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Coordination, timing and common knowledge[☆]



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ABSTRACT

There is tight link between coordination and common knowledge. The role of higher order beliefs in static incomplete information games has been widely studied. In particular, information frictions break down common knowledge. A large body of literature in economics examine dynamic coordination problems when there are timing frictions, in the sense that players do not all move at once. Timing frictions in dynamic coordination games play a role that is closely analogous to information frictions in static coordination games.

This paper makes explicit the role of higher order beliefs about timing in dynamic coordination games with timing frictions. An event is said to be effectively known if a player knew the event when he last had an option to change his behavior. The lack of effective common knowledge of the time drives results of dynamic coordination games.

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1. Introduction

We sometimes choose to do things only because other people are doing them. David Hume referred to the behavior chosen in such situations as *conventions*. A leading example of a convention for Hume and philosophers following him was language. Since everyone else uses the word “cat” to refer to a cat and the word “dog” to refer to a dog, I also do so, but if everyone was using “cat” to refer to a dog and “dog” to refer to a cat, then I would surely have an incentive to switch to the latter usage. Modern game theory describes such situations as *coordination games* and the existence of multiple stable conventions as multiple equilibria.

Coordination problems are ubiquitous in many important social contexts. The importance of incentives to coordinate, or strategic complementarities, has been highlighted in recent years by the global financial crisis. Lenders from the shadow banking sector had an incentive to lend less if others were lending less (via various economic channels), giving rise to strategic complementarities, or coordination incentives, that exacerbated the market freeze. Holders of European sovereign debt had an incentive to hold less debt if others were holding less, since this would drive up interest rates and thus increase the likelihood of default.

Foundational analysis of coordination games has long noted the importance of “higher order beliefs” in understanding coordination. Higher order beliefs refer to each individual's beliefs and knowledge not only about directly payoff relevant events, but also about the beliefs and knowledge of others, their beliefs about others' beliefs, and so on. The philosopher

[☆] The paper was prepared for the June 2103 Ca' Foscari University of Venice International Workshop in Economic Theory on “Where Do We Stand”? I am grateful to Yoram Moses for a conversation where he first explained the relation between timing and common knowledge to me, and for comments on this paper. This conversation led me to write a 1995 unpublished paper on “Co-operation and Timing” (Morris, 1995); this paper encompasses and generalizes that working paper.

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David Lewis emphasized in Lewis (1969) that for language to work, it is necessary not only for participants in a conversation to share vocabulary, so that we all use “cat” to refer to cat, etc., but it is also necessary for all of us to know that we all use “cat” to refer to cat, and so on. Lewis used the expression “common knowledge” to refer to such infinite chains of iterated knowledge. When Aumann (1976) introduced the formal study of common knowledge into economics, he credited Lewis with introducing this formulation. A large body of literature has grown up showing how coordination can be tightly related to relevant higher order beliefs. While common knowledge may be necessary for perfect coordination, as suggested by Lewis, one general insight is that “approximate common knowledge” generally suffices (Monderer and Samet, 1989). Specifically, say that an event is p -believed if at least proportion p believe it with probability at least p . Say that an event is “common p -belief” if it is p -believed, it is p -believed that it is p -believed, and so on.¹ For any given coordination game, behavior that would be an equilibrium if there was common knowledge of the payoffs of that game will still be an equilibrium if there is only common p -belief as long as p is close enough to one. How close p needs to be to 1 depends on how strong the incentives are in the coordination game. This observation highlights that public or almost public events play a key role in coordinating behavior. We also know that approximate common knowledge in this sense is a stringent requirement (Rubinstein, 1989; Weinstein and Yildiz, 2007). In the “global games” literature (Carlsson and van Damme, 1993), a small amount of noise about payoffs implies a dramatic failure of approximate common knowledge, and leads to a prediction that strategically safe behavior will be played; thus in symmetric binary action coordination games, each player will end up playing the “Laplacian” equilibrium corresponding to each player choosing a best response to a diffuse (i.e., uniform) belief over the proportion of his opponents who will choose each action (Morris and Shin, 2003).²

