

ZERO LOWER BOUND DEPRESSIONS

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I study a flexible price economy where heterogeneous agents use money for trading in decentralized capital markets and intermediaries match investors and producers. I apply it to a classic topic: the relation between output, inflation and interest rates; and show some novel theoretical results at the zero nominal interest rate in stationary equilibria where a form of *money non-satiation* arises. When some agents money hoarding behavior interacts with intermediation frictions, known forces behind the costs of high inflation turn out to be relevant for shaping zero lower bound depressions: situations characterized by low inflation and output, low money velocity and high interest rate spreads. I show how to implement constrained efficient allocations and that the optimal inflation rate increases as intermediation frictions worsen. The dynamic adjustment to shocks features a non-linear amplification mechanism around the zero lower bound. A worsening of intermediation frictions generates deflation and a steep and persistent output drop when the zero lower bound binds, whereas the response is much smoother if the nominal interest rate remains positive. Traditional monetary policy focused on monetary aggregates is helpful in taking the economy out of a zero lower bound depression.

1. INTRODUCTION

The broad purpose of this paper is to describe economies in which an asset, that I will call money, is used for trading in decentralized markets where intermediaries match "buyers" and "sellers". The market that I will use to illustrate the main ideas will be that of a factor of production: capital. The role of buyers

*I thank Fernando Alvarez, Veronica Guerrieri and Erik Hurst for their constant encouragement, patience and many hours of insightful discussions. For helpful comments, I thank Harald Uhlig, Lars Hansen, Martin Schneider, Pete Klenow, Nicholas Bloom, Klaus Desmet, Sebastian Di Tella, Pablo Kurlat, Roberto Robatto, Juliette Caminade, Gaston Navarro, Julio Andres Blanco and seminar participants at the University of Chicago Economic Dynamics and Applied Macroeconomic Theory Working Groups and Stanford Macro Bag Lunch.

will be played by heterogenous entrepreneurs with production opportunities. That of sellers by investors. I will call the intermediaries, bankers, and allow them to open markets where they transfer capital from investors to producers that direct their search towards such markets. While I will conduct all the analysis in such context, I believe many insights would carry over to other environments where intermediation is present to match two sides of a market. Intermediaries could be retailers in product markets, real state brokers or market-makers for some classes of assets.

The specific tasks that will occupy us is an application of such framework to classic issues regarding the determinants of inflation at the zero nominal interest rate; the link between low inflation and depressions¹; the welfare costs of high inflation; and the optimal monetary response to intermediation frictions shocks.

The literature has accumulated convincing evidence and theoretical explanations on the long run trade-off between output and inflation *in economies where the nominal interest rate is positive*. Anticipated monetary expansions are welfare reducing through inflation tax effects; as summarized in Lucas (2000) and Berentsen, Menzio, and Wright (2011).² Then again, sustained deflation have been held accountable for economic depressions such as the Great Depression and Japan's Lost Decade. A view best exemplified by Friedman and Schwartz (1963) seminal book or Fed's Chairman Ben Bernanke's statements³ that has received renewed interest in the Liquidity Trap literature.⁴

However, we cannot just take an off the shelf cash-in-advance, shopping time or money in the utility function model and discuss the effects of money supply changes at low inflation. There is simply nothing to talk about once the nominal interest rate zero lower bound is hit. Agents are not cash constrained anymore. While these have proven useful constructs for describing inflation tax effects at positive nominal interest rates; for sufficiently low inflation environments the same is no longer true. Faced with this issue, the literature has turn to frameworks featuring keynesian liquidity traps at the zero nominal interest rate where money demand considerations are completely absent. But a situation where the nominal interest rate has been driven to zero is also known to be the optimal monetary policy in a large class of models i.e. the Friedman Rule. As a matter of fact and theory, it is hard to say whether such policy results in a state of monetary bliss or an alarming keynesian trap. The two are rarely analyzed in the same context.

This paper asks whether some of the forces behind the cost of inflation when the nominal interest rate is positive are the same ones that shape Zero Lower Bound depressions. I discuss one of them in particular: portfolio choice. Inflation affects relative returns between money and alternative forms of investment and, potentially, distort real allocations when the nominal interest rate is positive. Severe deflation, on the other hand, might lower individual agents capital demand and make them pass on production projects in order to

¹See Atkeson and Kehoe (2004) or Benhabib and Spiegel (2009).

²"...Anticipated monetary expansions have inflation tax effects (...), but they are not associated with (...) stimulus to employment and production..." Robert E. Lucas, Jr., August, 1996

³"...Sustained deflation can be highly destructive to a modern economy and should be strongly resisted..."

Ben S. Bernanke, November, 2002

⁴See for instance, Krugman (1998), Eggertsson and Woodford (2003) or Guerrieri and Lorenzoni (2011).

take advantage of the high real return offered by money. Notwithstanding, if economic agents are *money satiated* at the zero nominal interest rate, as described in [Cole and Kocherlakota \(1998\)](#), this portfolio composition considerations become entirely irrelevant.

I take a different route here and emphasize a form of *monetary non-satiation* that arises in decentralized capital markets when money becomes a complement factor in production. Intuitively, a fraction of the agents in an economy will always be money constrained when the gains from holding money are not subject to decreasing marginal returns.⁵ This is the case, for instance, when profits are linear in the factor acquired using money.

I show that, in stationary equilibria, inflation is still determined by money supply growth rates even at the zero nominal interest rate. Moreover, there is a bounded continuum of inflation/money growth rates consistent with a zero nominal interest rate; while there is a one-to-one relationship between money growth rates and nominal interest rates when inflation is above a certain threshold and the nominal interest rate is positive. As far as I know, this result is new to the literature and opens the door for the analysis of non-neutral changes in money supply at the zero lower bound that motivates the paper and is described in what follows.

The second set of results concerns the behavior of aggregate output across stationary equilibria with different inflation rates. When the nominal interest rate is positive, capital misallocation worsens since some agents decide not to undertake projects that would be profitable if the opportunity cost of holding money was eliminated. Thus, as inflation rises output decreases. While the exact mechanism is somewhat different than in other papers studying the costs of inflation, the result and intuition are similar. For reasonable parameterizations this effect is quantitatively small. A 10 percent annual inflation rate is associated with an output loss in the order of 1 or 2 percent. This is very much in line with the estimates in, for example, [Lucas \(2000\)](#) and [Berentsen et al. \(2011\)](#)

The comparative statics at the zero nominal interest rate are novel. For sufficiently low inflation rates, sustained anticipated decreases in inflation result in lower aggregate output. However, in the absence of intermediation frictions or under free entry and exit of intermediaries in the decentralized market, aggregate output always increases as inflation decreases. Thus, the positive relation between output and inflation at the zero nominal interest rate should apply to the short to medium run of an economy where the measure of intermediaries cannot adjust. In the long run, as intermediaries are able to enter and exit the market, the relation between output and inflation should always be negative instead. In contrast to the output costs of high inflation, the output losses brought about by too low inflation are substantial. A 1 percent annual deflation results in output losses of approximately 10 percent.

The zero lower bound on the nominal interest rate effectively imposes a lower bound on the real interest for a given inflation rate. What other variable adjust to equate capital supply and demand when

⁵Models with money in the utility function or where cash is used for consumption almost never feature this property, as utility functions are assumed to be concave.

this constraint is binding? Hall (2011) and Kocherlakota (2012), among others, also note this tension when dealing with the zero lower bound. In the context of the heterogenous agent economy in this paper there is no need to make any necessarily ad-hoc decisions. It is the fraction of agents holding money that adjusts. Aggregate capital supply becomes perfectly elastic at the inflation determined real interest rate and some indifferent agents hoard money, i.e. use money as a savings vehicle and not for transactions. As inflation decreases and the real returns to hoarding money (or equivalent interest bearing assets at the zero nominal interest rate) increase, some agents decide to stop producing and become investors alone. Conversely, only a small measure of the most productive agents find it profitable to undertake production. Competition amongst a fixed measure of intermediaries facing a smaller pool of agents with profitable production opportunities results in *congestion* in the decentralized market where investors and producers are matched. The combination of a congestion induced thinner market and a higher real interest rate payed to investors drives up the marginal cost of a unit of capital charged to producers when intermediaries are required to pay a fixed cost before a match is realized. For low enough inflation this *interest rate* effect dominates an offsetting *selection* effect that comes about because active producers are on average more productive; reducing capital, labor demand and aggregate output.⁶ Interestingly, the spread between the interest rate payed to investors and the marginal cost of capital shows up as an aggregate investment wedge that decreases with inflation at the zero nominal interest rate. I will call a Zero Lower Bound Depression such a situation that is characterized by low inflation and output, low money velocity (since some agents hoard money) and high interest rate spreads (because of congestion in the decentralized market).

The comparative static predictions of the model helps shed light on the empirical literature studying nonlinearities in the relation between output and inflation. Atkeson and Kehoe (2004) find that deflation and output growth are negatively correlated only during the 1930s and in Japan during the 1990s (although they interpret the later as a long run secular trend) in a study comparing 17 countries in more than 100 years. Benhabib and Spiegel (2009) redo their exercise by allowing non-linearities dependent on inflation levels and obtain a large, positive and statistically significant estimate of the relationship between inflation and growth in a range of moderate to negative inflation, and a negative or insignificant relation for higher inflation. The model is consistent with this nonlinearity and, in fact, makes an even sharper prediction. The exact threshold inflation rate at which the relationship between inflation and output changes sign depends on an economy's technological and preference parameters. Most notably the discount rate and the degree of intermediation frictions which makes this non-linearity hard to identify as Benhabib and Spiegel (2009) note (their estimates are quiet noisy and inflation has a low explanatory power). But the link between depressions and low inflation should only be observed if simultaneously the nominal interest rate is zero. Taken at face value, this is why Atkeson and Kehoe (2004) find that such link is only evident during the Great Depression and Japan's Lost Decade. Two episodes where the countries in question where at the

⁶The *congestion* effect is absent under free entry of intermediaries or when there is no fixed cost. Without it, the *selection* effect always dominates the *interest rate* effect and decreasing inflation unambiguously increases output.

zero lower bound. I conjecture that by controlling for such an event would tighten the estimates and add explanatory power to said regressions. In addition, extending their samples to include observations after 2008 would feature many countries experiencing negative output growth, low inflation and a zero nominal interest rate.

The analysis so far has concentrated on describing a decentralized equilibrium for this economy. I turn next to issues of optimality and implementation which concurrently help in clarifying the forces at play in equilibrium. I study the problem of planner who maximizes an ex-ante utilitarian welfare function and is only able to choose whether an entrepreneur holds money or claims to capital that offer a return. In other words, is constrained to altering the cross-sectional distribution of assets in the economy while satisfying all other equilibrium optimal policies and feasibility conditions. The definition of constrained optimality is in the same spirit as that in [Davila, Hong, Krusell, and RiosRull \(2012\)](#). I show that the equilibrium is ex-ante constrained inefficient because heterogenous entrepreneurs do not internalize the change in prices when deciding on their optimal portfolio allocations which affects other agents. The constrained efficient allocation requires the elimination of inflation tax type distortions as well as money hoarding: agents holding money should only do so for transaction purposes, not for savings, and no agent should pass on profitable production opportunities because of the opportunity cost of holding money. Given the theoretical results previously presented, this implies that: (i) a zero nominal interest rate is a necessary condition for the equilibrium to be constrained optimal but it is not sufficient; (ii) the constrained efficient allocation can be implemented in a stationary equilibrium by setting the money supply growth rate equal to the maximum growth rate consistent with a zero nominal interest rate and (iii) the optimal inflation rate in a stationary equilibrium increases as intermediation frictions (the fixed cost paid by intermediaries in the decentralized capital market) worsen.

Both the equilibrium and optimal allocations characterization uncover an interesting interaction between the zero lower bound, intermediation frictions and money supply growth. To illustrate it, I move away from stationary theoretical analysis and numerically study the economy's dynamic adjustment to one time unanticipated intermediation frictions and monetary shocks. The model features a non-linear amplification mechanism around the zero lower bound. For example, when starting from a low inflation steady state, a permanent increase in the intermediaries' fixed cost results in a zero nominal interest rate, persistent deflation and a steep output drop followed by a fast recovery. On the other hand, an equivalent permanent shock starting from a high inflation steady state such that the zero lower bound never binds, results in some inflation and a smooth and persistent output decline towards the new steady state. Finally, a permanent increase in the money supply growth rate when the zero lower bound binds, increases output on impact and makes its transition dynamics similar to the smooth case where the nominal interest rate is positive. Inflation overshoots and converges from above to its higher steady state value. [Guerrieri and Lorenzoni \(2011\)](#) also find that a financial friction shock, in their case a tightening of borrowing constraints in a heterogenous agent economy, can push the economy into zero lower bound territory and generate amplification. The literature studying liquidity traps and monetary policy in New Keynesian models e.g.

