 23%, and the interest rate is 4.2%. With incomplete markets, but for plausible  $\sigma$ , these values change, but not much.

<sup>21</sup>Note that, under Assumption A1,  $\text{Var}_t[\ln r(A_{t+1}, \omega_{t+1})] = \sigma^2$ .

<sup>22</sup>For further details on these facts, see Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002).

$\sigma$	$\gamma$	$\theta$	$\beta$	$\alpha$	$\delta$	$\xi$	$\Delta(\text{Saving Rate})$	$\Delta(\text{GDP})$	<b>Interest Rate</b>	<b>Private Premium</b>
20%	3	1	.96	.36	.08	.60	<b>-3.8%</b>	<b>-9%</b>	<b>3.44%</b>	<b>2.97%</b>
40%							-8.4%	-22%	2.81%	7.92%
20%						0	-2.6%	-6%	3.96%	1.66%
40%						-6.4%	-16%	3.64%	4.96%	
20%	5	2	.94	.50	.05	.60	-5.3%	-13%	3.19%	4.42%
	1						-1.5%	-4%	3.85%	1.15%
							-4.1%	-10%	3.6%	3.04%
						1/3	-2.6%	-6%	2.91%	2.72%
							-3.3%	-9%	4.86%	3.62%
							-6.1%	-19%	2.67%	4.23%
						-4.2%	-13%	3.16%	3.47%	

Table 1: Steady-state effects in the benchmark model.

parametrization, but also consider  $\sigma = 40\%$  for comparison.<sup>23</sup>

The results are reported in Table 1. Each row of the table gives the reduction in the saving rate and in the level of output relative to complete markets, the interest rate, and the associated premium on private investment.<sup>24</sup>

Under the baseline parametrization (first row in Table 1), the steady-state saving rate falls by 3.8 percentage points, from 23.6% under complete markets to 19.8% under incomplete markets. Consequently, aggregate capital falls by 24% and aggregate income by 9%, while the risk-free rate falls from 4.2% to 3.4%. Finally, the associated risk premium on private equity is 3%. and hence the marginal product of capital is now 6.4%, whereas it was 4.2% under complete markets.

Not surprisingly, the effects are much stronger if  $\sigma = 40\%$ : the saving rate falls by 8.4 percentage

<sup>23</sup>One way to appreciate that these values are modest is to translate them to a comparable measure of labor income risk. For this purpose, introduce a small measure of workers in the model whose earnings follow an AR(1) process with mean equal to the steady-state wage  $\omega$ , and let  $\rho_L$  denote the serial correlation of earnings and  $\sigma_L$  the standard deviation of the innovation in earnings—this is the process assumed in Aiyagari (1994). Fix  $\rho_L = .3$  (which, as discussed in Aiyagari, is broadly consistent with available empirical evidence) and let  $\bar{\sigma}_H(\sigma_L)$  denote the conditional variance of human wealth for the mean worker (i.e., for a worker whose labor-income realization at  $t$  is equal to the steady-state  $\omega$ ). Similarly, let  $\bar{\sigma}_K(\sigma)$  denote the conditional variance of financial wealth for the mean investor (i.e., for an investor whose capital at  $t$  is equal to the steady-state  $K$ ). Now ask the following question: Given any  $\sigma$ , what is the value for  $\sigma_L$  for which  $\bar{\sigma}_H(\sigma_L) = \bar{\sigma}_K(\sigma)$ ? That is, what is the level of labor-income risk that generates a comparable measure of wealth risk for the average worker as  $\sigma$  generates for the average investor? When  $\sigma = 20\%$ , the answer is a  $\sigma_L$  equal to about 25% of the average wage. Aiyagari (1994), based on US data, calibrates the value of  $\sigma_L$  between 20% and 40%. Hence, although the above translation is very crude, it gives a sense that the level of investment risk assumed here is comparable to the level of labor income risk assumed in Aiyagari.

<sup>24</sup>For all quantitative results, income is measured by  $GDP \equiv \hat{f}(K) \equiv f(K) - (1 - \delta)K = K^\alpha L^{1-\alpha}$ ; the risk-free rate and the mean excess return on private equity by  $R - 1$  and  $\bar{r} - R$ , respectively; and the saving rate by  $s = I/GDP = \delta K/GDP$ . For comparison, note that under complete markets  $s = 23.6\%$  and  $R - 1 = \bar{r} - 1 = 4.2\%$ .

points and the level of income by 22%, while the risk-free rate drops to 2.8% and the private premium jumps to 7.9%. On the other hand, the effects are weaker if we assume no hand-to-mouth workers ( $\xi = 0$ ): with  $\sigma = 20\%$ , the saving rate falls by 2.6 percentage points, the level of income falls by 6%, the interest rate is 4% and the private equity premium is only 1.7%. This is because when  $\xi = 0$  the “investors” have more human wealth and hence they are more willing to take risk.

The losses in aggregate savings and income are higher the higher risk aversion, but remain significant even when  $\gamma$  is as low as 1. The losses are also higher if the depreciation rate  $\delta$  is 5% instead of 8%, but remain virtually the same if the discount rate is raised from 4.2% to 6%. Moreover, the losses are almost doubled if the income share of capital  $\alpha$  is raised from 36% to 50%; to the extent that a higher  $\alpha$  is interpreted as a broader definition of capital, this is indicative of the potential importance of idiosyncratic risk in entrepreneurial human capital.

Perhaps more surprising, given the theoretical importance of the EIS, is the finding that the losses remain significant even when  $\theta$  is as low as  $1/3$ . This suggests that the impact of the EIS is highly non-linear: the losses are close to zero when  $\theta$  is close to  $\underline{\theta}$  (which in the baseline parametrization is about 0.2), but they do not vary that much as  $\theta$  varies between  $1/3$  and 2. Since a value of  $1/3$  is a lower bound for almost all empirical estimates of the EIS, and a value of 2 is closer to the IV estimates of Mulligan (2002) and Gruber (2005), these findings suggest that significant quantitative effects are comfortably obtained within the model for the empirically plausible range of parameters.

