Online Appendix for "Equilibrium Allocations under Alternative Waitlist Designs: Evidence from Deceased Donor Kidneys Kidneys"

Nikhil Agarwal, Itai Ashlagi, Michael Rees, Paulo Somaini, Daniel Waldinger.

A Detailed Estimation Results

Positive Crossmatch Probability

Not all accepted offers result in transplantation because additional testing may yield a positive crossmatch indicating that the patient is likely to develop an immune response to the donor's kidney. These transplants are not carried out, and if possible the organ is placed with another patient. To account for positive crossmatches when computing value functions and conducting counterfactual simulations, we estimate a probit model to predict the probability that a patient has a positive crossmatch with an organ they have accepted. The specification includes interactions between the patient's CPRA and the number of HLA mismatches with the donor, in addition to controls for patient age and number of years on dialysis. We use a subset of the variables included in the CCP model to avoid overfitting. Coefficient estimates and standard errors are displayed in Table A.1. The results are intuitive and consistent with medical knowledge. For example, higher CPRA is associated with a higher positive crossmatch probability, as are more tissue-type dissimilarities (as measured by DR or HLA mismatches). This is consistent with the view that patients with more sensitized immune systems may be more likely to test positive against foreign antibodies, even if they have not tested positive in the past.

Maximum Number of Offers and Discards

Some organs are not offered to all compatible patients in NYRT. This usually occurs either because an organ becomes unsuitable for transplantation or because the organ is accepted by a patient in another OPO. We call these events "timeouts."

We model the maximum number of offers that can be made for a given organ using a censored exponential hazards model. Duration is the number of observed offers. Censoring occurs if the organ is placed, or if it is discarded after being offered to all compatible NYRT patients. The hazard function is

Table A.1: Positive Crossmatch Model

CPRA	1.025	(0.152)
0 or 1 HLA Mismatches	-1.374	(0.474)
2 or 3 HLA Mismatches	0.199	(0.0856)
0 DR Mismatches	-0.449	(0.0930)
CPRA x 1{0 or 1 HLA Mismatches}	-0.590	(0.684)
CPRA x 1{2 or 3 HLA Mismatches}	-0.477	(0.169)
CPRA 0	-0.587	(0.0827)
CPRA - 0.8 if CPRA > 0.8	-3.389	(0.811)
Log Dialysis Time at Registration (Years)	-0.0325	(0.00846)
Log Dialysis Time at Registration x 1{Over 5 Years}	1.035	(0.0812)
Patient Age at Registration (Years)	0.0108	(0.00490)
Age at Registration - 35 if Age > 35	-0.0272	(0.00628)
Constant	-0.254	(0.170)
Observations	3876	

Notes: coefficient estimates from a probit regression of positive crossmatch on patient CPRA, the number of tissue type mismatches, patient age, and years on dialysis at registration. The sample is all offers accepted by NYRT patients between 2010 and 2013. Positive crossmatches are identified by the appropriate refusal code in the PTR data. CPRA is measured on a [0,1] scale.

$$\lambda_o(z) = \lambda_o \exp(z\beta) \tag{A.1}$$

where z are characteristics of the donor, β is a vector of coefficients, and λ_0 is the constant baseline hazard rate. We allow the hazard to depend on geography and indicators of donor quality. Specifically, we control for whether the donor is an expanded criteria donor (ECD); the donor's cause of death (DCD); and whether the donor was recovered in NYRT, as well as interactions among these variables. The estimated timeout hazards are inputs in the counterfactual exercises.

Kidneys that reach the maximum number of offers can be discarded or allocated to a patient outside NYRT. We model the probability that a donor's unallocated kidneys are discarded using a probit model that includes the same set of covariates used to estimate the maximum number of potential offers. This part of the model does not influence allocation and incentives for patients in NYRT. It is used to properly account for changes in discards for kidneys not allocated to patients in NYRT.