Eggertsson and Woodford (2003); Christiano, Eichenbaum, and Rebelo (2011) or Werning (2011); resort to preference shocks in order to produce a binding zero lower bound and amplification. To the best of my knowledge, all these and other papers studying amplification at the zero lower bound have done so in economies where some sort of nominal rigidity is present. On empirical grounds, critics of this approach point out that the depth and persistence of economic depressions are hard to reconcile with the relatively high frequency of price adjustments if nominal rigidities were the main relevant friction. On theoretical grounds, Cochrane (2013) cautions us against taking the predictions of these later class of models at face value because of equilibrium selection concerns. The dynamics of the model in this paper show that even in the absence of any nominal rigidities, the zero lower bound can generate substantial amplification; and that traditional monetary policy focused on monetary aggregates, as opposed to *forward-guidance*, can be important in a Zero Lower Bound Depression.

Finally, I extend the benchmark model in two directions. First, I introduce an imperfect substitute to money for transactions by allowing entrepreneurs to issue debt that is collateralized with capital claims or bonds. This turns out to be entirely irrelevant at the zero nominal interest rate since no agent decides to finance their projects with debt. When the nominal interest rate is positive, the money demand elasticity with respect to the nominal interest rate increases and the costs of inflation are reduced as agents can substitute away from money and escape the inflation tax. Furthermore, it is possible to derive a clean expression for the velocity of money as a function of the nominal interest rate and technological parameters indexing the degree of substitutability and the distribution of productivity shocks.

Secondly, I consider the effects of introducing money in the economy through open market operations instead of the standard helicopter drop. Aggregate quantities and ex-ante welfare are not affected in a stationary equilibrium when the nominal interest rate is zero for a given inflation rate. However, the set of inflation rates consistent with a zero nominal interest rate expands, or equivalently, the constrained optimal money growth rate increases.

The paper is organized as follows. Section 2. presents the benchmark model and characterizes individual policies. Section 3. defines a Competitive Search Equilibrium in the spot decentralized capital market and a Stationary Equilibrium for the economy as a whole. In Section 4. I state the main theoretical propositions and in Section 5. I discuss constrained efficiency. Section 6. introduces the dynamic equilibrium response to shocks. Section 7. describes extensions and I conclude in Section 8..

2. THE MODEL

Consider an economy composed of a continuum of islands inhabited by entrepreneurs, workers and risk-neutral bankers. Time is discrete, agents are infinitely lived and there is no aggregate uncertainty.

Entrepreneurs are island specific; consume and produce a homogenous good by renting capital and hiring workers; potentially accumulate interest bearing assets a and a non-interest bearing asset m that I will call money. Moreover, they are hit by productivity shocks z and are endowed with a Cobb-Douglas

production technology $(zk)^{\alpha}l^{1-\alpha}$. Hence, an island/entrepreneur is identified by a portfolio-productivity vector $\{a, m, z\}$

Workers and bankers, on the other hand, are freely mobile across islands and hand-to-mouth.

There are centralized assets, goods and labor markets that are competitive and frictionless. The capital rental market is decentralized and subject to frictions.

At the end of each period the asset market opens, where entrepreneurs trade consumption units for money m at price ϕ or claims to a mutual fund a that will pay the risk-free interest rate r next period. Bankers intermediate capital between the mutual fund and entrepreneurs demanding capital for production in a decentralized spot market that opens at the beginning of the period.⁷ There is a one-to-one technology for transforming units of consumption into capital and bankers can draw from the mutual fund, paying r per unit withdrawn.

In the decentralized spot market, bankers can open sub-markets by paying a fixed cost and posting a capital rental rate \tilde{r} . Entrepreneurs direct their search towards these and matching within any given sub-market occurs as follows. Let θ be the thickness of sub-market \tilde{r} , i.e. the ratio of entrepreneurs to bankers. Whenever θ is finite, all entrepreneurs are matched with a banker. If $\theta < 1$, some bankers are left idle; when $\theta > 1$ the extra measure of entrepreneurs is divided proportionally across all bankers. Later on I define a competitive search equilibrium in this market and describe the trade process in more detail.

The goods market opens after the capital rental market closes. As a consequence, there is a timing mismatch in the entrepreneurial cash-flow between payments to bankers and revenues from sales. I assume it is impossible to enforce any within-period debt contract⁸. Thus, entrepreneurs need to use accumulated money to pay the bankers, giving rise to a cash-in-advance constraint for production.

It is worth noting that the intermediation structure I have just described results in entrepreneurs facing no idiosyncratic risk in the capital market. Neither in their investor role, as they can participate in the mutual fund; nor in their producer role, as they will always be matched in equilibrium with a banker able to satisfy their capital rental demands.

By going a little deeper in the trade process in the capital market, dispensing of the Walrasian auctioneer, a role for intermediation arises. The combination of a mutual fund and risk-neutral bankers successfully insures entrepreneurs against matching risk when exchange is done in a decentralized fashion. Directed search with price posting is a convenient and tractable way of modelling such environment.

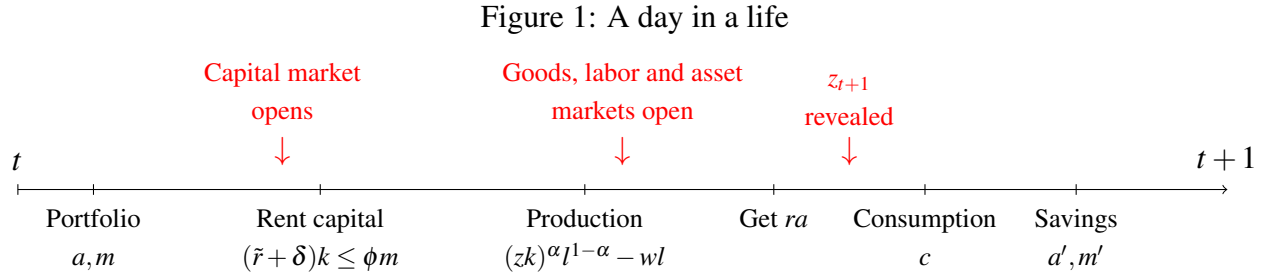
⁷If entrepreneurs had to intermediate capital by themselves they would be subject to matching uncertainty. The mutual fund eliminates it and is able to offer a risk free return by effectively pooling this risk.

⁸In [Section 7](#). I extend the benchmark model by allowing entrepreneurs to issue debt using assets a as collateral. This is a form of secured credit line. The results at the zero nominal interest rate which are the focus of this paper are unchanged since entrepreneurs decide not to issue any debt in equilibrium. However, when the nominal interest rate is positive some entrepreneurs will decide to hold no money and finance themselves with debt alone. This reduces the effects of inflation on output, as entrepreneurs find it possible to substitute away from money and partly avoid the inflation tax.

2.A. *Entrepreneurs*

2.A.i. *Timing and island structure*

Figure 1 below describes the timing of events and decisions taken by an entrepreneur within a period



The entrepreneur holds claims and money a, m at the beginning of a period. As mentioned, there is a mismatch between the time when the capital rental market opens and the goods market opens. The entrepreneur finances her capital rental expenditures $(r + \delta)k$ with accumulated money.

Later, hiring decisions l are taken given a real wage w ; production takes place using technology $(zk)^{\alpha} l^{1-\alpha}$ and the asset market opens and claims on mutual fund returns ra are collected.

I will let wealth n be the consolidated result of the above transactions. At this point, next period's productivity shock z_{t+1} is revealed and consumption and savings/portfolio decisions are made.

By assuming there is no idiosyncratic uncertainty, nor in the form of matching risk in the capital market neither as unknown productivity, I am excluding precautionary motives for holding either money or assets that are traditional in the literature. Moreover, money is used for consumption related transactions as well as capital expenditures. The purpose of this paper is to highlight the later and, thus, I abstract completely from the first.⁹

2.A.ii. *Formal Problem*

Let $V(n, z)$ be the value function of an entrepreneur in an island with period utility of consumption $\log(c)$, holding real wealth n and with next period's productivity shock z with transition density $\rho(z'|z)$. Formally,

⁹The benefits of this simplification is being able to derive sharper predictions at the cost of some realism. Nonetheless, it is rather straightforward to extend the model to include a consumption cash-in-advance constraint for workers. Doing so for entrepreneurs might take some effort and may result in only being able to solve the model numerically. I believe it would not qualitatively change the new theoretical results when inflation is low since at the zero nominal interest rate standard consumption cash-in-advance models have a non-binding constraint.

the problem is:

$$\begin{aligned}
 V(n, z) &= \max_{a', m', c} \log(c) + \beta \mathbb{E} [V(n'(a', m', z), z') | z] \\
 & \quad s.t. \\
 c + a' + \phi m' &\leq n \\
 & \quad \text{where} \\
 n'(a', m', z) &= \max_{l, k} a'(1 + r') + \phi' m' + (zk)^\alpha l^{1-\alpha} - w'l - (\tilde{r}' + \delta)k \\
 & \quad s.t. \\
 (\tilde{r}' + \delta)k &\leq \phi' m' \\
 m &\geq 0 \\
 a &\geq 0
 \end{aligned}$$

The no shorting constraint $a \geq 0$ prevents high productivity entrepreneurs from accumulating an infinite amount of money by borrowing at the risk free rate. Low productivity entrepreneurs, on the other hand, will always be unconstrained. In [Appendix ??](#) I extend the model by allowing entrepreneurs to borrow up to a proportion of their wealth i.e. $a \geq -\vartheta n$ and show that the qualitative equilibrium properties remain unchanged. [Claim 1](#) below characterizes the optimal solution to the second stage intratemporal problem.

CLAIM 1. An entrepreneur with assets a' , money m' and productivity z has strictly positive capital and labor demands only if $z \geq \bar{z} \equiv \frac{1}{\alpha}(\tilde{r}' + \delta) \left(\frac{w'}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}$. In particular, these are:

$$\begin{aligned}
 k(a', m', z) &= \frac{1}{\tilde{r}' + \delta} \phi' m' \\
 l(a', m', z) &= \left(\frac{(1-\alpha)}{w'}\right)^{\frac{1}{\alpha}} z k(a', m', z)
 \end{aligned}$$

Proof. The proof is straightforward. Once labor demand is optimally set, the entrepreneur's problem has linear objective and constraints in k . In turn, corner solutions arise where some entrepreneurs will produce, borrow money and rent as much capital as allowed by their leverage constraints and others will not demand any capital at all. ■

The linearity of capital demand makes the first stage problem very tractable. This idea has been used by [Angeletos \(2007\)](#), [Moll \(2011\)](#) and [Kiyotaki and Moore \(2012\)](#) in other contexts. Using [Claim 1](#), we

can re-write the problem as

$$\begin{aligned}
 V(n, z) &= \max_{a', m', c} \log(c) + \beta \mathbb{E} [V(n'(a', m', z), z') | z] \\
 & \text{s.t.} \\
 c + a' + \phi m' &\leq n \\
 n'(a', m', z) &= a'(1 + r') + \phi' m'(1 + \max\{\frac{z}{z'} - 1, 0\})
 \end{aligned}$$

This is isomorphic to the optimal consumption and portfolio allocation problem studied in the seminal work by [Merton \(1969\)](#). For some productivity shocks it will be money that dominates claims on the mutual fund.

Let us turn now to the characterization of the intertemporal allocation of consumption and savings. As is the case in [Merton \(1969\)](#), with constant relative risk aversion, the share of wealth allocated to each type of asset is independent of wealth and only depends on the returns process and, in particular, the assumption of log-utility implies the savings rate over wealth is equal to the discount factor. The following lemma formally characterizes the the solution.

Let the inflation rate be $\pi' \equiv \frac{\phi}{\phi'} - 1$ and the nominal interest rate $i \equiv (1 + \pi')(1 + r') - 1$.