But timing is also important for coordination. I am often concerned not only to take the same action as others, but also to take it at the same time. This suggests that higher order beliefs about timing should also matter in coordination problems. It is often argued that events which are particularly “public”, and thus approximate common knowledge, are important in coordinating a switch from one equilibrium to another. Thus – to stick to topical examples – Blanchard (2013) and others have highlighted the importance of the public announcement of the European Central Bank’s outright monetary transaction program in triggering a switch to a “new equilibrium” in the European sovereign debt crisis. Romer (2013) argues about the importance of public communication in generating a shift of equilibria both for Roosevelt at the start of the New Deal and for Abe in the Japanese monetary policy in 2013. Fearon (2011) argues that revolutions often follow elections that have been stolen by a dictator, not because the theft reveals any new information (it was commonly known that the election would be stolen), but because revolutions are extreme coordination problems (no one wants to be a part of a small minority who revolt) and the stealing of the election provides the public event to coordinate the timing of the revolution.

But while the higher order belief foundations of coordination in static games have been much studied, the higher order belief foundations of coordination and timing have not (at least in economics). In particular, there are large and important bodies of literature, which will be reviewed in Section 3, that introduce timing frictions into economic models. The effect of timing frictions is to make it harder to coordinate behavior, because they break down approximate common knowledge of time. The main purpose of this paper is to present such higher order belief foundations.

To understand the issues, it is useful to consider a common knowledge “paradox”. Many entertaining common knowledge paradoxes, concerning, for example, cheating spouses, hats and railway carriages (Geanakoplos, 1992; Fagin et al., 1995), have been used to illustrate the logical importance of higher order beliefs. But the following paradox concerns timing.³ There is a continuum of individuals whose clocks are not perfectly synchronized. In particular, they are slow by an amount between 0 and 4 min relative to the “true time”, with the delay uniformly distributed in the population. Each individual does not know how slow his clock is and has a uniform belief about the delay. At what time does it become common knowledge that the true time is, say, 8:00 a.m. or later? The answer is never. Only when the true time reaches 8:04 does everyone know that the true time has reached 8:00. Only at 8:08 does everyone know that everyone knows that it is past 8:00, and so on. Thus it never becomes common knowledge.

But the paradox gets worse. When does it become common $3/4$ -belief – in the sense described above – that the true time has reached 8:00? An individual only assigns probability $3/4$ to the true time being after 8:00 when his own clock reaches 7:59. At this point, the true time is after 8:00 as long as his clock is delayed by at least 1 min, a $3/4$ probability event. It is not until a true time of 8:02 that proportion $3/4$ of individuals observe a time after 7:59: this is because at true time 8:02, individual clock times are uniformly distributed between 7:58 and 8:02. Thus only at 8:02 is it $3/4$ -believed – i.e., $3/4$ of the population assign probability at least $3/4$ – that the true time is after 8:00. It is only at 8:04 that it is $3/4$ -believed that it is $3/4$ -believed that the time is after 8:00, and so on. So it is also never common $3/4$ -belief that the time is after 8:00. We will formalize and generalize this argument and verify that for any $p > 1/2$ it is never common p -belief that the true time is after 8:00. But, for any $p \leq 1/2$, there is common p -belief that the true time is after 8:00 from 8:00 on.

¹ This definition is a variant of the definition in Monderer and Samet (1989): their definition corresponds to replacing “at least proportion p ” with “everyone”. The modified definition is relevant for analysis in this paper.

² The selection of the Laplacian equilibrium is an implication of the more general result that potential maximizing equilibria are selected in global games if the underlying game has a potential.

³ This paradox, together with a formal analysis of knowledge and common knowledge about timing, appears in Section 8 of Halpern and Moses (1990). A more entertaining cheating spouse version of this paradox appears in the Stanford Ph.D. thesis, Moses (1986), of Yoram Moses. The thesis also discusses atemporal and temporal versions of approximate common knowledge.