Finally, the effects reported here are, not only qualitatively different, but also quantitatively stronger than those reported in Aiyagari (1994) for labor-income risk. For example, consider one of the baseline specifications in Aiyagari, which sets  $\alpha = .36, \delta = .08, \beta = .96$ , and  $\gamma = 3 = 1/\theta$ , and calibrates an AR(1) process for labor earnings with a serial correlation of 0.3 and a standard deviation of 20%. The implied increase in the saving rate reported in Aiyagari is around 0.24 percentage points. In contrast, for a comparable specification (see the 8th row in Table 1, the one for  $\theta = 1/3$ ) the reduction in the saving rate predicted by my model is around 2.6 percentage points—a change, not only of the opposite sign, but also ten-fold bigger.

Of course, my model and that of Aiyagari are not directly comparable. However, as noted earlier in footnote 23, the levels of effective wealth risk assumed in the two cases are quite similar. Hence, the reason that I find bigger steady-state effects is not that I assumed bigger exogenous risks. Rather, the explanation is likely to lie on the combination of two equilibrium forces that operate in my economy but not in that of Aiyagari: first, exposure to risk does not vanish as agents accumulate wealth; and second, as explained earlier, the general-equilibrium interaction between wealth of risk-taking amplifies the steady-state effects of incomplete markets. How important these forces might be in richer quantitative models remains an open question.

## 5 Two Sectors: Private and Public Equity

The analysis so far has assumed that all investment is subject to idiosyncratic risk. This is not necessarily a bad benchmark for less developed economies, in which production is dominated by privately-held firms. Nevertheless, it is important to understand the robustness of the results to the availability of a safe asset that is in positive net supply. For this purpose, I first consider a short-cut where the safe asset is a Lucas tree; I then examine there is a second sector of production, to be identified with public equity.

### 5.1 Equilibrium with a Lucas tree

Suppose there is a Lucas tree that produces a fixed dividend  $D > 0$  each period and let  $p_t$  denote the period- $t$  price of a claim on this tree. The budget constraint of agent  $i$  is now given by

$$c_t^i + k_{t+1}^i + b_{t+1}^i + p_t x_{t+1}^i = \pi_t^i + R_t b_t^i + (D + p_t)x_t^i + \omega_t,$$

where  $x_t^i$  denotes the fraction of tree claims held by agent  $i$  in period  $t$  (with  $\int_i x_t^i = 1$ ). In equilibrium, the price of the tree must equal the present value of its dividend:

$$p_t = \sum_{j=1}^{\infty} \frac{D}{R_{t+1} \dots R_{t+j}}.$$

Let  $P_t \equiv p_t + D$  denote the value of the tree inclusive of the current dividend. Proposition 1 continues to hold once we replace  $H_t$  in conditions (17) and (19) with  $H_t + P_t$ . That is, we simply need to include the value of the tree in effective wealth. Propositions 2 and 3 extend in a similar way: the threshold for the EIS remains  $\underline{\theta} \approx \phi/(2 - \phi)$ , but now  $\phi$  satisfies  $(1 - \phi)/\phi = (H + P)/K$ , with  $P = RD/(R - 1)$ . Equivalently, condition (26) becomes

$$\phi \approx \left( (1 - \xi) \left( \frac{\alpha - s}{1 - \alpha} \right)^{-1} + \left( \frac{\kappa}{1 - \kappa} \right)^{-1} \right)^{-1} \leq \min \left\{ \kappa, \left( (1 - \xi) \left( \frac{\alpha - s}{1 - s} \right)^{-1} + \xi \right)^{-1} \right\}, \quad (27)$$

where  $\kappa \equiv K/(K + P)$  denotes the ratio of private equity to financial wealth.

Although  $\kappa$  is endogenously determined in the model, it can be directly calibrated to available data. As mentioned earlier, private equity is about one half of financial wealth in the United States, so that  $\kappa \approx 50\%$ . Together with  $s \approx 23\%$  and  $\alpha \approx 36\%$ , this implies that the threshold for the EIS is  $\underline{\theta} \approx 0.08$  when  $\xi = 0$  (no hand-to-mouth workers) and  $\underline{\theta} \approx 0.14$  when  $\xi = 0.6$  (hand-to-mouth workers that account for one half of aggregate consumption). By comparison, in the benchmark model ( $\kappa = 1$ ), the threshold for the EIS was  $\underline{\theta} \approx 0.09$  when  $\xi = 0$  and  $\underline{\theta} \approx 0.2$  when  $\xi = 0.6$ .

That  $\underline{\theta}$  is always *lower* with the tree than without it is evident in condition (27). This is because the tree reduces the contribution of risky private equity to total wealth, making it easier for the substitution effect of higher risk to dominate the corresponding wealth effect. In other words, the introduction of a safe asset only makes it more likely that  $K$  is lower under incomplete markets.

The exercise conducted here, however, exogenously fixes the supply of the safe asset. This might be an important limitation if the level of risk in the economy affects the size of this asset—as in the case of Aiyagari (1994). For this reason, I now replace the exogenous tree with a second sector of production, which employs both capital and labor but is not subject to any undiversified investment risk. I henceforth identify this sector with public equity.

## 5.2 Equilibrium with public equity

Let  $X_t$  and  $L_t$  denote the total capital and labor allocated to the public-equity sector in period  $t$ . Total output for this sector is given by  $G(X_t, L_t)$ , where  $G$  is a neoclassical production function. Clearly, in equilibrium the wage and the risk-free rate must satisfy  $\omega_t = G_L(X_t, L_t)$  and  $R_t = G_X(X_t, L_t)$ . The rest of the equilibrium characterization is as in the benchmark model, replacing bond holdings with the sum of bond and public-equity holdings. We thus obtain the following variant of Proposition 1.