LEMMA 1. [There is a threshold \$z^*\$ such that an entrepreneur's optimal portfolio and consumption allocation is:](#)

$$\begin{aligned}
 c &= (1 - \beta)n \\
 a' &= \beta n \mathbb{I}_{z < z^*} \\
 \phi m' &= \beta n (1 - \mathbb{I}_{z < z^*})
 \end{aligned}$$

[Moreover, if \$i > 0\$, then \$z^* = \bar{z}'\(1 + i\)\$ and if \$i = 0\$, then \$z^* \leq \bar{z}'\$.](#)

When $i > 0$ it is easily seen by comparing the returns of a and m that a threshold $z^* = \bar{z}'(1 + i)$ exists and identifies a marginal entrepreneur that is indifferent between both types of assets. Thus even when some entrepreneurs with $z > \bar{z}'$ would be deriving positive period profits from production, they decide not to produce because of the opportunity cost of holding money. When $i = 0$, all entrepreneurs with $z > \bar{z}'$ strictly prefer money to assets and thus produce. However, entrepreneurs with $z \leq \bar{z}'$ are indifferent between holding claims or money. Without loss of generality, I assume there is a $z^* \leq \bar{z}'$ such that amongst the indifferent entrepreneurs those with $z \geq z^*$ decide to hold money and those with $z < z^*$ hold assets.¹⁰

Because static profits are linear in capital, the gains from holding money are not subject to decreasing marginal returns which makes some entrepreneurs *money non-satiated* even at the zero nominal interest

¹⁰In any equilibrium with positive capital supply, we must have $i \geq 0$. If i was negative none of the entrepreneurs would find it optimal to hold claims a and aggregate output would be zero. I will concentrate in the former class of equilibria.

rate. While relatively low productivity entrepreneurs will hoard money, using it as a savings vehicle alone, other higher productivity entrepreneurs will still use it for production purposes when transacting in the capital market. This is in contrast to many other models of money demand and will be largely responsible for the new results in this paper. Having money being a complement in production when the technology is subject to constant returns to scale is only one way, although perhaps an attractive one, of introducing non-satiation without recurring to strange utility functions.

2.B. *Workers*

There is a measure one of workers that are hand-to-mouth and have optimal labor supply $l = \min\{w^v, 1\}$. Moreover, they receive transfers from the government (the inflation tax payed by entrepreneurs). Hence, worker consumption is $c_w = \min\{w^v, 1\}w + \phi T$.

I choose to transfer the inflation tax revenues to workers instead of entrepreneurs for tractability purposes alone, since it keeps individual policy functions linear in wealth. Alternatively, we can think the government uses the revenues to finance some exogenous stream of government expenditures. In [Section 7.B](#). I entertain the possibility that government introduces money in the economy via open market operations.

The assumption that workers are excluded from capital markets will not be consequential for the long run implications of the model. If we introduced borrowing constraints alone, they would still decide not to save in a stationary equilibrium given that the real interest rate is lower than the discount rate.

3. STATIONARY EQUILIBRIUM

We are now ready to define a stationary equilibrium for this economy. I do so in two steps. First, I start by defining a competitive search equilibrium with a fixed number of bankers in the decentralized spot capital rental market. Later, armed with the characterization of rental rates \tilde{r} , I present the complete definition of a stationary equilibrium for the economy as a whole.

3.A. *Equilibrium in the decentralized capital market*

Each period a market for intermediating funds invested in the mutual fund to entrepreneurs opens. Bankers finance themselves at rate r and lend at rate \tilde{r} . Submarkets are indexed by the rental rate \tilde{r} , thickness θ and expected capital demand $k^d(\tilde{r})$

A banker that has been matched with an entrepreneur but is uncertain about the entrepreneurial type obtains expected profits $(\tilde{r} - r)k^d(\tilde{r})$, where $k^d(\tilde{r})$ is expected capital demand taken over all types (z, n) that decide to participate in sub-market \tilde{r} .

Bankers can open sub-markets by paying a fixed cost and posting \tilde{r} .¹¹

ASSUMPTION 1. The fixed cost is $qk^d(\tilde{r})$.

ASSUMPTION 2. There is a fixed measure of bankers b .

The assumption that search costs are proportional to $k^d(\tilde{r})$ is convenient as it implies that the intermediation technology exhibits constant return to scale and prevents equilibrium multiplicity. Equivalently, the average cost (the ratio of total cost to total intermediated capital) is decreasing in the number of projects/entrepreneurs financed but not in the size of the projects. I find this appealing since an intermediary with a given number of clients is just a scaled up version of another intermediary with the same number of clients but where the average client capital demand is lower.

Assuming there is no free entry and exit of bankers implies that one should think of the theoretical results that follow as applying to the short to medium run of economies where the measure of intermediaries cannot adjust. Nevertheless, I discuss the implications of relaxing this assumption as well and note that these should apply in the long run.

Within a given sub-market matching proceeds as follows. Entrepreneurs can costlessly find a banker with probability one and thus are indifferent about the thickness of the market as long as there is a strictly positive measure of bankers in the sub-market. On the other hand, since bankers have constant returns to scale in the supply of capital, they can serve more than one entrepreneur at a time in the case that $\theta > 1$. If $\theta < 1$, some bankers are left idle. I assume that projects in each submarket are distributed equally across all bankers and thus expected profits from opening sub-market $\{\tilde{r}, \theta, k^d(\tilde{r})\}$ are:

$$\Gamma(\tilde{r}, \theta, k^d(\tilde{r})) = (\theta(\tilde{r} - r) - q)k^d(\tilde{r}).$$

The matching technology is somewhat different than in traditional search models.¹² Usually, matching is bilateral and sellers can only serve one buyer at a time, which amounts to one side always being rationed (except for the knife edge case where there is exactly an equal measure of both buyers and sellers). As I have described it, the matching and banker's intermediation technology allows them to serve more than one entrepreneur at a time; resulting in entrepreneurs never being rationed. Formally, the traditional matching technology specifies a probability which has to be between zero and one, while the technology in this paper only posits a positive measure.

Profits for an entrepreneur holding money $m(z, n)$ and directing her search towards sub-market $\{\tilde{r}, \theta, k^d(\tilde{r})\}$

¹¹If the entrepreneurial types are private information it could potentially make menus of a rental rate and a capital quantity profitable for screening purposes. By only allowing bankers to post \tilde{r} we could be missing an important class of equilibria. In [Appendix F](#) I argue that such menus are never optimal for the banker and thus nothing is lost by restricting the set of banker strategies to price posting alone, resulting in all types being pooled.

¹²See [Moen \(1997\)](#); [Guerrieri, Shimer, and Wright \(2010\)](#) and [Rocheteau and Wright \(2005\)](#).

are:

$$\Pi(z, n, \tilde{r}, \theta) = \mathbb{I}_{\theta < \infty} \left(\alpha \left(\frac{(1-\alpha)}{w} \right)^{\frac{1-\alpha}{\alpha}} \frac{z}{\tilde{r} + \delta} - 1 \right) \phi m(z, n)$$

DEFINITION 1. For a given measure of bankers b , real money policy function $\phi m(z, n)$, entrepreneurial types distribution $\Psi(z, n)$ and prices $\{r, w\}$; a *Competitive Search Equilibrium* (CSE) is an allocation $\{V, \iota, B, \tilde{R}, \Theta, K^d\}$ with $V : Z \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\iota : Z \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $B \in \mathbb{R}_+$, $\tilde{R} \subset \mathbb{R}_+$, $\Theta : \mathbb{R}_+ \rightarrow [0, \infty]$, $K^d : \mathbb{R}_+ \rightarrow [0, \infty]$ satisfying:

1. Entrepreneurs optimality: for any $\tilde{r} \in \mathbb{R}_+$ and for all z, n

$$\Pi(z, n, \tilde{r}, \Theta(\tilde{r})) \leq V(z, n)$$

with equality if $\Theta(\tilde{r}) < \infty$, where

$$\iota(z, n) = \operatorname{argmax}_{\tilde{r} \in \mathbb{R}_+} \Pi(z, n, \tilde{r}, \Theta(\tilde{r}))$$

$$V(z, n) = \Pi(z, n, \iota(z, n), \Theta(\iota(z, n)))$$

2. Bankers optimality: for any $\tilde{r} \in \mathbb{R}_+$

$$\Gamma(\tilde{r}, \Theta(\tilde{r}), K^d(\tilde{r})) \leq B$$

with equality if $\Theta(\tilde{r}) > 0$, where

$$B = \max_{\tilde{r} \in \mathbb{R}_+} \Gamma(\tilde{r}, \Theta(\tilde{r}), K^d(\tilde{r}))$$

3. Market clearing:

$$b = \int_{\tilde{r} \in \tilde{R}} \int_{Z \times \mathbb{R}_+} \frac{\mathbb{I}_{\iota(z, n) = \tilde{r}}}{\Theta(\tilde{r})} d\Psi(z, n) d\tilde{r}$$

$$K^d(\tilde{r}) = \int_{Z \times \mathbb{R}_+} \frac{\phi m(z, n)}{\tilde{r} + \delta} \mathbb{I}_{\iota(z, n) = \tilde{r}} d\Psi(z, n) = k^d(\tilde{r}) \quad \forall \tilde{r} \in \tilde{R}$$

Condition 1. requires that entrepreneurs direct their search towards the market that maximizes their gains from trade given a set of beliefs about the distribution of interest rates and thickness in each market. Similarly, condition 2. requires that intermediaries post interest rates in order to maximize profits. Condition 3. is a consistency requirement: the aggregate measure of bankers has to be equal to the sum over all bankers in each open sub-market and the aggregate capital intermediated should be consistent with individual capital demands for given money holdings distribution and bankers' beliefs.

The following lemma characterizes the rental rate in the decentralized spot capital market and shows that competition amongst a fixed measure of intermediaries facing a given pool of entrepreneurs with profitable production opportunities results in *congestion* driving a wedge between the real interest rate paid to investors and the marginal cost of capital paid by producers. Traditionally equilibria with directed search and price posting do not exhibit this type of externalities, in contrast to random search equilibria.¹³ The key difference here is that there is free entry on the side of the market searching but not on the side posting prices and the specific matching technology that makes the thickness of a market almost irrelevant for the searchers. Once we allow for free entry on the intermediaries side, as is done in **Lemma 3**, the externality disappears.

LEMMA 2. In a CSE there is a unique sub-market open, where $\tilde{r} = r + \frac{q}{\theta}$ and $\theta = \frac{1}{b} \int_{z > \max\{\bar{z}, z^*\}} d\Psi(z, n)$

Proof. Entrepreneurs are indifferent about the thickness of each submarket as long as $\Theta(\tilde{r}') < \infty$ and direct their search towards the market with the lowest \tilde{r}' . Thus any market with $\tilde{r}' > \tilde{r}$, necessarily has $\Theta(\tilde{r}') = 0$. The market clearing conditions together with the policy functions $m(z, n)$ then imply that $\Theta(\tilde{r}) = \frac{1}{b} \int_{z > \max\{\bar{z}, z^*\}} d\Psi(z, n)$. Finally, in the unique open sub-market Bertrand-style competition pushes $B = 0$ and, hence, $\tilde{r} = r + \frac{q}{\theta}$. ■

LEMMA 3. The equilibrium with free entry would have $b = \int_{z > \max\{\bar{z}, z^*\}} d\Psi(z, n)$ and is equivalent to an equilibrium with no search frictions, $q = 0$, and a higher depreciation rate δ .

3.B. Full equilibrium definition

We are now ready to define a Stationary Equilibrium for this economy. Let N be aggregate wealth, Y aggregate output and $\frac{\partial \Phi(z)}{\partial z} = \frac{\int n \psi(n, z) dn}{N}$ the wealth density of islands with productivity shock z . When productivity shocks are iid the distribution of z and n are independent and $\Phi(z)$ is equal to the exogenous distribution of z . I will assume this is the case throughout the paper and leave the case where shocks are persistent for future research.¹⁴

Let the money supply growth rate μ be exogenously determined and \tilde{m} be aggregate real money balances.

DEFINITION 2. For given μ , a *Stationary Equilibrium* is a price system \tilde{r}, r, w, π, i ; thresholds \bar{z}, z^* ; strictly positive constant aggregates N, Y, \tilde{m} ; individual policy functions $c(z, n), k(z, n), l(z, n), m(z, n), a(z, n)$, a law of motion for individual wealth $n'(z, n)$ such that:

1. Given prices, the policy functions and wealth law of motion are optimal for the entrepreneur.

¹³See Moen (1997) and Mortensen and Pissarides (1994) for examples of each.

¹⁴Kiyotaki (1998) and Moll (2011) also use a similar density definition in characterizing equilibria. Given that all policy functions are linear in wealth n this is a convenient normalization for characterizing the aggregate economy, even though the distribution of wealth is undetermined.