To see the strategic implications of these higher order beliefs, suppose that the continuum population chooses whether to invest or not invest at each instant using only his own clock (and so, in particular, not observing others' actions). They will receive flow payoffs depending on the actions in the population and the time.⁴ The payoff to not investing is always 0. The payoff to investing is $1 - p$ if at least proportion p of players invest and the time is after 8:00. The payoff to investing is $-p$ if the true time is before 8:00 or if less than proportion p of the population invest. Thus we can think of p as being the cost of investing and investment only succeeds (giving a payoff of 1) if the time is after 8:00 and at least proportion p invest. It will be shown that common p -belief that the time is after 8:00 is necessary and sufficient for investment to succeed in this example, and thus investment never occurs if $p > 1/2$.

This argument very closely parallels global game arguments. In particular, investment is the Laplacian equilibrium – i.e., each player is choosing a best response to a uniform belief over others' actions – only if $p \leq 1/2$. Thus this argument illustrates the connection between higher order beliefs about payoffs in static coordination games and higher order beliefs about timing in dynamic coordination games. But the form of the timing friction – unsynchronized clocks and no information about others' actions – is an extreme and unnatural one for economic applications. So an attention friction, versions of which are widely used in economics, will also be analyzed to verify the more general applicability of the connection. Suppose now that the same game is being played, but that everyone observes the true time (so there is common knowledge of the time) and they also get to observe others' past actions. But now suppose that each player only gets to switch his action every 4 min. In particular, suppose we index players by a real number between 0 and 4 and the player indexed by i only gets to switch his action at i minutes past 8:00, i minutes past 8:04, i minutes past 8:08 and so on. He also had the opportunity to switch his action at i minutes past 7:56, i minutes past 7:52, and so on. What are the relevant higher order beliefs about timing in this modified problem? In this case, what matters is not players' actual beliefs, but what their beliefs were at their last opportunity to switch actions. So let an “event” be a set of times. We can think of an event corresponding to a property that is only true at certain times. Thus an example of an event is “the time is after 8:00” and we are interested in this event because it corresponds to the property that investment is potentially successful. Now say that an event “effectively p -believed” if at least proportion p of the population, when they last had an opportunity to switch actions, expected to spend proportion p of the time until their next revision opportunity, in that event. When is there effective common $3/4$ -belief that the time is past 8:00? A player choosing at 7:59 knows that the time will be after 8:00 for $3/4$ of the time his current choice is fixed. Only at 8:02 will $3/4$ of the population have made their choices after 7:59. Thus only at 8:02 is it effectively $3/4$ -believed that the time is after 8:00. Only at 8:04 is it effectively $3/4$ -believed that it is effectively $3/4$ -believed that the time is after 8:00, and so on. So it is never effective common $3/4$ -belief that the time is after 8:00. It will be shown that effective higher order beliefs under the attention friction exactly match higher order beliefs under the synchronization friction, and the strategic analysis is also identical. So, if $p > 1/2$, there is never investment in the timing friction problem.

Connections between information frictions and timing frictions have been studied before (see, for example, discussions in Morris and Shin (2003, 2005)). The purpose of this paper is to explicitly describe the relation between higher order beliefs about timing and coordination in a canonical setting. There has been little formal analysis of higher order beliefs and timing in economics. An important exception is the paper by Dasgupta et al. (2012) who study how relaxing the need for perfect synchronicity in action choices weakens the nature of the approximate common knowledge of timing required for coordination. This connection between higher order knowledge and timing has been extensively studied in the computer science literature; see Halpern and Moses (1990), Fagin et al. (1995, Chapter 8) and references therein. Recent work in the computer science literature (Ben-Zvi and Moses, 2013 and Gonczarowski and Moses, 2013), like Dasgupta et al. (2012), examines how changing the degree of synchronicity of coordination required changes in the higher order knowledge requirements.⁵

The formal model is presented in Section 2. Work on timing frictions in economics is reviewed in Section 3. Section 4 concludes the paper.