**Proposition 4 (General Equilibrium)** *In an equilibrium in which both sectors are active, the aggregate dynamics satisfy*

$$C_t + K_{t+1} + X_{t+1} = Y_t = F(K_t, N_t, 1) + G(X_t, L_t) \quad (28)$$

$$C_t = (1 - \varsigma_t)(Y_t + H_t) \quad (29)$$

$$(1 - \varsigma_t)^{-1} = 1 + \beta^\theta \rho_t^{\theta-1} (1 - \varsigma_{t+1})^{-1} \quad (30)$$

$$R_t = G_X(X_t, L_t) \quad \omega_t = G_L(X_t, L_t) \quad (31)$$

$$K_{t+1} = \phi_t \varsigma_t (Y_t + H_t) \quad N_t = \bar{n}(\omega_t) K_t \quad (32)$$

$$N_t + L_t = 1 \quad (33)$$

$$H_t = (\omega_{t+1} + H_{t+1}) / R_{t+1} \quad (34)$$

where  $\rho_t = \rho(\omega_{t+1}, R_{t+1})$  and  $\phi_t = \phi(\omega_{t+1}, R_{t+1})$ .

The system is again recursive, as in the benchmark model. The difference is that the system is now four-dimensional (in  $K_t, X_t, H_t$ , and  $\varsigma_t$ ) instead of three-dimensional, due to the addition of  $X_t$  as a relevant state variable.<sup>25</sup> The interpretation of the conditions is also similar. Condition (28) is the resource constraint of the economy. Conditions (29) and (30) give the equilibrium consumption and the Euler condition. Condition (31) characterizes the equilibrium capital and employment in public equity, whereas condition (32) gives the analogues for private equity. Finally, condition (33) is the clearing condition for the labor market, and condition (34) gives the present value of aggregate labor income in recursive form.

A steady state in which both sectors are active is a fixed point of the dynamic system (28)-(34). To simplify the analysis, it is useful to assume that the capital intensity of the technology used by

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<sup>25</sup>Once again, the dimensionality of the system can be reduced by one if  $\theta = 1$ , for then  $s_t = \beta$  of all  $t$ .

public-equity firms is identical to the one in privately-held firms, in which case the productivity difference between the two sectors is parameterized by a single scalar.

**Assumption A2.**  $G(X, L) = F(X, L, 1/\mu)$  for some  $\mu > 1$ .

The restriction  $\mu > 1$  simply means that (risky) private equity has a higher mean return than (riskless) public equity, which is necessary for a positive amount of private equity to be held in equilibrium. In what follows, I focus on the case where both sectors are active in equilibrium, which is the case as long as  $\mu$  is too high relative to  $\sigma$ ; otherwise, the equilibrium would involve zero resources allocated to the public-equity sector, and the analysis would simply reduce back to the one-sector benchmark examined before.

Letting  $R(\omega) \equiv \max_l [G(1, l) - \omega l]$  and  $l(\omega) \equiv \arg \max_l [G(1, l) - \omega l]$ , we can restate condition (31) as  $R_t = R(\omega_t)$  and  $L_t = l(\omega_t) X_t$ . Moreover, Assumptions A1 and A2 give  $\bar{r}(\omega) = \mu R(\omega)$ , so that  $\mu$  pins down the risk premium on private equity. It follows that  $\phi(\omega, R) = \varphi$  and  $\rho(\omega, R) = \varrho R$ , where  $\varphi$  and  $\varrho$  are determined by the exogenous parameters  $\mu$ ,  $\sigma$  and  $\gamma$  alone:

$$\varphi \equiv \arg \max_{\phi \in [0,1]} \mathbb{C}\mathbb{E}_t(\phi \mu A_{t+1} + 1 - \phi) \quad \text{and} \quad \varrho \equiv \max_{\phi \in [0,1]} \mathbb{C}\mathbb{E}_t(\phi \mu A_{t+1} + 1 - \phi). \quad (35)$$

These properties greatly simplify the characterization of the steady state.

**Proposition 5 (Steady State)** *In a steady state in which both sectors are active, the following are true:*

(i) *The interest rate is given by*

$$R = \beta^{-1} \varrho^{1/\theta-1} (\varphi \mu + 1 - \varphi)^{-1/\theta} < 1/\beta \quad (36)$$

where  $\varphi$  and  $\varrho$  are given by condition (35); the wage rate is then given by the solution to  $R(\omega) = R$  and the capital stocks by

$$K = \frac{1/l(\omega) + \omega/(R-1)}{\mu + 1/\varphi - 1} \quad \text{and} \quad X = 1/l(\omega) - \mu K. \quad (37)$$

(ii) *There exists  $\underline{\theta} = \underline{\theta}(\sigma, \gamma, \mu) < 1$  such that  $\theta > \underline{\theta}$  suffices for a local increase in  $\sigma$  to raise the interest rate, to reduce aggregate investment in private equity, and to reduce the aggregate levels of TFP, output, and consumption.*

As compared to the benchmark model, public equity has three novel implications. First, an Aiyagari-like effect applies to public equity: since the risk-free rate is necessarily lower than the discount rate, the capital-labor ratio in publicly-traded firms is unambiguously higher than that under complete markets. As a result, a higher  $\sigma$  can possibly lead to higher aggregate savings even when it leads to less investment in private equity.

Second, an increase in idiosyncratic risk triggers a reallocation of resources (both capital and labor) from the more risky but more productive sector (private equity) to the less risky but less

productive one (public equity), thus causing a reduction in aggregate total factor productivity. As a result, aggregate output can fall with  $\sigma$  even when aggregate capital does not.

Third, even though the risk-free rate is always below the discount rate, an increase in  $\sigma$  may locally *increase* the risk-free rate when both private and public equity are held. This is unlike either the Bewley class of models or the one-sector model of the previous section. The reason for this new effect is that the technology in the public equity sector imposes a negative relation between the wage rate and the interest rate, namely the relation implied by the equation of the input price ratio with the marginal rate of technical substitution. When an increase in  $\sigma$  causes a reallocation of resources from private to public equity, thus reducing aggregate productivity and wages, the reduction in wages is necessarily associated with an increase in interest rates. In the Bewley class of models, the same negative relation between wages and interest rates is present, since all capital is public, but it works the other way around: higher labor-income risk leads to a lower interest rate and thereby to a higher capital-labor ratio and a higher wage rate. In the one-sector benchmark, on the other hand, the negative relation between wages and interest rates was broken because the interest rate is not equated to the marginal product of capital.