2. Markets clear:

- $\min\{w^v, 1\} = N \int_{\max\{\bar{z}, z^*\}} l(z, 1) d\Phi(z)$ (Labor)
- $\int^{z^*} a(z, 1) d\Phi(z) = \int_{\max\{\bar{z}, z^*\}} k(z, 1) d\Phi(z)$ (Capital)
- $1 = \int n'(z, 1) d\Phi(z)$ (Goods/Euler)
- $\tilde{m} = \frac{1}{1+\mu} N \int_{z^*} m(z, 1) d\Phi(z)$ (Money)
- $Y = \frac{w \min\{w^v, 1\}}{1-\alpha}$ (Output)

3. $1 + i = (1 + r)(1 + \pi)$ (Fisher equation)

4. $1 + \pi = 1 + \mu$ (Constant real money)

5. Thresholds satisfy:

- $\bar{z} = \frac{\tilde{r} + \delta}{\alpha} \left(\frac{w}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}}$
- $z^* = \bar{z}(1 + i)$ if $i > 0$ and $z^* \leq \bar{z}$ if $i = 0$

6. The rental rate in the decentralized capital market is consistent with a CSE:

- $\tilde{r} = r + \frac{qb}{1-\Phi(\max\{\bar{z}, z^*\})}$ (CSE)

The definition of a Stationary Equilibrium is rather standard. Two comments are in order though. First, condition 6. closes the system by specifying an expression for the marginal cost of capital that is consistent with a competitive search equilibrium in the spot capital market. Second, condition 4. says that we are going to be looking for an equilibrium where aggregate real money balances are constant. Because some entrepreneurs will always be *money non-satiated* this requires that the inflation rate and the aggregate money supply growth rate coincide.

4. INFLATION, OUTPUT AND THE NOMINAL INTEREST RATE

This section characterizes the Stationary Equilibrium more sharply, presents existence and uniqueness proofs and discusses the paper's main results.

PROPOSITION 1. There exists a unique threshold money growth rate $\bar{\mu} : i = 0 \iff \mu \in [\beta - 1, \bar{\mu}]$ and $i > 0 \iff \mu > \bar{\mu}$.

COROLLARY 1. If $\mu > \beta - 1$ a Stationary Equilibrium exists and is unique.

Proof. See [Appendix A](#)

The prediction that a zero nominal interest rate is consistent with several inflation rates, whereas the relationship is one-to-one when the nominal interest rate is strictly positive; is independent of the decentralized capital market structure and the assumption of fixed entry. It would still hold even if $q = 0$ or with free entry of bankers as it is a direct consequence of agent heterogeneity, together with a positive measure of agents still being liquidity constrained when the nominal interest rate is zero.

The zero lower bound on the nominal interest rate effectively imposes a lower bound on the real interest for a given inflation rate since $r = -\frac{\pi}{1+\pi}$ when $i = 0$. How do we make consistent an already determined system of equation characterizing an equilibrium with the introduction of a new equation setting the nominal interest rate to zero. Which equation, if any, should be dropped? These are the questions asked in Hall (2011) and Kocherlakota (2012). Such is not an issue in the present model as there is a natural equation that is not an equilibrium condition anymore when $i = 0$; that is $z^* = \bar{z}(1+i)$. When the nominal interest rate is positive some agents strictly prefer to hold claims on the mutual fund, others strictly prefer to hold money and aggregate capital supply per unit of wealth is less than perfectly elastic. However, when the nominal interest rate is zero, agents who have low productivity shocks and only desire to hold assets for savings purposes are indifferent between claims and money. As a result, aggregate capital supply per unit of wealth becomes perfectly elastic at the real interest rate r when $i = 0$ and aggregate capital in equilibrium is demand determined.

Proposition 1 is in sharp contrast with Cole and Kocherlakota (1998). They conclude that “...while inflation is a monetary phenomenon for any suboptimal monetary policy, inflation is entirely a real phenomenon for any optimal monetary policy (because the rate of deflation equals the real rate of interest).” and “...the optimality of monetary policy can be verified only by looking at interest rates, not by looking at the growth rates of the money supply.”

By a suboptimal monetary policy they mean any policy that is not consistent with a zero nominal interest rate. The sufficiency of a zero nominal interest rate for optimality and the disconnect between inflation and money growth rates hinge entirely on a non-binding cash-in-advance constraint at the zero nominal interest rate, which is true for all the class of models exhibiting *satiation*, and the assumption of a representative agent.

Proposition 1 together with the definition of Stationary Equilibrium establish that inflation is still a monetary phenomenon even at the zero nominal interest rate precisely because some agents continue to be money constrained.

While the non-optimality of a zero nominal interest rate in heterogeneous agent economies due to redistributive concerns has already been established ¹⁵, **Proposition 2** and its corollary describe first order aggregate output losses at the zero lower bound. Of course, the fact that aggregate output is not maximized is not sufficient for the non-optimality of a positive nominal interest rate. In **Section 5**. I show that a zero nominal interest is a necessary but not a sufficient condition for constrained efficiency, precisely

¹⁵See...some references

because of these losses.

PROPOSITION 2. In a Stationary Equilibrium, output Y is decreasing in μ for all $\mu > \bar{\mu}$. Also, there exists $\underline{\mu} \leq \bar{\mu}$ such that Y is increasing in μ for all $\mu \in [\beta - 1, \underline{\mu})$.

COROLLARY 2. The output maximizing money growth rate $\mu^* \in [\underline{\mu}, \bar{\mu}]$, with $\underline{\mu}$ and $\bar{\mu}$ increasing in q . Only if $q = 0$ or under banker's free entry, $\mu^* = \underline{\mu} = \beta - 1$.

Proof. See [Appendix B](#).

When the nominal interest rate is positive, capital misallocation worsens since some agents decide not to undertake projects that would be profitable if the opportunity cost of holding money was eliminated. This is clearly seen from equation $z^* = \bar{z}(1 + i)$. On the other hand, when the nominal interest rate is zero and inflation is low enough, entrepreneurs' decision to hoard money instead of devoting resources to low productivity projects results in congestion in the spot capital market and a higher marginal cost of capital.

To gain some intuition on the mechanics behind the comparative statics it is useful to discuss the partial equilibrium effect of a change in the real interest rate r . Consider, for instance, an increase in r . Some marginal entrepreneurs decide to stop producing as the return of holding claims in the mutual fund increased in comparison to holding money and renting capital. This *selection effect* decreases the extensive margin of labor demand but increases the intensive margin since the average entrepreneur is more productive. At the same time, the intermediary's marginal financing cost has increased and the probability of finding an entrepreneur decreased. Competition in the capital rental market pushes the marginal cost of capital up and the intensive margin of capital and labor demand decrease because of this *congestion effect*. The total effect on wages (and aggregate output) depend on the relative magnitudes of both effects on labor demand.

How does a change in r come about in equilibrium? At the zero lower bound reductions in inflation/money growth pass through directly to increases in the real rate, whereas at a positive nominal rate the effect is divided between a decrease in the nominal rate and an increase in the real rate. When $q = 0$ or under banker's free entry aggregate labor demand always increases as inflation decreases and r increases; the output maximizing real rate is equal to the discount rate. In an economy with no search frictions, the inflation tax negative effect on the intensive margin of capital demand is too strong and dominates the positive effect due to lower real interest rates because the congestion effect is absent. When $q > 0$ and banker's are in fixed supply, labor demand increases with the real rate (decreases with inflation) as long as the nominal interest rate is positive, again as a result of eliminating the opportunity cost of holding money and inflation tax type distortions. However, at the zero lower bound and when inflation is low enough, the probability of finding an entrepreneur is so low that a small reduction in real interest rates (rise in inflation) brings in a relatively large measure of new, less efficient producers, raises the probability of finding an entrepreneur and decreases the marginal cost of capital by a significant amount. As a consequence, labor demand and output increase. For intermediate inflation rates, the total effect is ambiguous.

Corollary 2 formally states this and characterizes a nonempty interval containing the output maximizing inflation. The interval ends rise with q and thus, typically, the output maximizing money growth rate would increase as well. However, this is not guaranteed without restricting the shock distribution $\Phi(z)$.

I conclude the section with some further comparative statics regarding the effects of inflation on interest rates and money velocity that hold under some restrictions on the shock distribution detailed in the appendix.

Let money velocity be $V \equiv \frac{Y}{\bar{m}}$. The model delivers an interest-elastic velocity that is increasing in inflation and the nominal interest rate ¹⁶.

CLAIM 2. Money velocity is:

$$V(i, \bar{z}, z^*) = \begin{cases} (1+i) \frac{\mathbb{E}[z|z>z^*]}{\alpha z^*} & i \geq 0 \\ \frac{1-\Phi(\bar{z})}{1-\Phi(z^*)} \frac{\mathbb{E}[z|z>\bar{z}]}{\alpha \bar{z}} & i = 0 \end{cases}$$

The endogenous elasticity is akin to models with segmented assets markets in the tradition of **Alvarez, Lucas, and Weber (2001)** where the extensive margin of money demand determines velocity. While in most of these models segmentation occurs as a consequence of fixed costs or idiosyncratic preference shocks; in the model I have described, it results from idiosyncratic productivity shocks separating entrepreneurs into holding only one type of asset. In **Section 7.A.**, I extend the model by allowing agents to issue collateralized debt for financing production which further affects velocity as entrepreneurs are able to substitute away from money and escape the inflation tax.

Let $\tilde{i} \equiv (1 + \tilde{r})(1 + \mu) - 1$ be the nominal interest rate paid by entrepreneurs in the capital rental market and $s = \alpha \frac{Y}{K} - (r + \delta)$ the investment wedge as defined in **Chari, Kehoe, and McGrattan (2007)** stated as an interest rate spread which is a crude measure of distortions in the economy.

CLAIM 3. \tilde{i} is decreasing in μ for all $\mu \in [\beta - 1, \bar{\mu})$ and increasing for all $\mu > \bar{\mu}$

CLAIM 4. The investment wedge is $s = (\tilde{r} + \delta) \frac{\mathbb{E}[z|z>\max\{\bar{z}, z^*\}]}{\max\{\bar{z}, z^*\}} (1 + i) - (r + \delta)$.

Proof. See **Appendix C**

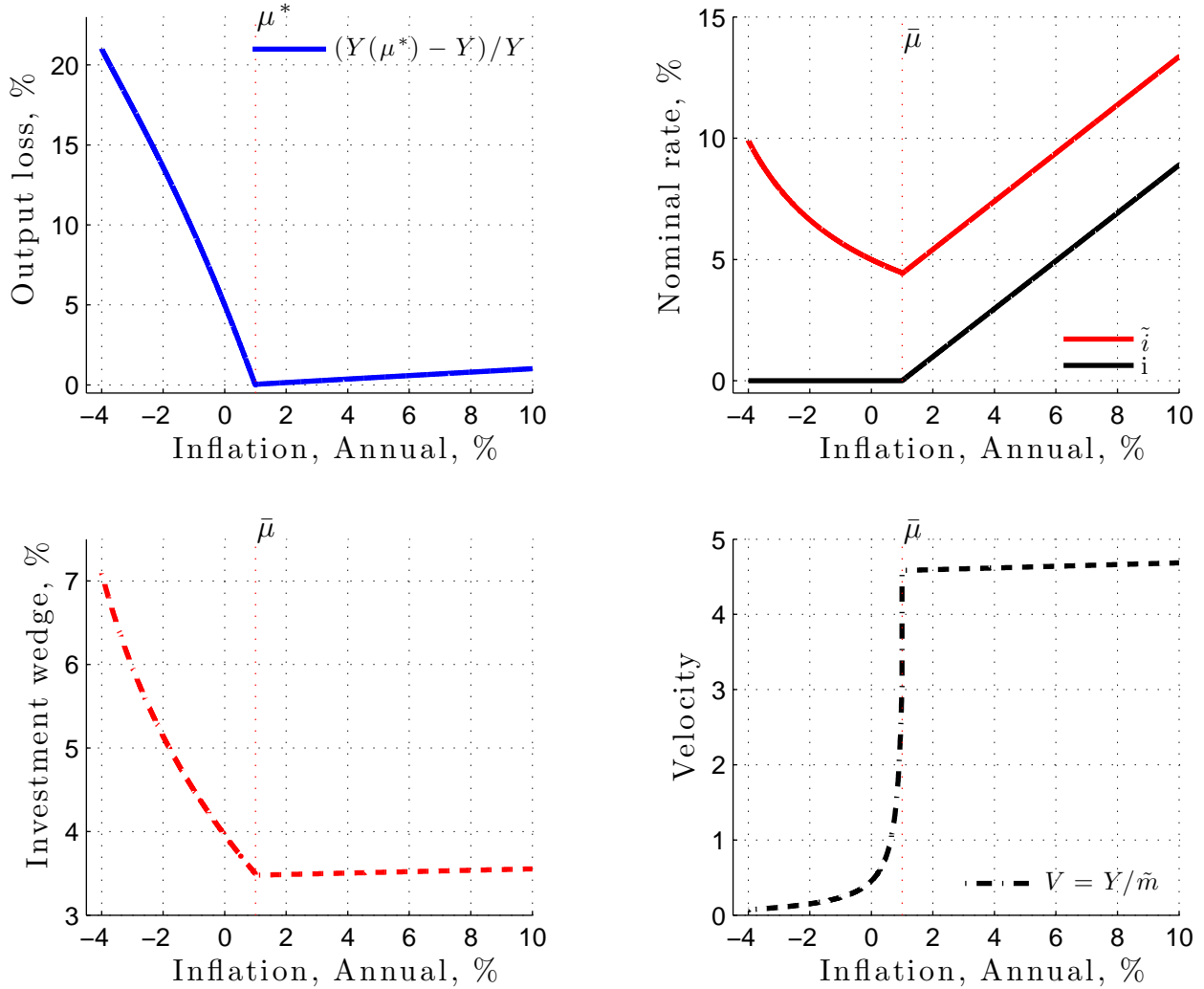
4.A. A numerical example

To illustrate the propositions characterizing the behavior of interest rates, output and money demand in a Stationary Equilibrium, I parametrize the shocks to be Pareto distributed. The details of the calibration and

¹⁶To obtain the result, It is sufficient that $\frac{\mathbb{E}[z|z>z]}{z}$ be weakly decreasing, since $\bar{z}(z^*)$ is decreasing (increasing) in μ when $i = 0$; and z^* is decreasing when $i > 0$.

robustness to parameter changes are in [Section G](#) in the Appendix. The first panel in [Figure 2](#) presents the output losses associated to deviations from the optimal money growth rate μ^* . The second panel shows the comparative statics of producer payed, \tilde{i} , and investor perceived, i , nominal interest rates. The last panel describes money velocity.