2. Formal modelling of coordination, timing and higher order beliefs

We will focus our analysis on a simple game. Players received a flow payoff at each point in time $t \in \mathbb{R}$. It is convenient to think of time extending indefinitely both into the future and into the past. A continuum of players indexed by $i \in [0, 1]$ must choose an action, “Not Invest” or “Invest,” at each instant of time. The payoff to not investing is always 0. There is a (closed) subset of time periods $E^* \subseteq \mathbb{R}$ when there are potential gains from investment. We say that “investment succeeds” if both the time is contained in E^* and at least proportion $q \in (0, 1)$ of the population invest. If either the time is not contained in E^* or less than proportion q of the population invest, then “investment fails”. There is always a cost $p \in (0, 1)$ to each player who invests. There is a return 1 to those players investing only if investment succeeds. Thus a player who invests receives net payoff $1 - p$ if investment succeeds and payoff $-p$ if investment fails.

This game is a stripped down version of the coordination game studied in Morris and Shin (1998). In the working paper on which this paper builds, Morris (1995), a finite player version of this problem was studied, where all players had to invest

⁴ These payoffs must be received after play of the game, in order for players not to learn about others' actions.

⁵ Morris and Shin (1997) discuss similarities and dissimilarities between the economics and computer science literatures on coordination and higher order beliefs.

to get a successful investment. The game discussed in the introduction corresponds to the case where $q=p$ and the event E^* corresponds to the time being after 8:00.

If there were no timing frictions, then players would be playing a coordination game at each point in time. For times t that are not in E^* , each player has a dominant strategy to not invest. If t is in E^* , there are multiple equilibria: thus we could either have all players investing or all players not investing at any time in E^* , and arbitrary switching between the two. We will consider two timing frictions in turn.

2.1. Synchronicity friction

We first suppose that there is a lack of synchronization. Each player i has his own clock which is not synchronized with the true time t . We assume that the time on player i 's clock is $t-i$. Thus player i 's clock is slow by i minutes. We assume that each player does not know his delay i , so if his clock says τ , he has a uniform belief about the true time over the interval $[\tau, \tau+1]$.

Now necessary conditions for investment to succeed are that we are at a time t when

1. $t \in E^*$ and at least proportion q of players assign probability at least p to the true time being in E^*
2. $t \in E^*$ and proportion q of players assign probability at least p to the true time satisfying (1)
3. $t \in E^*$ and proportion q of players assign probability at least p to the true time satisfying (2)
4. and so on.

Even if each player incorrectly anticipated that others would always invest, condition (1) would be necessary for investment to succeed. If each player incorrectly anticipated that other players would invest whenever they assign probability at least p to condition (1) being true, condition (2) would be necessary for investment to succeed, and so on.

A sufficient condition for investment to succeed is that we can find a set of times F with the property that, at each time in F , at least proportion q of players assign probability at least p to the true time being in F . The necessary conditions and sufficient condition are both statements about higher order beliefs about the true time. Thus we will introduce a formal language to discuss higher order beliefs about the true time and use it to show the equivalence between the necessary conditions and the sufficient condition.

2.1.1. Beliefs and higher order beliefs about clock times

An event will be a collection of times, $E \subseteq \mathbb{R}$. It will be technically convenient to maintain the assumption that events are closed. Now to say that event E is true is to say that the true time is an element of E .