### 5.3 Numerical simulation

I now explore the quantitative importance of introducing public equity into the model. The new parameter that needs to be calibrated is  $\mu$ . Other things equal,  $\mu$  determines the allocation of resources between the two sectors. As mentioned earlier, private and public equity each claim roughly one half of aggregate wealth in the United States. For any given set of values for  $(\sigma, \beta, \gamma, \theta, \alpha, \delta)$ , I thus calibrate  $\mu$  so that the implied steady-state shares of private and public equity in the aggregate capital stock are 50% each. The results are reported in Table 2.

In the baseline parametrization (first row of Table 2), the reduction in the saving rate is 2 percentage points with public equity as compared to 3.8 percentage points in the benchmark model. Similarly, the reduction in the capital stock is now 16% as compared to 24% before. On the other hand, the reduction in the steady-state level of income is 8% as compared to 9% before. Hence, the impact of incomplete markets on aggregate savings is significantly mitigated by the introduction of public equity, but the effect on aggregate income remains strong. As anticipated earlier, the reason for this is that incomplete markets now also distort aggregate total factor productivity.

This finding is quite robust to different parameter specifications (rest of Table 2). The premium on private equity, on the other hand, is considerably lower. In the baseline parametrization, for example, the private premium is 1.7%, compared to 3% without public equity. This is simply because the excess return required for holding a risky asset (here private equity) is lower the lower the fraction of savings invested in this asset.

In conclusion, significant negative effects on aggregate savings and income are both empirically plausible and consistent with low private premia. Therefore, large aggregative effects are not necessarily at odds with the low estimates of private premia reported in Moskowitz and Vissing-

$\sigma$	$\gamma$	$\theta$	$\beta$	$\alpha$	$\delta$	$\xi$	$\Delta(\text{Saving Rate})$	$\Delta(\text{GDP})$	<b>Interest Rate</b>	<b>Private Premium</b>				
20%	3	1	.96	.36	.08	.60	<b>-2%</b>	<b>-8%</b>	<b>3.94%</b>	<b>1.71%</b>				
40%										-5.3%	-21%	3.57%	5.27%	
20%									0	-1.2%	-5%	4.13%	0.9%	
40%										-3.6%	-14%	4%	2.97%	
20%	5	2				.60	-3%	-12%	3.82%	2.68%				
	1									-0.7%	-3%	4.1%	0.61%	
										-2.1%	-9%	4.01%	1.72%	
						1/3				-1.6%	-7%	3.72%	1.66%	
							.94				-1.8%	-9%	5.62%	2.07%
								.50			-3.3%	-18%	3.61%	2.61%
					.05					-2.4%	-12%	3.83%	2.07%	

Table 2: Steady-state effects in the model with public equity.

Jørgensen (2002). For example, when  $\xi = 0$  the private premium becomes as low as 0.9%, while the reduction in income remains as high as 5%.

## 6 Extensions

### 6.1 Stationary wealth distribution

The analysis has focused on the steady state of aggregate dynamics, completely ignoring what happens at the cross section. It is easy to see that, at steady-state prices, individual effective wealth (i.e.,  $w_t + h_t$ ) follows a geometric random walk. As a result, although the steady state is well behaved in terms of aggregates, there is no stationary distribution for the cross section.

If one is concerned about the robustness of the results to perturbations of the model that admit a stationary distribution of wealth, then an easy ad-hoc fix is available. Suppose that a fraction  $\eta$  of the population, randomly selected from the entire population, is replaced each period with new households. The new households are endowed with the mean wealth of the exiting households. This introduces mean-reversion in the cross-sectional dynamics and ensures that a stationary distribution exists for every  $\eta \in (0, 1)$ . At the same time, none of the aggregative results of this paper are greatly affected (especially for  $\eta$  close to zero).

Of course, a more natural way to introduce mean-reversion in individual wealth is to allow for labor-income risk, borrowing constraints, or diminishing returns at the individual level. These extensions are beyond the scope of this paper; they would also break the tractability of the model.

Nevertheless, a potentially useful hint that comes out from the results of this paper is that the random-walk component introduced by idiosyncratic capital-income (or rate-of-return) risk may help understand the skewness of observed wealth inequality.

## 6.2 Transitional dynamics and business cycles

The model abstracts from aggregate uncertainty; but the structure of transitional dynamics can provide insight into the business-cycle implications of idiosyncratic investment risk.

For this purpose, it is useful to extend the model so as to allow for variation in both total factor productivity and the level of risk. Let  $Z_t$  denote aggregate labor productivity—with a Cobb-Douglas production function, variation in  $Z_t$  is equivalent to variation in total factor productivity for both sectors—and assume that  $Z_t$  follows the deterministic analogue of an  $AR(1)$  process:

$$\ln Z_{t+1} = \rho \ln Z_t, \tag{38}$$

where  $\rho \in [0, 1)$  measures the persistence of productivity. Next, let the level of idiosyncratic investment risk co-vary with  $Z_t$ :

$$\sigma_t \equiv [\text{Var}_t(\ln A_{t+1}^i)]^{1/2} = \sigma [1 - \eta \ln Z_t], \tag{39}$$

where  $\sigma \geq 0$  and  $\eta \geq 0$  parameterize, respectively, the mean level and the cyclical elasticity of idiosyncratic risk. Starting with any initial values for  $K_0, X_0$ , and  $Z_0$ , the equilibrium dynamics are easily computed using the recursion in Proposition 4 together with conditions (38) and (39).

One can then mimic a negative productivity shock with a reduction in  $Z_0$  starting from steady state. Preliminary simulations suggest that the response of the economy is significantly amplified when  $\eta$ , the cyclical elasticity of idiosyncratic investment risk, is sufficiently high.<sup>26</sup> The intuition for this finding is simple. If recessions are accompanied by an increase in the level of risk, then a flight to quality (a reallocation of resources from private to public equity) during recessions generates an endogenous reduction in aggregate productivity (the Solow residual). This, in turn, amplifies the reaction of aggregate output, investment and consumption to the exogenous productivity shock. As a result, for the same exogenous shocks, the incomplete-markets economy exhibits larger fluctuations than its complete-markets variant.