Figure 2: Comparative statics



For this particular calibration the output maximizing money growth rate μ^* and the upper bound money growth rate consistent with a zero lender nominal interest rate $\bar{\mu}$ are equalized.¹⁷ Moreover, the implied real interest rate is always positive since $\bar{\mu}$ is negative.

It is worth noting the strong asymmetry for low and high inflation rates. When $\mu < \bar{\mu}$ the Stationary equilibrium features the hallmarks of a deflationary depression. Output drops sharply, the nominal interest rate paid by producers increases while the nominal interest rate perceived by investors is stuck at zero, the investment wedge increases and money velocity significantly decreases. For example, going from 1

¹⁷This implies μ^* is also constrained efficient as defined in [Section 5](#).

percent inflation to 1 percent deflation decreases output by 10 percent while the spread between interest rates increases less than 1 percent and money velocity converges to zero. When $\mu > \bar{\mu}$ the nominal interest rate turns positive, the interest rate spread remains roughly constant and we see a very modest effect of inflation on output. The rising opportunity cost of holding money increases velocity, but the elasticity is somewhat low. This is far from surprising. Given the lack of money substitutes for production most of the action comes from the extensive margin. [Section 7.A.](#) entertains the possibility of carrying out transactions with imperfect money substitutes, which makes money demand more elastic, and presents a very clean expression for this elasticity as a function of technological parameters alone.

5. CONSTRAINED EFFICIENCY

In a Stationary Equilibrium there is a pecuniary externality. When entrepreneurs choose their portfolio they do not take into account the effect on prices r, \tilde{r}, w and how they affect other entrepreneurs and workers.

The presence of a pecuniary externality is not sufficient to make the equilibrium ex ante inefficient. However, this turns out to be the case. I formalize the idea by studying the problem of planner that is not able to alter consumption, savings, capital and labor decisions by individual entrepreneurs and workers.¹⁸ I only allow the constrained planner to choose the measure of agents that hold either money or claims conditional on the type (z, n) in order to maximize an ex-ante utilitarian welfare function.

I show the equilibrium is not constrained efficient in the spirit of [Davila, Hong, Krusell, and RiosRull \(2012\)](#) and discuss how a benevolent government could implement the constrained efficient allocation by setting the nominal interest rate to zero and choosing the inflation rate appropriately.

To gain some intuition and see how the market allocation can be improved upon, consider the following thought experiment when $i = 0$.

The private cost of financing a project with $z < \bar{z}$ is $\frac{\beta n}{1+\pi}(1 - \frac{z}{\bar{z}})$. Now again, if ALL entrepreneurs with productivity $z \in (\bar{z} - \varepsilon, \bar{z})$ simultaneously decided to produce, the new threshold would be $\bar{z}' = \bar{z} - \varepsilon$; the benefit for all $z \in (\bar{z}', \bar{z})$ would be $\frac{\beta n}{1+\pi}(\frac{z}{\bar{z}} - 1)$ and for all $z > \bar{z}$ it would be $\frac{\beta n}{1+\pi} \frac{z}{\bar{z}^2} \varepsilon$.

Hence, all entrepreneurs benefit from the decrease in \bar{z} . What about workers?

The wage change is $dw = \varepsilon \frac{\alpha w}{(\alpha-1)\bar{z}} \left(1 - q \frac{\Phi'(\bar{z})}{(\frac{1-\alpha}{w})^{\frac{1-\alpha}{\alpha}} (1-\Phi(\bar{z}))} \right)$, which will be positive when q is not too small. Thus, if all marginal entrepreneurs that find it unprofitable to finance their projects were forced to do so, all agents would be made better off.

[Appendix D](#) presents the formal planning problem characterizing the constrained efficient allocation that leads to the following proposition,¹⁹

¹⁸The planner would also desire to alter this if possible. While it could achieve it by direct taxation, the issue is beyond the purpose of this paper.

¹⁹For simplicity, I only study the case where labor supply is perfectly inelastic and worker utility is logarithmic. The constrained inefficiency proposition still holds if labor supply is elastic and under general utility functions. The exact characterization of the monetary policy that implements the constrained efficient allocation becomes more involved and may require

PROPOSITION 3. If $\vartheta \geq \frac{1-\alpha}{\alpha(1-\beta)}$ (i.e worker welfare is valued), a constrained efficient allocation requires $\bar{z}_t = z_t^* \implies i_t = 0$.

COROLLARY 3. Setting $\mu = \bar{\mu}$ implements it in a Stationary Equilibrium. Also, $\bar{\mu}$ is increasing in q .

As in [Davila et al. \(2012\)](#) the object of study is the stationary outcome that is optimal when taking discounting into account. In other words, the constrained efficient steady state is not derived by maximizing steady-state utility. Instead, I characterize the stationary outcome that is achieved when the planner sequentially optimizes the utilitarian welfare function. It turns out that such problem can be reduced to a sequence of static problems when productivity shocks are iid, greatly simplifying the analysis. Given a welfare weight ϑ and functionals $\{\bar{z}(\gamma), i(\gamma), r(\gamma), w(\gamma)\}$ consistent with an equilibrium, the planner chooses a measure $\gamma(z) \in [0, 1]$ each period to maximize the welfare function:

$$\frac{1}{1-\beta} \left[\int \log \left(\frac{1 + \max\{\frac{z}{\bar{z}(\gamma)} - 1, 0\}}{1 + i(\gamma)} \right) \gamma(z) d\Phi(z) + \log(1 + r(\gamma)) \right] + \vartheta \log(w(\gamma))$$

There is a trade-off between the static gains to workers of a one time change in wages and the dynamic gains to entrepreneurs of a permanent change in returns per unit of accumulated wealth. By shifting the money distribution towards relatively more productive entrepreneurs, the planner increases the average productivity of producers and, ultimately, wages. On the other hand, returns decrease as aggregate capital supply per unit of wealth and the mass of entrepreneurs holding mutual fund claims increase. The constrained efficient allocation is characterized by a threshold z^* above which entrepreneurs are chosen to hold money and below which they hold claims. Furthermore, the planner chooses to equalize z^* and \bar{z} at all times if $\vartheta \geq \frac{1-\alpha}{\alpha(1-\beta)}$ and we only consider allocations where $i \geq 0$. Intuitively, it is optimal to eliminate the distortions coming from some entrepreneurs passing on projects because of the opportunity cost of holding money and the distortions resulting from money hoarding.

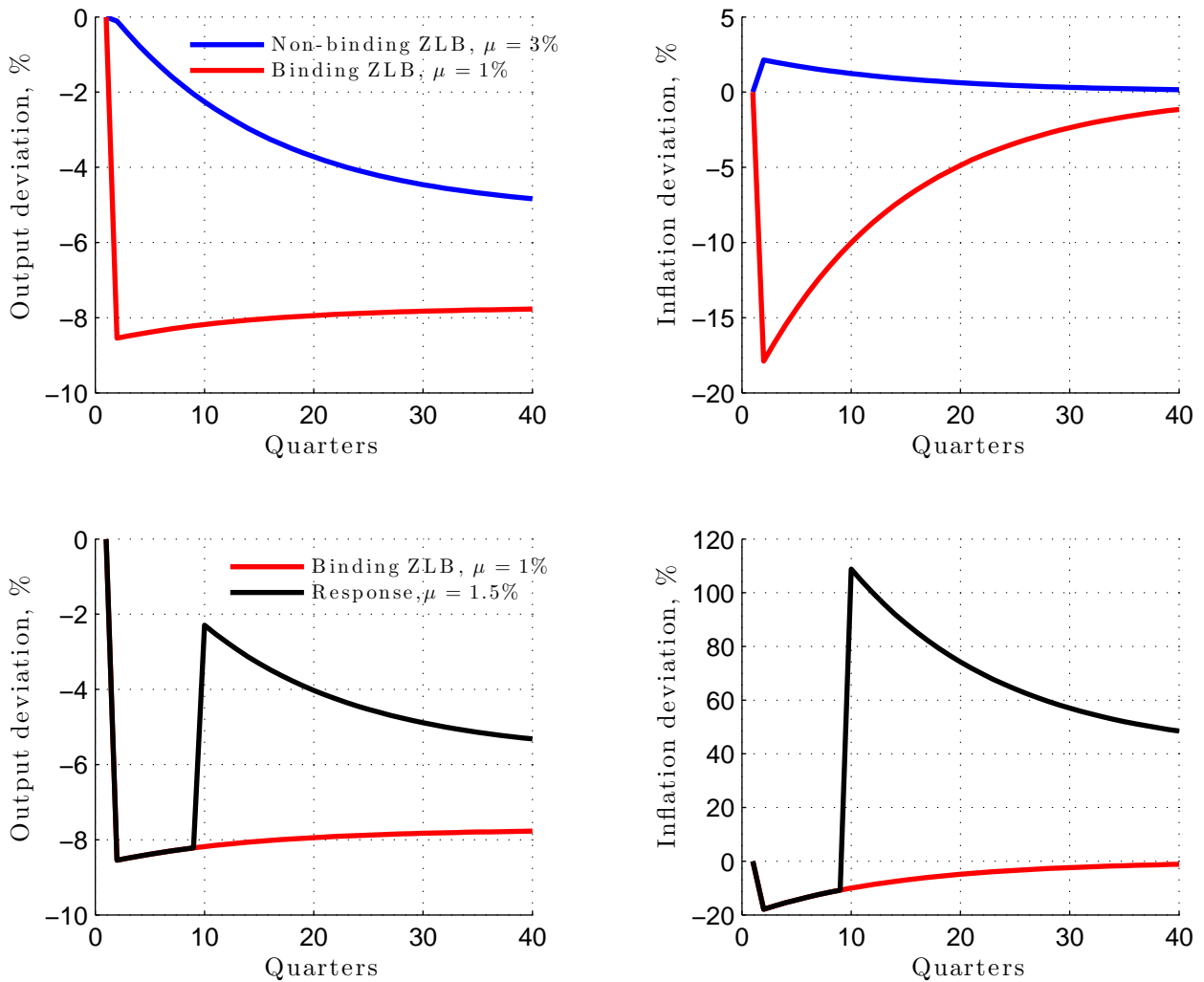
While in the transition the constrained efficient \bar{z} will generally depend on the money growth rate and inflation, we have shown in [Proposition 1](#) that there is only one inflation rate consistent with $z^* = \bar{z}$ and constant aggregates and prices. Then, it is easy to see that setting $i = 0$ is necessary but not sufficient to implement the constrained efficient allocation in a Stationary Equilibrium. The money growth rate has to be set at $\mu = \bar{\mu}$ in order to do so, which in turn implies that $i = 0$. Finally, the optimal money growth rate is increasing in the degree of intermediation frictions parametrized by the fixed cost parameter q . Because an increase in q decreases aggregate capital demand per unit of wealth for all inflation rates without affecting aggregate capital supply, it expands the set of inflation rates for which the zero lower bound binds.

some qualifications.

6. DYNAMICS

I move away from stationary analysis and characterize the dynamics of the economy to better illustrate the interaction between the zero lower bound, intermediation frictions and money supply growth. The derivation can be found in [Appendix E](#) where I setup the aggregate dynamic system of equations and then log-linearize around the steady state. I conduct three exercises: (1) a permanent increase in q starting from a steady state where μ is high and the zero lower bound never binds, (2) a permanent increase in q starting from a steady state where μ is low and the zero lower bound binds after the shock; (3) a permanent increase in μ after a q shock makes the zero lower bound binds. [Figure 3](#) present the impulse responses.