A player with time τ on his clock will believe E with probability at least p if

$$\lambda(E \cap [\tau, \tau+1]) \geq p,$$

where we write $\lambda(E)$ for the Lebesgue measure of a set on the real line. Thus if the true time is t , the player with delay i will observe time $\tau = t-i$ on his clock and will then assign probability at least p to E if

$$\lambda(E \cap [t-i, t-i+1]) \geq p.$$

Thus the proportion of players who p -believe event E at time t will be

$$\lambda[\{i \in [0, 1] | \lambda(E \cap [t-i, t-i+1]) \geq p\}].$$

Say that E is (p,q) -believed at those times in E when the proportion of players who p -believe event E is at least q .⁶ We will write $B^{p,q}(E)$ for this event, so

$$B^{p,q}(E) = \{t \in E | \lambda[\{i \in [0, 1] | \lambda(E \cap [t-i, t-i+1]) \geq p\}] \geq q\}.$$

This operator can be iterated. Thus the set of times in E where the proportion of players who p -believe that E is (p,q) -believed is at least q corresponds to

$$B^{p,q}(B^{p,q}(E)),$$

which will sometimes be written as

$$[B^{p,q}]^2(E).$$

Define

$$C^{p,q}(E) = \bigcap_{n \geq 1} [B^{p,q}]^n(E).$$

If $t \in C^{p,q}(E)$, then there is said to be *common* (p,q) -*belief* of event E . Notice that $C^{p,q}(E)$ is exactly the set of times when the

⁶ Notice that we thus impose the requirement that E is true when it is (p,q) -believed. This simplifies the statement of results, but they could be re-formulated using belief operators without this requirement.

necessary conditions above are satisfied. Also say that event E is (p, q) -evident if

$$E \subseteq B^{p,q}(E).$$

Now we have the following proposition.

Proposition 1. *Event E is common (p,q) -belief if and only if there exists a (p,q) -evident event F that implies E . Formally,*

$$t \in C^{p,q}(E) \Leftrightarrow \exists (p, q)\text{-evident } F \text{ such that } t \in F \subseteq E.$$

Proof. The operator $B^{p,q}$ is a decreasing operator (under set inclusion) on closed subsets of \mathbb{R} . If $t \in F \subseteq E$ and $F \subseteq B^{p,q}(F)$, then $t \in F \subseteq B^{p,q}(F) \subseteq B^{p,q}(E)$ and, by induction, $t \in F \subseteq [B^{p,q}]^2(F) \subseteq [B^{p,q}]^n(E)$ and thus $t \in C^{p,q}(E)$. For the converse, observe that $C^{p,q}(E)$ is (p,q) -evident. To see why, observe that $[B^{p,q}]^n(E)$ is a decreasing sequence of events and thus

$$\lambda[[B^{p,q}]^n(E) \cap [\tau, \tau + 1]] \rightarrow \lambda[C^{p,q}(E) \cap [\tau, \tau + 1]]$$

for each τ , and

$$\lambda\{[i \in [0, 1] | \lambda([B^{p,q}]^n(E) \cap [t - i, t - i + 1]) \geq p]\} \rightarrow \lambda\{[i \in [0, 1] | \lambda(C^{p,q}(E) \cap [t - i, t - i + 1]) \geq p]\}$$

for each t ; so

$$t \in B^{p,q}([B^{p,q}]^n(E)) \text{ for each } n \Rightarrow t \in B^{p,q}(C^{p,q}(E))$$

and so

$$C^{p,q}(E) \subseteq B^{p,q}(C^{p,q}(E)). \quad \square$$

The proposition belongs to a family of results relating iterated definitions of (approximate) common knowledge to fixed point definitions. [Aumann \(1976\)](#) proved such a result for common knowledge. [Monderer and Samet \(1989\)](#) proved the analogous result for common p -belief. [Morris and Shin \(2007\)](#) proved a version for some more general belief operators.