This finding highlights the differential role that uninsured idiosyncratic investment risk can play along the business cycle. Whereas cyclical labor-income risk is likely to have a mitigating effect by contributing to higher savings during a recession, cyclical investment risk is likely to amplify the transitional dynamics by reducing the demand for investment during a recession. Further exploring the business-cycle implications of different types of idiosyncratic risk is an interesting direction for future research.

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<sup>26</sup>See the working-paper version (Angeletos, 2005) for detailed discussion and numerical simulations.

### 6.3 Fiscal policy

Another possible extension of the model is to introduce a government; the presence of investment risk may have non-trivial implications for fiscal policy.

Consider, first, the macroeconomic effects of government spending. Suppose that government spending is financed by lump-sum taxation. As long as the only borrowing constraint agents face is the natural-solvency constraint, Ricardian equivalence continues to hold despite the lack of perfect risk sharing; this is because agents can freely trade the riskless bond. However, the supply-side effect of a change in the level of government spending can be quite different than in the complete-markets paradigm. There, the steady-state level of the capital-labor ratio is pinned down by the equality of the marginal product of capital with the discount rate, and hence it is independent of the level of government spending. Here, instead, an increase in the level of government spending means a reduction in the net-of-taxes wealth of the agents, which reduces risk tolerance and thereby discourages investment. As a result, in the incomplete-markets model of this paper, the steady-state level of the capital-labor ratio (and hence of labor productivity) is a decreasing function of the size of government spending, even when taxation is non-distortionary.

Consider, next, optimal taxation. A cornerstone result in the Ramsey paradigm of optimal taxation is that the optimal tax on capital income is zero under complete markets (Chamley, 1986; Judd, 1985; Atkeson, Chari and Kehoe, 1999). It is known that idiosyncratic labor-income risk justifies a positive tax on capital (Aiyagari, 1995). But it remains an open question how optimal capital taxation is affected by idiosyncratic investment risk. The tractability of the framework may help provide a first answer to this question.<sup>27</sup>

### 6.4 Open economy

In this paper I focused on a closed economy; but the framework can be applied to a small, open economy as well. Consider, for example, the benchmark model and suppose that the economy is open to an international market for the riskless bond. Production continues to take place in privately-held businesses, but now the domestic bond market need not clear. Instead, the interest rate  $R$  is exogenously given and  $B_t \equiv \int_i b_t^i$  is the net foreign asset position of the country.

As long as  $1 < R < 1/\beta$  (i.e., as long as the international interest rate is positive and less than the domestic discount rate), the steady-state levels of domestic capital, GDP, foreign debt, GNP, and consumption are all uniquely determined, and are independent of initial conditions. This is in contrast to the complete-markets case, where the steady state is indeterminate—under complete markets, the steady-state levels of foreign debt, GNP and consumption move one-to-one with the initial debt position of the economy.

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<sup>27</sup>The dynamic Mirrlees literature has also focused on labor-income risk. Whether the optimality of a positive wedge on savings derived in this literature (e.g., Golosov, Kocherlakota and Tsyvinski, 2003) survives in the presence of idiosyncratic investment risk is another open question. For some work in this direction, see Angeletos and Werning (2006) and Albanesi (2006).

The model can thus generate a stationary distribution of wealth and capital in a cross-section of countries. Suppose, for example, that the world consists of a large number of countries, all with positive but possibly different levels of idiosyncratic risk. There is then a non-degenerate stationary distribution in the cross-section of countries, with inequality across countries reflecting (partly or entirely) differences in the degree of within-country risk-sharing: countries with higher  $\sigma$  have lower levels of capital and income, even though they can freely borrow from richer countries.

The model can also be used to study the reaction of domestic investment, savings, and current-account deficits to aggregate productivity shocks. Unlike the case of complete markets, here fluctuations can be persistent, and investment and savings can be positively correlated, even if the shock is entirely transitory. The reason is that domestic investment tends to track domestic wealth due to the dependence of risk tolerance on wealth.

## 7 Concluding Remarks

This paper modified the standard neoclassical growth model in a *single* dimension: it introduced uninsured idiosyncratic investment risk and examined its impact on aggregate savings and income.

An alternative approach often taken in the literature is to use richer models that better capture agent heterogeneity and wealth inequality by introducing additional frictions, such as borrowing constraints, or additional aspects of entrepreneurial activity, such as occupational choice. This approach inevitably constrains the analysis to a limited set of numerical exercises, which may be fine for certain purposes but would have been premature for the purposes of this paper. This is because the question of interest here was not only quantitative but also qualitative: the impact of investment risk is ambiguous in general. A clear theoretical benchmark was thus in due before any quantitative exercise.

The main contribution of this paper was to offer such a benchmark, and thereby to provide guidance on the conditions under which idiosyncratic investment risk leads to lower aggregate savings and income. Once this goal was achieved, a first attempt at quantifying the effects was also made. This paper thus highlighted that idiosyncratic investment risk can have significant negative effects on aggregate savings and income—which contrast to the positive effects of labor-income risk in Bewley-type models. Further work on the role of private investment and entrepreneurial risk-sharing is thus needed before we understand the implications of incomplete markets for capital accumulation, business cycles, and welfare.