Figure 3: Permanent increase in q



Explanation...

7. EXTENSIONS

7.A. *Money substitutes*

Assume we allow the entrepreneurs to issue within period debt d up to a fraction $\lambda < 1$ of their claims holdings a . This is basically a secured credit line. The constraints in the static profit maximization problem become:

$$\begin{aligned}(\tilde{r} + \delta)k &\leq \phi m + d \\ d &\leq \lambda a\end{aligned}$$

Essentially the claims on the mutual fund become an imperfect substitute for money in carrying out transactions. It is straightforward to show that when $i = 0$ no agent will decide to issue debt since $\lambda < 1$ and the returns on m and a are the same when used for savings. Thus, all the results from the previous sections carry over to an environment with imperfect money substitutes at the zero lower bound.

When the nominal interest rate is positive some agents will decide to hold claims instead of money for renting capital as they can escape the inflation tax. As a consequence, money demand elasticity with respect to the nominal interest rate increases when $i > 0$. To see this, in the case when the shock distribution is Pareto with tail index η , we can derive an intuitive expression for money velocity:

$$V = \frac{1}{\alpha} \frac{\eta}{\eta - 1} \left(1 + i + \lambda(1 + \pi) \left(\left(1 + \frac{i}{(1 - \lambda(1 + \pi))} \right)^\eta - 1 \right) \right)$$

The benchmark case is $\lambda = 0$ where velocity is a linear function of the interest rate. When $1 > \lambda > 0$ agents with intermediate productivity shocks find the returns to claims higher than the returns to holding money and issue some debt. The higher the nominal interest rate, the stronger this substitution effect as can be seen in the term $\left(1 + \frac{i}{(1 - \lambda(1 + \pi))} \right)^\eta$.

7.B. *Open market operations*

Consider a government that issues nominal bonds and money. The government budget constraint is:

$$M_{t+1} - M_t = B_{t+1} - B_t(1 + i_t) \iff \tilde{m}_t \mu_t = \tilde{b}_{t+1}(1 + \pi_{t+1}) - \tilde{b}_t(1 + i_t)$$

Basically, the government buys part of the output produced by entrepreneurs and invest it all in the mutual fund (the opposite than hand-to-mouth agent). In a Stationary Equilibrium, $b = \tilde{m} \frac{\mu}{\mu - i}$. Then, the capital market clearing equation becomes,

$$\Phi(z^*) + \frac{1 - \Phi(z^*)}{1 + \mu} \frac{\mu}{\mu - i} = \frac{1}{\tilde{r} + \delta} \frac{1 - \Phi(\max\{\bar{z}, z^*\})}{1 + \mu}$$

Consider the determination of $\bar{\mu}$. We now have,

$$\Phi(\bar{z}(\bar{\mu})) + \frac{1 - \Phi(\bar{z}(\bar{\mu}))}{1 + \bar{\mu}} = \frac{1 - \Phi(\bar{z}(\bar{\mu}))}{\delta - \bar{\mu}(1 - \delta) + \frac{qb(1 + \bar{\mu})}{1 - \Phi(\bar{z}(\bar{\mu}))}}$$

Since the LHS is larger for all μ when monetary policy is conducted through open market operations, then $\bar{\mu}$ is higher as well. Thus open market operations expand the set of inflation rates consistent with $i = 0$ and increases the constrained optimal inflation rate.

Nevertheless, for a given inflation rate and when the nominal interest rate is zero, aggregate output and ex-ante welfare remain unchanged since only z^* is affected.

8. CONCLUSIONS

I presented a theory of the costs of high and low inflation when money is a complement in production and intermediaries match investors and producers in a decentralized capital market. I showed that *too high* inflation exacerbates capital misallocation and produces a situation akin to *Stagflation* similar to the US in the late 1970's. *Too low* inflation potentially result in Zero Lower Bound Depressions as some agents' money hoarding behavior generate congestion in decentralized markets. By studying the dynamics and the implementation of optimal allocation in such economy, I illustrate how worsening intermediation frictions can push the economy into zero lower bound territory and that higher money supply growth rates are the constrained optimal response in this event...

APPENDIX

A EXISTENCE AND UNIQUENESS OF A STATIONARY EQUILIBRIUM

1.A. Case $i = 0$ and $z^* \leq \bar{z}$

The system of equation characterizing $\{z^*, \bar{z}, r, \tilde{r}\}$ for given π is:

$$\begin{aligned} 1 &= \frac{\beta}{1+\pi} \left(\Phi(\bar{z}) + \int_{z>\bar{z}} \frac{z}{\bar{z}} d\Phi(z) \right) \\ r &= -\frac{\pi}{1+\pi} \\ \tilde{r} &= r + \frac{qb}{1-\Phi(\bar{z})} \\ \Phi(z^*) &= \frac{1}{\tilde{r} + \delta} \frac{1-\Phi(\bar{z})}{1+\pi} \end{aligned}$$

From the first equation it is obvious that a solution for $\bar{z}(\pi)$ with $\bar{z}'(\pi) < 0$, is unique and exists only if $\pi \geq \beta - 1$. Moreover, in order for $z^* \leq \bar{z}$ it has to be that $\Phi(\bar{z}(\pi)) \geq \frac{1-\Phi(\bar{z}(\pi))}{\delta - \pi(1-\delta) + \frac{qb}{1-\Phi(\bar{z}(\pi))}}$. The LHS is a decreasing function of π and equals 1 when $\pi = \beta - 1$. The following assumption ensures the RHS is strictly increasing whenever $\tilde{r} + \delta > 0$ (which is the only case studied here):

ASSUMPTION 3. $\frac{\partial \frac{\mathbb{E}[z|z_0]}{z_0}}{\partial z_0} \geq -\frac{\Phi'(z_0)}{(1-\Phi(z_0))^2}$ or q small.

Furthermore, the RHS equals 0 when $\pi = \beta - 1$. By Bolzano's Theorem there exists a unique cutoff value $\bar{\mu}$ such that $z^* < \bar{z}$ whenever $\pi < \bar{\mu}$, characterized as the solution to:

$$\Phi(\bar{z}(\bar{\mu})) = \frac{1-\Phi(\bar{z}(\bar{\mu}))}{\delta - \bar{\mu}(1-\delta) + \frac{qb(1+\bar{\mu})}{1-\Phi(\bar{z}(\bar{\mu}))}}$$

We conclude that an equilibrium with $i = 0$ exists and is unique if and only if $\pi \in [\beta - 1, \bar{\mu}]$.

1.B. Case $i > 0$ and $z^* = \bar{z}(1+i)$

The system of equation characterizing $\{z^*, \bar{z}, r, \tilde{r}, i\}$ for given π is:

$$\begin{aligned} 1 &= \beta(1+r) \left(\Phi(z^*) + \int_{z>z^*} \frac{z}{z^*} d\Phi(z) \right) \\ 1+i &= (1+r)(1+\pi) \\ \tilde{r} &= r + \frac{qb}{1-\Phi(z^*)} \\ \Phi(z^*) &= \frac{1}{\tilde{r}+\delta} \frac{1-\Phi(z^*)}{1+\pi} \\ z^* &= \bar{z}(1+i) \end{aligned}$$

From the first equation it is obvious that a solution for $r(z^*)$ with $r'(z^*) > 0$. Using the last equation we obtain:

$$g(z^*, \pi) \equiv \frac{\Phi(z^*)}{1-\Phi(z^*)} (1+\pi) = \frac{1}{r(z^*) + \delta + \frac{qb}{1-\Phi(z^*)}} \equiv f(z^*)$$

where $g_z > 0, g_\pi > 0, g(z_{min}, \pi) = 0, g(z_{max}, \pi) = \infty$ and $f_z < 0, f(z_{max}) = 0$. Note that f could have a vertical asymptote at z_a when $r(z_a) + \delta + \frac{qb}{1-\Phi(z_a)} = 0$. If $f(z_{min}) < 0$ or $z_a < z_{min}$ then the above equation has a unique solution z^* . Otherwise, there are two solutions $z_0^* < z_1^*$. However, we can dismiss the solution z_0^* because it implies $\tilde{r} + \delta < 0$. The following assumption is sufficient to ensure that the system of equations characterizes the unique solution without having to worry about $\tilde{r} + \delta$ being negative. Moreover, the solution is such that $\frac{\partial z^*}{\partial \pi} < 0$.

ASSUMPTION 4. $f(z_{min}) = \frac{1}{\frac{1}{\beta\mathbb{E}[z]} + \delta - 1 + qb} < 0$

We are only left to show that at the solution $z^*(\pi)$, the equilibrium features $i(\pi) > 0$. We have shown already that there is a unique $\bar{\mu}$ such that $i(\bar{\mu}) = 0$. Then to show that $i(\pi) > 0$ if and only if $\pi > \bar{\mu}$, it suffices to show that $\frac{\partial i}{\partial \pi}|_{\pi=\bar{\mu}} > 0$. This is equivalent to:

$$\frac{\partial i}{\partial \pi}|_{\pi=\bar{\mu}} = 1 + r(\bar{z}(\bar{\mu})) - (\tilde{r}(\bar{z}(\bar{\mu})) + \delta) \frac{r'(\bar{z}(\bar{\mu}))}{r'(\bar{z}(\bar{\mu})) + \Phi'(\bar{z}(\bar{\mu})) \left(\frac{qb}{1-\Phi(\bar{z}(\bar{\mu}))} + \frac{1}{(1+\bar{\mu})\Phi(\bar{z}(\bar{\mu}))} \right)} > 0$$

Since $r'(\cdot) > 0$, a sufficient condition is given by the following assumption, which will be true in any reasonable calibration:

ASSUMPTION 5. $\tilde{r}(\bar{z}(\bar{\mu})) - r(\bar{z}(\bar{\mu})) < 1 - \delta$

We conclude that under **Assumption 4** and **Assumption 5**, an equilibrium with $i > 0$ exists and is unique if and only if $\pi > \bar{\mu}$.

B COMPARATIVE STATICS OF OUTPUT AND INFLATION

We have that output Y is strictly increasing in w and that $\log(w) \propto \log\left(\frac{\bar{z}}{\bar{r}+\delta}\right)$. Thus by establishing properties of this expression we establish the properties of Y .

2.A. Case $\mu > \bar{\mu}$

Using the system of equation characterizing the equilibrium we obtain:

$$\frac{\bar{z}}{\bar{r}+\delta} = \beta \left(\Phi(z^*)^2 \frac{z^*}{1-\Phi(z^*)} + \mathbb{E}[z|z > z^*] \Phi(z^*) \right)$$

Then $\frac{d\log(w)}{dz^*} > 0$ and, since we have shown before that $\frac{dz^*}{d\mu} < 0$ for all $\mu > \bar{\mu}$, we can conclude that $\frac{d\log(Y)}{d\mu} < 0$ in this case as well.

2.B. Case $\mu \leq \bar{\mu}$ and $q = 0$

Using the system of equation characterizing the equilibrium we obtain:

$$\frac{\bar{z}}{\bar{r}+\delta} = f(\mu, q) \equiv \frac{\bar{z}(\mu)(1-\Phi(\bar{z}(\mu)))}{\left(-\frac{\mu}{1+\mu} + \delta\right)(1-\Phi(\bar{z}(\mu))) + q}$$

When $q = 0$, then $\frac{d\log(f)}{d\mu} = -\frac{1}{1+\mu-\beta\Phi(\bar{z})} + \frac{1}{(1+\mu)(\delta-\mu(1-\delta))}$. Note that, $\frac{d\log(f)}{d\mu}|_{\mu=\beta-1} = -\infty$. Also, $\frac{d^2\log(f)}{d^2\mu} < 0$ only if $(1+2\mu)(1-\delta) - \delta < 0$. The lowest value this can take is $2\beta(1-\delta) - 1$, which will be positive for any reasonable calibration.

ASSUMPTION 6. $2\beta(1-\delta) > 1$ and $\frac{z_{max}}{\frac{1-\beta}{\beta}+\delta} > \frac{\bar{z}(\bar{\mu})}{-\frac{\bar{\mu}}{1+\bar{\mu}}+\delta}$

The first condition ensures that $\frac{d^2\log(f)}{d^2\mu} > 0$. Then, either $f(\mu, 0)$ is always decreasing for all $\mu \in [\beta-1, \bar{\mu}]$ or it has a unique minimum in the interval. Finally, if such minimum exists, the second condition ensures that z_{max} is large enough such that the output maximizing money growth rate is $\mu^* = \beta - 1$.