We can illustrate the result (and verify the analysis of the examples in the introduction) by considering the case where the set of times where investment might succeed consists of all dates after a given threshold. Write E_r^+ for the set of dates r or later:

$$E_r^+ = \{t \in \mathbb{R} | t \geq r\}.$$

Suppose now that investment can only succeed from time t^* onward, so that

$$E^* = E_{t^*}^+.$$

A player with τ on his clock p -believes E_r^+ if

$$\tau + 1 - r \geq p$$

and so if

$$\tau \geq r - (1 - p)$$

Thus the player with actual delay i p -believes E_r^+ at true time t if

$$t - i \geq r - (1 - p)$$

or if

$$i \leq (t - r) + (1 - p)$$

So the proportion of players who p -believe E_r^+ at time t is now

$$\lambda\{[i \in [0, 1] | i \leq (t - r) + (1 - p)]\} = \begin{cases} 0, & \text{if } t \leq r + p - 1 \\ (t - r) + (1 - p), & \text{if } r + p - 1 \leq t \leq r + p \\ 1, & t \geq r + p \end{cases}$$

So

$$B^{p,q}(E_r^+) = \{t \in E_r^+ | (t - r) + (1 - p) \geq q\} = \{t \in E_r^+ | t \geq r + p + q - 1\} = E_{\max(r, r + p + q - 1)}^+$$

Iterating this expression, we see that

$$[B^{p,q}]^n(E_r^+) = E_{\max(r, r + (p + q - 1)n)}^+$$

and thus

$$C^{p,q}(E^*) = C^{p,q}(E_{t^*}^+) = \begin{cases} E_{t^*}^+, & \text{if } p + q \leq 1 \\ \emptyset, & \text{if } p + q > 1 \end{cases}$$

Thus if $p+q > 1$, there is never common (p,q) -belief that the time is after t^* . But if $p+q \leq 1$, then there is common (p,q) -belief that the time is after t^* from time t^* onward.

2.1.2. Back to the investment game

We now return to the analysis of strategic behavior in the investment game. A strategy for any player is a mapping from times on his clock to a probability distribution over actions. For technical convenience, let us assume that (i) players follow a pure strategy; (ii) players follow the same strategy; and (iii) if indifferent, the player will invest. These assumptions simplify the analysis but are not necessary for the main conclusions. So a symmetric strategy is represented by a closed set of own clock times $I \subseteq \mathbb{R}$ at which players invest. The strategy I will imply a set of true times where investment succeeds, $F \subseteq \mathbb{R}$, given by

$$F = \{t \in E^* \mid \lambda[\{i \in [0, 1] \mid t - i \in I\}] \geq q\}. \tag{1}$$

For strategy I to be a best response given F , we must have that

$$I = \{\tau \in \mathbb{R} \mid \lambda(F \cap [\tau, \tau + 1]) \geq p\}. \tag{2}$$

But substituting (2) into (1), we have that

$$F = \{t \in E^* \mid \lambda[\{i \in [0, 1] \mid \lambda(F \cap [t - i, t - i + 1]) \geq p\}] \geq q\}.$$

But this condition is equivalent to the requirements that

$$F = B^{p,q}(F) \text{ and } F \subseteq E^*. \tag{3}$$

Thus the set of equilibria is characterized by the set of fixed points of $B^{p,q}$ contained in E^* . Thus there is an equilibrium where investment never takes place, since

$$\emptyset = B^{p,q}(\emptyset);$$

and there is an equilibrium where investment takes place only if there is common (p,q) -belief of E^* , since

$$C^{p,q}(E^*) = B^{p,q}(C^{p,q}(E^*)).$$

But \emptyset and $C^{p,q}(E^*)$ are the smallest and largest events, respectively, that characterize equilibria. So we have the following proposition.

Proposition 2. *There is an equilibrium where investment is successful at times in $C^{p,q}(E^*)$; each player invests if the time on his clock is in*

$$I^* = \{\tau \in \mathbb{R} \mid \lambda(C^{p,q}(E^*) \cap [\tau, \tau + 1]) \geq p\}.$$

In every equilibrium, investment does not succeed at times outside $C^{p,q}(E^)$; and players never invest outside the set I^* .*

We conclude that in the case where $p+q \leq 1$ and $E^* = E_{t^*}^+$, there is an equilibrium where investment succeeds from time t^* on and players start investing when the time on their clocks reaches $t^* + (1 - p)$. If $p+q > 1$, then there is never investment in equilibrium. Note that investing is a Laplacian equilibrium at states in E^* exactly if $p+q \leq 1$.