## Appendix: Proofs

**Proof of Lemma 1.** By the linear homogeneity of  $F(K, L, A)$  in  $(K, L)$ ,

$$\frac{\pi_t^i}{k_t^i} = F\left(1, \frac{n_t^i}{k_t^i}, A_t^i\right) - \omega_t \frac{n_t^i}{k_t^i}. \quad (40)$$

Since  $k_t^i$  and  $A_t^i$  are known when  $n_t^i$  is chosen, the optimal  $n_t^i/k_t^i$  maximizes (40) for any  $A_t^i$ , which gives (6). By definition of  $n(\cdot)$  and  $r(\cdot)$ ,  $F_L(1, n(A, \omega), A) \equiv \omega$  and  $r(A, \omega) \equiv F(1, n(A, \omega), A) - \omega n(A, \omega)$ . Hence,  $F(K, L, 0) = 0$  implies  $n(0, \cdot) = r(0, \cdot) = 0$ , whereas  $n(A, \cdot) > 0$  and  $r(A, \cdot) > 0$  for  $A > 0$ . Applying the implicit function theorem and using  $F_L > 0$ ,  $F_{LL} < 0$ ,  $F_A > 0$ , and  $F_{LA} > 0$ , we infer  $n_\omega < 0 < n_A$  and  $r_\omega < 0 < r_A$ . Finally, the Inada conditions imply, for any  $A > 0$ ,  $\lim_{\omega \rightarrow 0} n(A, \omega) = \lim_{\omega \rightarrow 0} r(A, \omega) = \infty$  and  $\lim_{\omega \rightarrow \infty} n(A, \omega) = \lim_{\omega \rightarrow \infty} r(A, \omega) = 0$ . ■

**Proof of Lemma 2.** For notational simplicity, I drop the superscript  $i$  and use  $r_{t+1}$  as a short-cut for  $r(A_{t+1}, \omega_{t+1})$ . I propose—and then verify—the following solution:

$$V(w; t) = U(a_t(w + h_t)), \quad c(w; t) = (1 - \varsigma_t)(w + h_t), \quad k(w; t) = \phi_t \varsigma_t(w + h_t), \quad (41)$$

where  $a_t$ ,  $\varsigma_t$ , and  $\phi_t$  are coefficients (time-varying but non-stochastic) to be determined. From the budget constraint and (41), we then infer  $b(w; t) = (1 - \phi_t)\varsigma_t(w + h_t) - h_t$ . From (2) and (41), the certainty equivalent of the value of wealth is

$$\mathbb{C}\mathbb{E}_t [U^{-1}V_{t+1}(w_{t+1})] = \Upsilon^{-1} (\mathbb{E}_t [\Upsilon U^{-1}V_{t+1}(w_{t+1})]) = a_{t+1} [\mathbb{E}_t (w_{t+1} + h_{t+1})^{1-\gamma}]^{1/(1-\gamma)}.$$

Hence, the first-order conditions with respect to  $k_{t+1}$  and  $b_{t+1}$  give:

$$(c_t)^{-1/\theta} = \beta a_{t+1}^{1-1/\theta} \cdot [\mathbb{E}_t (w_{t+1} + h_{t+1})^{1-\gamma}]^{(\gamma-1/\theta)/(1-\gamma)} \cdot \mathbb{E}_t [(w_{t+1} + h_{t+1})^{-\gamma} r_{t+1}], \quad (42)$$

$$(c_t)^{-1/\theta} = \beta a_{t+1}^{1-1/\theta} \cdot [\mathbb{E}_t (w_{t+1}^i + h_{t+1})^{1-\gamma}]^{(\gamma-1/\theta)/(1-\gamma)} \cdot \mathbb{E}_t [(w_{t+1} + h_{t+1})^{-\gamma} R_{t+1}]. \quad (43)$$

Combining the two conditions, and using

$$w_{t+1} = r_{t+1}k_{t+1} + R_{t+1}b_{t+1} + \omega_{t+1} = [\phi_t r_{t+1} + (1 - \phi_t)R_{t+1}] \varsigma_t(w_t + h_t) - h_{t+1}, \quad (44)$$

we get  $\mathbb{E}_t \{ [R_{t+1} + \phi_t(r_{t+1} - R_{t+1})]^{-\gamma} (r_{t+1} - R_{t+1}) \} = 0$ , or equivalently  $\phi_t = \phi(\omega_{t+1}, R_{t+1})$ . Next, the envelope condition,  $V'(w_t; t) = U'(c_t)$ , or equivalently  $a_t^{1-1/\theta} (w_t + h_t)^{-1/\theta} = (c_t)^{-1/\theta}$ , along with  $c_t = (1 - \varsigma_t)(w_t + h_t)$  from (41), implies

$$a_t^{1-1/\theta} = (1 - \varsigma_t)^{-1/\theta}. \quad (45)$$

Multiplying (42) and (43) with  $\phi_t$  and  $(1 - \phi_t)$ , respectively, summing up, substituting  $w_{t+1}$  in the

resulting relation from (44), and rearranging, gives the saving rate in recursive form:

$$(1 - \varsigma_t)^{-1} = 1 + \beta^\theta \rho_t^{\theta-1} (1 - \varsigma_{t+1})^{-1}, \quad (46)$$

where  $\rho_t = \rho(\omega_{t+1}, R_{t+1})$ . For any  $\{\omega_t, R_t\}_{t=0}^\infty$  that is part of an equilibrium,  $\sum_{s=t}^\infty \prod_{\tau=t}^s [\beta^\theta \rho_\tau^{\theta-1}]$  is finite. Forward iteration of (46) thus yields (12), with  $\varsigma_t \in (0, 1)$ . Using (41), we then verify that  $c_t > 0$ ,  $k_{t+1} > 0$ , and  $b_{t+1} > -h_t$ . Finally, we verify that (41) solves the Bellman equation. Substituting (41) into (8) gives

$$U(a_t(w_t + h_t)) = U((1 - \varsigma_t)(w_t + h_t)) + \beta U(a_{t+1}[\mathbb{E}_t(w_{t+1} + h_{t+1})^{1-\gamma}]^{1/(1-\gamma)}).$$

Dividing both sides by  $U(w_t + h_t)$  and using (44) and (46), the above reduces to  $a_t^{1-1/\theta} = (1 - \varsigma_t)^{-1/\theta} [(1 - \varsigma_t) + a_{t+1}^{1-1/\theta} (1 - \varsigma_{t+1})^{1/\theta} \varsigma_t]$ , which is satisfied by (45). ■

**Proof of Condition (15).** To simplify notation, let  $r_{t+1}^i = r(A_{t+1}^i, \omega_{t+1})$ ,  $\bar{r}_{t+1} = \mathbb{E}_t r_{t+1}^i$ , and  $\sigma_{t+1}^2 = \text{Var}_t[\ln r_{t+1}^i]$ . A second-order Taylor approximation for  $\ln \rho_t$  around  $\sigma_t = 0$  gives

$$\ln \rho_t \approx \phi_t \mathbb{E}_t[\ln r_{t+1}^i] + (1 - \phi_t) \ln R_{t+1} + \frac{1}{2} \phi_t (1 - \phi_t) \sigma_{t+1}^2 + \frac{1 - \gamma}{2} \phi_t^2 \sigma_{t+1}^2. \quad (47)$$