2.C. Case $\mu \leq \bar{\mu}$ and $q > 0$

Now we have:

$$\frac{d\log(f)}{d\mu} = \frac{1}{1+\mu-\beta\Phi(\bar{z})} \frac{1}{\bar{r}+\delta} \left(\frac{qb}{(1-\Phi(\bar{z}))} \left(\frac{\Phi'(\bar{z})\bar{z}}{1-\Phi(\bar{z})} - 1 \right) + (1-\delta) - \frac{\beta\Phi(\bar{z})}{(1+\mu)^2} \right)$$

The above equation might have several roots. However, let $\underline{\mu}$ be the lowest real root. It is easy to see that $\underline{\mu} > \beta - 1$ and that $\frac{d\log(f)}{d\mu} > 0$ for all $\mu \in [\beta - 1, \underline{\mu}]$. Moreover, $\underline{\mu}$ is increasing in q (as is $\bar{\mu}$).

We can conclude that the output maximizing money growth rate $\mu^* \in [\underline{\mu}, \bar{\mu}]$.

C COMPARATIVE STATICS OF \tilde{i}

We have that $\tilde{i} = i + \frac{qb(1+\mu)}{1-\Phi(\max\{\bar{z}, z^*\})}$. When $i = 0$, we have that:

$$\frac{d\tilde{i}}{d\mu} = \frac{qb}{1-\Phi(\bar{z})} \left(1 - \frac{(1+\mu) \frac{\Phi'(\bar{z})\bar{z}}{1-\Phi(\bar{z})}}{1+\mu-\beta\Phi(\bar{z})} \right)$$

ASSUMPTION 7. $\lim_{z \rightarrow z_m} \frac{\Phi'(\bar{z})\bar{z}}{1-\Phi(\bar{z})} > 1$ and $\frac{\Phi'(\bar{z})\bar{z}}{1-\Phi(\bar{z})}$ is weakly decreasing.

Then, $\frac{d\tilde{i}}{d\mu} < 0$ for all $\mu \in [\beta - 1, \bar{\mu})$.

When $\mu > \bar{\mu}$, under **Assumption 5**, since z^* is decreasing in μ , we have that i is increasing in μ . Then, under **Assumption 7**, we have that \tilde{i} is also increasing in μ .

D CONSTRAINED EFFICIENT ALLOCATION

The planner's problem consists in choosing at each point in time the agents that will hold money conditional on (z, n) , taking as given the consumption, savings, capital and labor demand policy functions.

$$\begin{aligned} & \max_{\mathbb{I}_m(z, n)} \sum_{t=0}^{\infty} \beta^t \int (\log((1-\beta)n_t) + \vartheta_t \log(w_t)) d\psi_t(z, n) \\ & n_{t+1} = \begin{cases} \beta n_t (1+r_{t+1}) & \mathbb{I}_m(z, n) = 0 \\ \beta n_t (1+r_{t+1}) \frac{1+\max\{\frac{z}{\bar{z}_{t+1}}-1, 0\}}{1+i_{t+1}} & \mathbb{I}_m(z, n) = 1 \end{cases} \\ & \alpha \bar{z}_{t+1} = (\tilde{r}_{t+1} + \delta) \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \\ & \tilde{r}_{t+1} = r_{t+1} + \frac{q}{\int_{z > \bar{z}_{t+1}} \mathbb{I}_m(z, n) d\psi_t(z, n)} \\ & \int_{z > \bar{z}_{t+1}} \mathbb{I}_m(z, n) d\psi_t(z, n) = (\tilde{r}_{t+1} + \delta)(1 + \pi_{t+1}) \int (1 - \mathbb{I}_m(z, n)) d\psi_t(z, n) \\ & 1 + i_{t+1} = (1 + r_{t+1})(1 + \pi_{t+1}) \end{aligned}$$

Defining $\gamma_t(z) \equiv \int \mathbb{I}_m(z, n) \psi_t(z, n) dn$ and using the independence of the z and n distributions we re-write the objective as:

$$\sum_{t=1}^{\infty} \beta^t \sum_{j=1}^t \int \log \left(\frac{1 + \max\{\frac{z}{\bar{z}_j} - 1, 0\}}{1 + i_j} \right) \gamma_t(z) d\Phi(z) + \sum_{t=1}^{\infty} \beta^t \left[\vartheta_t \log(w_t) + \sum_{j=1}^t \log(1 + r_j) \right] + \frac{\int \log((1 - \beta)n_0) dG_0(n)}{1 - \beta} + \frac{\log(\beta)}{(1 - \beta)^2} + \log(w_0)$$

Changing the summation order we can re-state the relevant term in the objective and see that that the maximization problem is just a sequence of static problems, given by

$$\begin{aligned} \max_{\gamma(z) \in [0, 1]} \sum_{t=1}^{\infty} \beta^t \left[\frac{1}{1 - \beta} \int \log \left(\frac{1 + \max\{\frac{z}{\bar{z}_t} - 1, 0\}}{1 + i_t} \right) \gamma(z) d\Phi(z) + \vartheta_t \log(w_t) + \frac{1}{1 - \beta} \log(1 + r_t) \right] \\ \alpha \bar{z}_t = (\tilde{r}_t + \delta) \left(\frac{w_t}{1 - \alpha} \right)^{\frac{1 - \alpha}{\alpha}} \\ \tilde{r}_t = r_t + \frac{q}{\int_{z > \bar{z}_t} \gamma(z) d\Phi(z)} \\ \int_{z > \bar{z}_t} \gamma(z) d\Phi(z) = (\tilde{r}_t + \delta)(1 + \pi_t) \int (1 - \gamma(z)) d\Phi(z) \\ 1 + i_t = (1 + r_t)(1 + \pi_t) \end{aligned}$$

The objective and constraints are all linear in $\gamma(z)$ and the FOC wrt to $\gamma(z)$ is weakly increasing in z . This implies that the static problem reduces to choosing a threshold z^* such that if $z \geq z^* \Rightarrow \gamma(z) = 1$:

$$\max_{z_t^*} \frac{1}{1 - \beta} \int_{z > \max\{z_t^*, \bar{z}_t\}} \log \left(\frac{z}{\bar{z}_t} \right) d\Phi(z) + \vartheta_t \log(w_t) + \frac{1}{1 - \beta} (\Phi(z_t^*) \log(1 + i_t) - \log(1 + \pi_t))$$

s.t.

$$\begin{aligned} \alpha \bar{z}_t &= (\tilde{r}_t + \delta) \left(\frac{w_t}{1 - \alpha} \right)^{\frac{1 - \alpha}{\alpha}} \\ \tilde{r}_t &= r_t + \frac{q}{1 - \Phi(\max\{z_t^*, \bar{z}_t\})} \end{aligned}$$

$$1 - \Phi(\max\{z_t^*, \bar{z}_t\}) = (\tilde{r}_t + \delta)(1 + \pi_t) \Phi(z_t^*)$$

$$1 + i_t = (1 + r_t)(1 + \pi_t)$$

ASSUMPTION 8. $\vartheta > \frac{1 - \alpha}{\alpha(1 - \beta)}$ and $i \geq 0$

LEMMA 4. If Assumption 8 is satisfied, then $\bar{z}^* = \bar{z}$ in the constrained efficient allocation.

Proof. First, consider whether $\bar{z}_t < z_t^*$ is ever optimal. Note that the maximization problem is isomorphic

in \bar{z}_t and z_t^* . Thus, the FOC wrt \bar{z}_t is

$$\frac{1}{\bar{z}_t} \left(\vartheta_t \frac{\alpha}{1-\alpha} - \frac{1-\Phi(z_t^*)}{1-\beta} \right)$$

The FOC is strictly positive and thus optimality requires $\bar{z}_t \geq z_t^*$. We restate the problem as,

$$\begin{aligned} & \max_{\bar{z}_t, z_t^*} \frac{1}{1-\beta} \int_{z > \bar{z}_t} \log \left(\frac{z}{\bar{z}_t} \right) d\Phi(z) + \vartheta_t \log(w_t) + \frac{1}{1-\beta} (\Phi(z_t^*) \log(1+i_t) - \log(1+\pi_t)) \\ & \text{s.t.} \\ & \alpha \bar{z}_t = (\tilde{r}_t + \delta) \left(\frac{w_t}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \\ & \tilde{r}_t = r_t + \frac{q}{1-\Phi(\bar{z}_t)} \\ & 1 - \Phi(\bar{z}_t) = (\tilde{r}_t + \delta)(1+\pi_t)\Phi(z_t^*) \\ & 1 + i_t = (1+r_t)(1+\pi_t) \\ & z_t^* \leq \bar{z}_t \end{aligned}$$

Consider the FOC wrt to z_t^*

$$\Phi'(z_t^*) \left(\frac{\log(1+i_t)}{1-\beta} + \frac{1}{\Phi(z_t^*)} \left(\vartheta_t \frac{\alpha}{1-\alpha} - \frac{1}{1+i_t} \frac{1-\Phi(\bar{z}_t)}{1-\beta} \right) \right)$$

The FOC is strictly positive and thus $z_t^* = \bar{z}_t$. ■

To completely characterize the constrained efficient allocation we need to choose a $\bar{z} = \bar{z}^*$ that maximizes the objective. However, remember **Proposition 1** states that $\bar{\mu}$ is the unique inflation rate consistent with an allocation with $z_t^* = \bar{z}_t$ and constant aggregate quantities. Hence, the planner is only free to choose $\bar{z} = \bar{z}^*$ during the transition as a function of the inflation rate; letting aggregate wealth growth be endogenously determined. To achieve a stationary allocation, \bar{z} is endogenously determined as a function of inflation and the planner is constrained to choosing zero aggregate wealth growth.

Thus, setting $\mu = \bar{\mu}$ implements the constrained efficient allocation in a Stationary Equilibrium.

E DYNAMICS

The dynamic system is:

$$\begin{aligned}
N_{t+1} &= \beta N_t (1 + r_{t+1}) \left[1 + \frac{1}{(\eta - 1)(\tilde{z}_{t+1})^\eta} \right] \\
\tilde{m}_{t+1} &= \tilde{m}_t (1 + \mu_t) \frac{\phi_{t+1}}{\phi_t} 1 - (z_{t+1}^*)^{-\eta} = \left(\frac{1}{(\tilde{r}_{t+1} + \delta)} \frac{\phi_{t+1}}{\phi_t} \right) \frac{1}{(\tilde{z}_{t+1})^\eta} \\
\tilde{r}_{t+1} &= r_{t+1} + q_{t+1} (\tilde{z}_{t+1})^\eta \\
\frac{\phi_{t+1}}{\phi_t} &= (z_{t+1}^*)^\eta \frac{\tilde{m}_{t+1}}{\beta N_t} \\
1 &= \frac{\beta}{\alpha} \frac{N_t}{(\tilde{z}_{t+1})^\eta} (1 + r_{t+1}) \frac{\eta}{\eta - 1} \\
\bar{z}_{t+1} &= \frac{\tilde{z}_{t+1}}{(1 + i_{t+1})} \\
z_{t+1}^* &\leq (1 + i_{t+1}) \bar{z}_{t+1} \\
1 + i_{t+1} &= \max \left\{ (1 + r_{t+1}) \frac{1}{\frac{\phi_{t+1}}{\phi_t}}, 1 \right\} \\
(1 - \alpha) Y_{t+1} = w_{t+1} &= (1 - \alpha) \left(\frac{\alpha \bar{z}_{t+1}}{\tilde{r}_{t+1} + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

States are $\{N_t, \mu_t, q_{t+1}\}$. Controls are $\{\tilde{m}_t, \tilde{z}_{t+1}, z_{t+1}^*, \bar{z}_{t+1}, r_{t+1}, \frac{\phi_{t+1}}{\phi_t}, i_{t+1}, \tilde{r}_{t+1}, w_{t+1}\}$.