2.2. Attention friction

We now suppose that all players know the true time and observe the past history of play. We assume that each player moves only once every time period. In particular, player $i \in [0, 1]$ can switch his action only at times of $z+i$ where z is an integer. One interpretation is that there is a technological constraint preventing players moving more often. Another interpretation – motivating the label “attention frictions” – is that players only pay attention to the action choice once every time period, and these times are not synchronized across players. Now necessary conditions for investment to succeed at time t are that

1. $t \in E^*$ and at least proportion q of players, when they chose the action they are currently taking, expected to be in E^* at least proportion p of the time until they had the opportunity to revise their behavior.
2. $t \in E^*$ and at least proportion q of players, when they chose the action they are currently taking, expected (1) to be true at least proportion p of the time until they had the opportunity to revise their behavior.
3. and so on.

A sufficient condition for investment to succeed is that we can find a set of times F with the property that, at each time in F , at least proportion q of players, when they chose the action they are currently choosing, expected (1) to be true at least proportion p of the time until they have the opportunity to revise their behavior.

It is now demonstrated that the analysis of the synchronicity friction remains unchanged with a new interpretation of the belief operator. For a player making a choice at time t , the proportion of time that event E is true until his next revision is

at least p if

$$\lambda(E \cap [t, t+1]) \geq p.$$

Thus the proportion of players who expected E to be true proportion at least p of the time will be

$$\lambda(\{i \in [0, 1] | \lambda(E \cap [t-i, t-i+1]) \geq p\}).$$

Say that E is effectively (p, q) -believed if E is true and the proportion of players who, when they last revised their action, expected E to be true proportion p of the time going forward is at least q . This is an alternative interpretation, based on the attention friction interpretation, of the same formal operator $B^{p,q}$ which we defined above for the synchronicity friction. Now a symmetric pure strategy is characterized by the set of times where the player choosing at that date invests. It is again the case that an equilibrium is characterized by a set of times where investment succeeds, satisfying the fixed point property $F = B^{p,q}(F)$, and Proposition 2 remains true as stated for this alternative timing friction.

3. Timing frictions in economics

There are many bodies of literature in economics that assume some version of timing frictions. The purpose of this section is to briefly review them. The focus will be on settings where players are small and so do not take into account their influence on others' actions in the future. There is also an important related body of literature on timing frictions with large players.⁷ The review will be carried out in such a way as to highlight a view of how these studies fit together and the connection to the earlier analysis. In doing so, many of the subtleties of the work will be glossed over.

A common modelling device for introducing timing frictions into dynamics is to assume that players cannot make choices instantaneously, but only when an opportunity to revise behavior arrives according to a Poisson process. This is a tractable way of introducing timing frictions while maintaining stationarity. Calvo (1983) introduced this modelling device into macroeconomics where it is extensively used and known as a “Calvo friction”. The impact is very similar to that of the uniform timing friction used in the above analysis. Mankiw and Reis (2002) study a variant when new information is only observed according to a Poisson clock.

Matsui and Matsuyama (1995) used Poisson clocks in their model of perfect foresight dynamics. They considered a continuum of players randomly matched to play a two player two action coordination game. They considered when one equilibrium of the underlying coordination game was “globally absorbing” in the sense that there always exists an equilibrium of the dynamic game where play converges to the static equilibrium. They showed that only the risk dominant equilibrium is globally absorbing. These results have been widely generalized with global game results paralleling perfect foresight dynamics results. Thus Laplacian actions are played (Kojima, 2006) and more generally, weak sufficient conditions identified for global game selection in Morris and Ui (2005) (monotone potential maximizers) are sufficient for global absorption (Oyama, Takahashi, and Hofbauer (2008)). Takahashi (2008) highlights the formal connection between global games and perfect foresight dynamics.