Since  $\phi_t$  maximizes  $\rho_t$ , the above also implies

$$\phi_t \approx \frac{\mathbb{E}_t \ln r_{t+1}^i - \ln R_{t+1} + \sigma_{t+1}^2/2}{\gamma \sigma_{t+1}^2}. \quad (48)$$

Combining the two conditions above and using  $\mathbb{E}_t \ln r_{t+1}^i \approx \ln \mathbb{E}_t r_{t+1}^i - \text{Var}_t[\ln r_{t+1}^i]/2 = \ln \bar{r}_{t+1} - \sigma_{t+1}^2/2$  gives (15). ■

**Proof of Proposition 1.** Note that  $\phi_t$  and  $\varsigma_t$  are identical across agents. Aggregating the conditions in Lemma 2 over all  $i$  and using the facts that  $A_t^i$  and  $k_t^i$  are independent and that  $\Pi_t + \omega_t = \bar{r}(\omega_t)K_t + \omega_t = f(K_t) = Y_t$ , we infer

$$W_t = \Pi_t + RB_t + \omega_t = f(K_t) + RB_t \quad (49)$$

$$C_t = (1 - \varsigma_t)(W_t + H_t) \quad (50)$$

$$K_{t+1} = \varsigma_t \phi_t (W_t + H_t) \quad (51)$$

$$B_{t+1} = \varsigma_t (1 - \phi_t)(W_t + H_t) - H_t \quad (52)$$

where  $B_t = \int b_t^i di$ . The bond market clears if and only if  $B_t = 0$  and therefore  $W_t = f(K_t) = Y_t$ . Along with (50) and (51), this immediately gives (17) and (18). Next, adding up (50)-(52) gives the resource constraint (16), whereas (19) follows directly from (5). Finally, the labor market clears if and only if  $1 = \int n_t^i = \bar{n}(\omega_t)K_t$ , which gives (21). ■

**Proof of Proposition 2.** Evaluating (49)-(52) [equivalently, (16)-(21)] in the steady state

and combining, we get  $K + H = \varsigma(W + H) = \varsigma[\bar{r}(\omega)K + RH] = \varsigma[\phi\bar{r}(\omega) + (1 - \phi)R](K + H)$ , or equivalently

$$1 = \varsigma[\phi\bar{r}(\omega) + (1 - \phi)R],$$

which is simply the stationarity condition for aggregate wealth. Substituting  $\varsigma = \beta^\theta \rho^{\theta-1}$  into the above gives condition (22) in the Proposition. Next, by (21), (51), (52), and the property that, under Assumption A1,  $\bar{r}(\omega) = F_K(K, 1, 1) = f'(K)$  and  $\omega = f(K) - f'(K)K$ , we have

$$\frac{H}{K} = \frac{1 - \phi}{\phi} \quad \text{and} \quad H = \frac{\omega}{R - 1} = \frac{f(K) - f'(K)K}{R - 1}.$$

Combining gives condition (23) in the Proposition. ■

**Proof of Proposition 3.** *Part (i).* By (23),  $\phi \in (0, 1)$ . For that to be true, it must be that  $f'(K) > R$ , or otherwise the bond would dominate private equity. By risk aversion, then,  $R < \rho < \phi f'(K) + (1 - \phi)R$ , which together with (22) gives also  $\rho < 1/\beta$ . Combining, we have

$$R < \rho < \phi f'(K) + (1 - \phi)R < f'(K) \quad \text{and} \quad R < \rho < 1/\beta. \quad (53)$$

Next, taking logarithms of (22) and rearranging gives

$$\theta \log[\beta f'(K)] = -\log[\phi + (1 - \phi)R/f'(K)] - (\theta - 1) \log[\rho/f'(K)].$$

It follows that  $\beta f'(K) > 1$  if and only if  $\theta > \underline{\theta}$  where

$$\underline{\theta} = 1 - \frac{\log[\phi + (1 - \phi)R/f'(K)]}{\log[\rho/f'(K)]}. \quad (54)$$

Note that  $\underline{\theta}$  above is expressed in terms of endogenous variables, but (53) ensures that  $\underline{\theta} < 1$ .

*Part (ii).* The proof of this part is available upon request.

*Part (iii).* Since  $\omega = f(K) - f'(K)K$  and  $R < f'(K)$ , we have that

$$\frac{1 - \phi}{\phi} = \frac{H}{K} = \frac{\omega}{(R - 1)K} = \frac{f(K) - f'(K)K}{(R - 1)K} > \frac{f(K)/K - f'(K)}{f'(K) - 1}.$$

Using this together with  $\hat{f}(K) = f(K) - (1 - \delta)K$ ,  $\alpha = \hat{f}'(K)K/\hat{f}(K)$ , and  $s = \delta K/\hat{f}(K)$ , we get

$$\phi < \frac{f'(K) - 1}{f(K)/K - 1} = \frac{\hat{f}'(K) - \delta}{\hat{f}(K)/K - \delta} = \frac{\alpha - s}{1 - s} \leq \alpha,$$

which completes the proof. ■

**Proof of Proposition 4.** The budget constraint of household  $i$  in period  $t$  reduces to

$$c_t^i + k_{t+1}^i + (x_{t+1}^i + b_{t+1}^i) \leq w_t^i = r(A_t^i, \omega_t)k_t^i + R_t(x_t^i + b_t^i) + \omega_t.$$

Hence, Lemma 2 continues to apply provided we replace  $b$  with  $x + b$ ; that is,

$$\begin{aligned} c_t^i &= (1 - \varsigma_t)(w_t^i + h_t) \\ k_{t+1}^i &= \varsigma_t \phi_t(w_t^i + h_t) \\ x_{t+1}^i + b_{t+1}^i &= \varsigma_t(1 - \phi_t)(w_t^i + h_t) - h_t \end{aligned}$$

where  $\phi_t$ ,  $\rho_t$ , and  $\varsigma_t$  are defined again as in Lemma 2. Conditions (28), (29), (32), and (34) then follow from aggregating across agents and using the bond market clearing condition, as in Proposition 1. Finally, (31) follows from profit maximization in the public-equity sector and (33) from labor market clearing. ■