First we have:

$$(z_{t+1}^*)^\eta = \frac{\beta N_t}{\tilde{m}_t (1 + \mu_t)}$$

Then,

$$\begin{aligned}
(1 + r_{t+1}) &= (\tilde{z}_{t+1})^\eta \frac{\alpha}{\beta N_t} \frac{\eta - 1}{\eta} \\
\tilde{r}_{t+1} + \delta &= (\tilde{z}_{t+1})^\eta \left(\frac{\alpha}{\beta N_t} \frac{\eta - 1}{\eta} + q_{t+1} \right) - (1 - \delta) \\
\frac{\phi_{t+1}}{\phi_t} &= (1 - (z_{t+1}^*)^{-\eta}) (\tilde{z}_{t+1})^\eta (\tilde{r}_{t+1} + \delta) \\
1 + i_{t+1} &= \frac{\alpha}{\beta N_t} \frac{\eta - 1}{\eta} \frac{1}{(1 - (z_{t+1}^*)^{-\eta}) (\tilde{r}_{t+1} + \delta)} \\
Y_{t+1} &= \left(\frac{\eta}{\eta - 1} \beta N_t \tilde{z}_{t+1} (1 - (z_{t+1}^*)^{-\eta}) \right)^{\frac{\alpha}{1-\alpha}} \\
\tilde{z}_{t+1} &= \begin{cases} z_{t+1}^* & i_{t+1} > 0 \\ \geq z_{t+1}^* & i_{t+1} = 0 \end{cases} \\
N_{t+1} &= \frac{\alpha}{\eta} [(\eta - 1) (\tilde{z}_{t+1})^\eta + 1] \\
\tilde{m}_{t+1} &= \tilde{m}_t (1 + \mu_t) \frac{\phi_{t+1}}{\phi_t}
\end{aligned}$$

So we can write the 2 dimensional system in $\{N_t, M_t \equiv (1 + \mu_t) \tilde{m}_t\}$:

$$\begin{aligned}
N_{t+1} &= \frac{\alpha}{\eta} \left[(\eta - 1) \frac{(\tilde{z}_{t+1})^\eta}{(z_{t+1}^*)^\eta} \frac{\beta N_t}{M_t} + 1 \right] \\
M_{t+1} &= (1 + \mu_{t+1}) \left(\frac{\beta N_t}{M_t} - 1 \right) \frac{(\tilde{z}_{t+1})^\eta}{(z_{t+1}^*)^\eta} \left(\frac{(\tilde{z}_{t+1})^\eta}{(z_{t+1}^*)^\eta} \left(\alpha \frac{\eta - 1}{\eta} + q_{t+1} \beta N_t \right) - (1 - \delta) M_t \right) \\
&\quad \frac{(\tilde{z}_{t+1})^\eta}{(z_{t+1}^*)^\eta} = \max \left\{ \frac{M_{t+1}}{1 + \mu_{t+1}} \frac{\eta}{\alpha(\eta - 1)}, 1 \right\}
\end{aligned}$$

The system when $i > 0$ is:

$$\begin{aligned}
N_{t+1} &= \frac{\alpha}{\eta} \left[(\eta - 1) \frac{\beta N_t}{m_t (1 + \mu_t)} + 1 \right] \\
m_{t+1} &= \left(\frac{\beta N_t}{m_t (1 + \mu_t)} - 1 \right) \left(\alpha \frac{\eta - 1}{\eta} + q_t \beta N_t - (1 - \delta) m_t (1 + \mu_t) \right) \\
\log(q_{t+1}) &= (1 - \rho_q) \log(q) + \rho_q \log(1 + q_t) \\
\log(1 + \mu_{t+1}) &= (1 - \rho_\mu) \log(1 + \mu) + \rho_\mu \log(1 + \mu_t)
\end{aligned}$$

And the loglinearized system around the steady state:

$$\begin{aligned}
\hat{N}_{t+1} &= \frac{\alpha(\eta-1)}{\eta} \frac{\beta}{m(1+\mu)} (\hat{N}_t - \hat{m}_t - \hat{\mu}_t) \\
\hat{m}_{t+1} &= \left(\frac{\beta N}{\beta N - m(1+\mu)} + \frac{\beta N}{m} \left(\frac{\beta N}{m(1+\mu)} - 1 \right) q \right) \hat{N}_t \\
&\quad - \left(\frac{\beta N}{\beta N - m(1+\mu)} + \left(\frac{\beta N}{m(1+\mu)} - 1 \right) (1-\delta)(1+\mu) \right) (\hat{m}_t + \hat{\mu}_t) \\
&\quad + \frac{\beta N}{m} \left(\frac{\beta N}{m(1+\mu)} - 1 \right) q \hat{q}_t \\
\hat{\mu}_{t+1} &= \rho \hat{\mu}_t \\
\hat{q}_{t+1} &= \rho \hat{q}_t \\
\hat{z}_{t+1}^* &= \frac{1}{\eta} (\hat{N}_t - \hat{m}_t - \hat{\mu}_t) \\
\hat{Y}_{t+1} &= \frac{\alpha}{1-\alpha} \left(\hat{N}_t + \left(1 + \eta \frac{1}{(z^*)^\eta - 1} \right) \hat{z}_{t+1}^* \right) \\
\hat{R}_{t+1} &= \frac{1+\tilde{r}}{\tilde{r}+\delta} \eta \hat{z}_{t+1}^* + \frac{q(z^*)^\eta}{\tilde{r}+\delta} \hat{q}_t - \frac{1+\tilde{r}-q(z^*)^\eta}{\tilde{r}+\delta} \hat{N}_t \\
\hat{\pi}_{t+1} &= -\hat{R}_{t+1} - \frac{(z^*)^\eta}{(z^*)^\eta - 1} \eta \hat{z}_{t+1}^* \\
\hat{i}_{t+1} &= \hat{\pi}_{t+1} - \hat{m}_t - \hat{\mu}_t
\end{aligned}$$

For reasonable parametrizations the eigenvalues satisfy $\lambda_m < -1 < 0 < \lambda_n < 1$. Hence there is a saddle path solution associated with λ_n where both n_t, m_t converge monotonically to the steady state. There is another solution where n_t converges monotonically to the steady state and m_t diverges with an oscillatory movement. The lower path in this second solution has $m_t < 0$ in finite time and thus can be ruled out.

The upper path has m_t increasing and at some point surpassing the threshold such that $i_t > 0$. As we will see, we can rule out this path as well using the transversality condition. The reason is that the equilibrium when $i = 0$ has $0 < \lambda_n < 1 < \lambda_m$ and so there are a convergent and divergent solution. However, we have shown already that a unique steady state exists and thus the convergent solution is not an equilibrium. In this way, following the upper path solution until the point where $i_t = 0$ results in m_t following the dynamics in this zone which eventually result in $m_t, n_t \rightarrow \infty$ corresponding to the paths other than the saddle path.

The system when $i = 0$ is:

$$\begin{aligned}
N_{t+1} &= \frac{\alpha}{\eta} [(\eta-1)(\tilde{z}_{t+1})^\eta + 1] \\
m_{t+1} &= (1+\mu_t)m_t \frac{\alpha}{\beta N_t} \frac{\eta-1}{\eta} (\tilde{z}_{t+1})^\eta
\end{aligned}$$

where

$$(\tilde{z}_{t+1})^\eta = \frac{\alpha \frac{\eta-1}{\eta} \frac{1}{\beta N_t - m_t(1+\mu_t)} + 1 - \delta}{\left(\frac{\alpha}{\beta N_t} \frac{\eta-1}{\eta} + q_{t+1} \right)}$$

Then the log-linearized system is:

$$\begin{aligned} \hat{N}_{t+1} &= \frac{\beta}{1+\mu} x_{t+1} \\ \hat{m}_{t+1} &= \hat{m}_t + \hat{\mu}_t - \hat{N}_t + x_{t+1} \\ x_{t+1} &= \frac{\left(\frac{\alpha}{1+\mu} \frac{\eta-1}{\eta} - (\tilde{r} + \delta)^2 \beta N \right)}{(1+\tilde{r})\alpha \frac{\eta-1}{\eta}} \hat{N}_t + \frac{(\tilde{r} + \delta)^2 m(1+\mu)}{(1+\tilde{r})\alpha \frac{\eta-1}{\eta}} (\hat{m}_t + \hat{\mu}_t) - \frac{q\beta N}{\left(\alpha \frac{\eta-1}{\eta} + q\beta N \right)} q_{t+1} \\ \hat{\pi}_{t+1} &= \hat{N}_t - x_{t+1} \\ \hat{z}_{t+1}^* &= \frac{1}{\eta} (\hat{N}_t - \hat{m}_t - \hat{\mu}_t) \\ \hat{Y}_{t+1} &= \frac{\alpha}{1-\alpha} (\hat{N}_t + \frac{1}{\eta} x_{t+1} + \frac{\eta}{(z^*)^\eta - 1} \hat{z}_{t+1}^*) \end{aligned}$$

As mentioned, the eigenvalues satisfy $0 < \lambda_n < 1 < \lambda_m$ for reasonable calibrations and thus there is a unique saddle path converging to the steady state. The upper path solutions result in $n_t \rightarrow \infty$ and violating transversality. There are below saddle path solutions where m_t decreases and enters the zone where $i_t > 0$. We can rule out this since eventually they imply that $m_t = 0$ or $n_t = 0$ by similar arguments as before.

F SCREENING

G CALIBRATION

$\beta = 0.98$	$r = 0$ when $\pi = 0$
$\delta = 0.02$	Investment capital ratio
$\alpha = 0.4$	Capital share
$\nu = 0.85$	Frisch labor elasticity
$q = 0.0003$	$\tilde{i} - i = 5\%$ annual when $\pi = 0$
$\eta = 2.2$	Pareto tail index

REFERENCES

- ALVAREZ, F., R. E. LUCAS, AND W. E. WEBER (2001): "Interest Rates and Inflation," *American Economic Review*, 91, 219–225.
- ANGELETOS, G.-M. (2007): "Uninsured Idiosyncratic Investment Risk and Aggregate Saving," *Review of Economic Dynamics*, 10, 1–30.
- ATKESON, A. AND P. J. KEHOE (2004): "Deflation and Depression: Is There an Empirical Link?" *American Economic Review*, 94, 99–103.
- BENHABIB, J. AND M. M. SPIEGEL (2009): "Moderate Inflation and the Deflation-Depression Link," *Journal of Money, Credit and Banking*, 41, 787–798.
- BERENTSEN, A., G. MENZIO, AND R. WRIGHT (2011): "Inflation and Unemployment in the Long Run," *American Economic Review*, 101, 371–98.
- CHARI, V. V., P. J. KEHOE, AND E. R. MCGRATTAN (2007): "Business Cycle Accounting," *Econometrica*, 75, 781–836.
- CHRISTIANO, L., M. EICHENBAUM, AND S. REBELO (2011): "When Is the Government Spending Multiplier Large?" *Journal of Political Economy*, 119, 78 – 121.
- COCHRANE, J. H. (2013): "The New-Keynesian Liquidity Trap," NBER Working Papers 19476, National Bureau of Economic Research, Inc.
- COLE, H. L. AND N. R. KOCHERLAKOTA (1998): "Zero nominal interest rates: why they're good and how to get them," *Quarterly Review*, 2–10.
- DAVILA, J., J. H. HONG, P. KRUSELL, AND J. V. RIOSRULL (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.
- EGGERTSSON, G. B. AND M. WOODFORD (2003): "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, 34, 139–235.
- FRIEDMAN, M. AND A. J. SCHWARTZ (1963): *A Monetary History of the United States, 1867-1960*, no. frie63-1 in NBER Books, National Bureau of Economic Research, Inc.
- GUERRIERI, V. AND G. LORENZONI (2011): "Credit Crises, Precautionary Savings, and the Liquidity Trap," NBER Working Papers 17583, National Bureau of Economic Research, Inc.

- GUERRIERI, V., R. SHIMER, AND R. WRIGHT (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78, 1823–1862.
- HALL, R. E. (2011): "Clashing Theories of Unemployment," NBER Working Papers 17179, National Bureau of Economic Research, Inc.
- KIYOTAKI, N. (1998): "Credit and Business Cycles," *The Japanese Economic Review*, 49, 1835.
- KIYOTAKI, N. AND J. MOORE (2012): "Liquidity, Business Cycles, and Monetary Policy," NBER Working Papers 17934, National Bureau of Economic Research, Inc.
- KOCHERLAKOTA, N. R. (2012): "Incomplete Labor Markets," Tech. rep.
- KRUGMAN, P. R. (1998): "It's Baaack: Japan's Slump and the Return of the Liquidity Trap," *Brookings Papers on Economic Activity*, 29, 137–206.
- LUCAS, R. E. (2000): "Inflation and Welfare," *Econometrica*, 68, 247–274.
- MERTON, R. C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, 51, 247–57.
- MOEN, E. R. (1997): "Competitive Search Equilibrium," *Journal of Political Economy*, 105, 385–411.
- MOLL, B. (2011): "Productivity Losses from Financial Frictions: Can Self-financing Undo Capital Misallocation?" Dissertation, University of Chicago.
- MORTENSEN, D. T. AND C. A. PISSARIDES (1994): "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, 61, 397–415.
- ROCHETEAU, G. AND R. WRIGHT (2005): "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 73, 175–202.
- WERNING, I. (2011): "Managing a Liquidity Trap: Monetary and Fiscal Policy," NBER Working Papers 17344, National Bureau of Economic Research, Inc.