Frankel and Pauzner (2000) and Burdzy, Frankel, and Pauzner (2001) study a setting where a payoff relevant state evolves according to a stochastic process and is publicly observed. A continuum population receive opportunities to revise their behavior according to a Poisson clock. Players are randomly matched to play a game at each point in time, and a player's payoff depends on his action, the population average action and payoff state. If the underlying game is a two player, two action, game, the risk dominant equilibrium is selected. This is an elegant framework for analysis of coordination in dynamic settings but has not been extended beyond simple games. We can think of the analysis of the game above as corresponding to the case where payoffs evolve deterministically between a good state (when the time is in E^*) and a bad state (when the time is not in E^*).

Abreu and Brunnermeier (2003) consider a situation where investors learn that conditions for an asset pricing bubble have arrived that will last for a finite length of time. Because they find out at slightly different times, there is no common knowledge of when the conditions will end. The lack of common knowledge of the end time allows an asset pricing bubble to persist. If investors could only condition their behavior on the length of time since they heard about the bubble, the timing structure would be similar to that described above. But in this context it is natural to assume that investors can also observe real time and use real time in coordinating behavior. This means that there is learning interacting with a persistent lack of common knowledge about when the bubble will end. Another distinct feature of this problem is that while investors want to take an action (getting out of the bubble) around the same time as others do, they do not want to take the action at exactly the same time, but rather would like to take it a small time before others, i.e., to get out before the crash. This pre-emption motive also creates an important variation on synchronous coordination payoffs.

The modelling features described above can be combined in various ways. Guimaraes (2006) and He and Xiong (2012) consider situations where, like Frankel and Pauzner (2000) and Burdzy et al. (2001), a payoff relevant state evolves according to some stochastic process and is publicly observed. In the currency attack model of Guimaraes (2006), investors observe the state of a national economy; in the debt rollover model of He and Xiong (2012), investors observe the state of a company whose debt they hold. A continuum of investors can then switch their action according to a Poisson clock.

⁷ See Gale (1995), Lagunoff and Matsui (1997), Fudenberg and Olszewski (2011), Dutta (2012) and Bhaskar et al. (2013).

In He and Xiong (2012), this assumption is very naturally motivated by the assumption that they are deciding whether to rollover short term debt with staggered maturities. Currency crises and debt rollover crises have been modelled using static global games (Morris and Shin, 1998, 2003) assuming that these are pure coordination games. But the natural assumptions for both applications are that decisions are made through time, there are timing frictions and that, as in Abreu and Brunnermeier (2003), investors want to preempt other investors, i.e., attack the currency or stop rolling over debt just before others. Now, in these models, it is the dramatic failure of effective approximate common knowledge of timing that pins down when the currency or funding crisis occurs.

Given the importance of approximate common knowledge in coordination, it is natural to study the role of individual learning and communication in attaining it. Cripps et al. (2008) show sufficient conditions for individual learning to achieve approximate common knowledge, i.e., “common learning”. A maintained assumption in this analysis is that the signal profile, correlated with a fixed payoff relevant state, that players observe in each period is drawn conditionally independent by across periods. Without this assumption, common learning is very hard to achieve exactly because the conditional dependence of signals across time allows a lack of common knowledge of the time at which learning occurs (Cripps et al., 2013). In Morris (2002) and Steiner and Stewart (2011), uncertainty about the time that a communication will take to arrive similarly breaks down common knowledge of timing and thus coordination.

4. Conclusion

Neither static coordination games with incomplete information nor dynamic coordination games with timing frictions require the study of implicit higher order beliefs in order to analyze them. However, understanding results in terms of the underlying higher order beliefs has been insightful in the former context and might add insight in the latter context. The purpose of this paper has been to highlight the fact that there are deep connections across a number of bodies of literature that have been only incompletely explored.

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