**Proof of Proposition 5.** We first prove that  $\rho(\omega, R) = \varrho R$  and  $\phi(\omega, R) = \varphi$ , where  $\varrho$  and  $\varphi$  are given by (35). Under A1,  $n(A, \omega) = A\bar{n}(\omega)$ ,  $r(A, \omega) = A\bar{r}(\omega)$ , and  $\bar{r}(\omega) = F_K(1, \bar{n}(\omega), 1)$ . It follows that

$$\begin{aligned} \phi(\omega, R) &= \arg \max_{\varphi} \{[\varphi A F_K(1, \bar{n}(\omega), 1) + (1 - \varphi)R]^{1-\gamma} \psi(A) dA\}^{1/(1-\gamma)} \\ \rho(\omega, R) &= \max_{\varphi} \{[\varphi A F_K(1, \bar{n}(\omega), 1) + (1 - \varphi)R]^{1-\gamma} \psi(A) dA\}^{1/(1-\gamma)} \end{aligned}$$

Under Assumption A2, on the other hand,  $R = G_K(1, l(\omega)) = F_K(1, \bar{n}(\omega), 1) / \mu$ . Combining gives the result. We now prove the proposition.

(i) As in the one-sector case, stationarity of aggregate savings requires  $\varsigma[\varphi\bar{r}(\omega) + (1 - \varphi)R] = 1$ , where  $\varsigma = \beta^\theta \rho^{\theta-1}$ . Using  $R = R(\omega)$  and  $\bar{r}(\omega) = \mu R(\omega)$ , we have  $\rho = \varrho R$  and  $[\varphi\bar{r}(\omega) + (1 - \varphi)R] = (\varphi\mu + 1 - \varphi)R$ , and therefore the stationarity condition reduces to (36). This together with  $R = R(\omega)$  gives a unique  $R$  and a unique  $\omega$ . Next, in steady state,  $K = \varphi\varsigma[W + H]$  and  $X + H = (1 - \varphi)[W + H]$ , and therefore  $(X + H)/K = (1 - \varphi)/\varphi$ . On the other hand, the clearing condition for the labor market gives  $\bar{n}(\omega)K + l(\omega)X = 1$ . Using  $\bar{n}(\omega) = \mu l(\omega)$ , and solving the above two conditions for  $K$  and  $X$ , we get

$$K = \frac{\varphi[1 + l(\omega)H]}{(\varphi\mu + 1 - \varphi)l(\omega)} \quad \text{and} \quad X = \frac{1 - \varphi - \varphi\mu l(\omega)H}{(\varphi\mu + 1 - \varphi)l(\omega)},$$

or equivalently (37). This completes the characterization of the steady state. Uniqueness is obvious. As for existence, note that any  $\mu > 1$  implies  $\varphi > 0$  and therefore  $K > 0$  necessarily. On the other hand,  $X > 0$  if and only if  $\varphi$  is sufficiently small, which is the case as long as  $\sigma$  is sufficiently large.

(ii) Since  $R'(\omega) < 0$ ,  $\omega$  decreases with  $\sigma$  if and only if  $R$  increases with  $\sigma$ . From condition (36),

$$\frac{d \ln R}{d\sigma} = -\frac{1}{\theta} \left[ (\theta - 1) \frac{d \ln \varrho}{d\sigma} + \frac{\mu - 1}{\varphi\mu + 1 - \varphi} \frac{d\varphi}{d\sigma} \right].$$

It follows that  $dR/d\sigma > 0$ , and therefore  $d\omega/d\sigma < 0$ , if and only if  $\theta > \underline{\theta}(\mu, \sigma, \gamma)$ , where

$$\underline{\theta}(\mu, \sigma, \gamma) \equiv 1 - \frac{\mu - 1}{\varphi\mu + 1 - \varphi} \frac{d\varphi/d\sigma}{d \ln \varrho/d\sigma}$$

Clearly,  $d\varphi/d\sigma < 0$ ,  $d\ln \varrho/d\sigma < 0$ , and therefore  $\underline{\theta} < 1$ . Next, since  $l'(\omega) < 0$  and  $R'(\omega) < 0$ , from (37) we infer that  $K$  is increasing in  $\omega$  and decreasing in  $\varphi$ , and  $X$  is increasing in  $\varphi$  but (possibly) non-monotonic in  $\omega$ . Since

$$K + X = 1/l(\omega) - (\mu - 1) \frac{1/l(\omega) + \omega/(R - 1)}{\mu + 1/\varphi - 1} = 1/l(\omega) - (\mu - 1)K,$$

total capital  $K + X$  is increasing in  $\varphi$  but non-monotonic in  $\omega$ . Finally, aggregate output is  $Y = F(1, \bar{n}(\omega), \bar{A})K + G(1, l(\omega))X$ . Using  $\bar{n}(\omega) = \mu l(\omega)$  and  $F(K/\mu, N, A) = F(K, N, A/\mu) = G(K, N)$ , we have  $F(1, \bar{n}(\omega), 1) = F(1, \mu l(\omega), 1) = G(1, l(\omega))\mu$  and therefore  $Y = G(1, l(\omega))[\mu K + X]$ , which together with (37) gives  $Y = G(1/l(\omega), 1)$ . Output thus increases with  $\omega$ , reflecting the fact that  $\omega$  increases if and only if resources are shifted from less productive public equity to more productive private equity. By implication,  $Y/(K + X)$  and  $C = Y - (K + X)$  also increase with  $\omega$  and decrease with  $\varphi$ . Hence,  $\theta > \underline{\theta}$  suffices for a higher  $\sigma$  to raise  $R$  and reduce  $\omega$ ,  $K$ ,  $Y$ ,  $Y/(N + L)$ , and  $Y/(K + X)$ . ■

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