Table A.2: Survival Model Estimates

	Gompertz	Weibull	Cox
	(1)	(2)	(3)
Diabetic Patient	0.0812	0.0739	0.0850
	(0.0336)	(0.0336)	(0.0336)
Bloodtype A Patient	0.159	0.127	0.165
	(0.0437)	(0.0436)	(0.0438)
Bloodtype O Patient	0.00394	0.00400	0.00385
	(0.0392)	(0.0392)	(0.0392)
Calculated Panel Reactive Antibodies (CPRA)	-0.000126	-0.000211	-0.000275
	(0.00150)	(0.00150)	(0.00150)
CPRA = 0	0.190	0.179	0.181
	(0.0738)	(0.0738)	(0.0739)
CPRA - 80 if CPRA>=80	-0.0230	-0.0204	-0.0225
	(0.00650)	(0.00650)	(0.00650)
Age (at Registration)	-0.0418	-0.0363	-0.0361
	(0.0150)	(0.0151)	(0.0151)
Age - 18 if Age>=18	0.0399	0.0356	0.0348
	(0.0184)	(0.0186)	(0.0186)
Age - 35 if Age>=35	-0.00988	-0.0121	-0.0104
	(0.00966)	(0.00966)	(0.00966)
Age - 50 if Age>=50	0.0236	0.0231	0.0242
	(0.00729)	(0.00728)	(0.00729)
Age - 65 if Age>=65	0.0241	0.0233	0.0238
	(0.00927)	(0.00926)	(0.00929)
Prior Transplant	0.0513	0.0590	0.0546
	(0.0552)	(0.0550)	(0.0552)
Body Mass Index (BMI)	-0.0155	-0.0145	-0.0156
	(0.00639)	(0.00639)	(0.00640)
Missing BMI	-0.0680	0.0736	-0.104
	(0.199)	(0.199)	(0.200)
BMI >= 18.5	-0.0382	-0.0450	-0.0356
	(0.106)	(0.106)	(0.106)
BMI >= 25	0.00882	0.00346	0.00918
	(0.0492)	(0.0492)	(0.0492)
BMI >= 30	0.0509	0.0429	0.0513
	(0.0595)	(0.0595)	(0.0595)
Total Serum Albumin	-0.163	-0.160	-0.156
	(0.0549)	(0.0550)	(0.0548)
Missing Total Serum Albumin	-0.533	-0.461	-0.490
	(0.189)	(0.189)	(0.189)
Total Serum Albumin >= 3.7	-0.0645	-0.0630	-0.0681
	(0.0591)	(0.0592)	(0.0591)
Total Serum Albumin >= 4.4	0.0512	0.0405	0.0505
	(0.0510)	(0.0509)	(0.0510)
On Dialysis at Registration	-0.149	-0.169	-0.142
	(0.113)	(0.113)	(0.113)
Log Years on Dialysis at Registration	-0.00139	0.00451	-0.00291
	(0.0185)	(0.0185)	(0.0185)
Log Years on Dialysis at Registration x 1{Over 5 Years}	0.187	0.181	0.181
	(0.110)	(0.110)	(0.110)
Constant	-5.870	-5.308	
	(0.342)	(0.352)	
Constant	0.0000922		
	(0.0000210)		
Constant		-0.0785	
		(0.0143)	
Observations	9623	9623	9623

Detailed CCP Estimates

Table A.3: Conditional Choice Probability of Acceptance (Detailed)

	Base Specification		Unobserved Heterog.		Waiting Time + UH	
	(1)		(2)		(3)	
Constant	-3.70	(0.02)	-4.47	(0.03)	-4.49	(0.05)
Patient Diabetic	-0.06	(0.01)	-0.05	(0.02)	-0.03	(0.02)
Calculated Panel Reactive Antibody (CPRA)	0.60	(0.05)	0.68	(0.06)	0.58	(0.09)
CPRA >= 0.8	0.27	(0.05)	0.10	(0.06)	0.12	(0.08)
CPRA = 0	-0.10	(0.02)	-0.02	(0.03)	-0.02	(0.03)
CPRA - 0.8 if CPRA >= 0.8	-0.37	(0.37)	-0.37	(0.48)	-0.56	(0.50)
Patient had Prior Transplant	0.38	(0.02)	0.36	(0.02)	0.14	(0.03)
Donor Age < 18	0.27	(0.10)	-0.09	(0.19)	-0.04	(0.20)
Donor Age 18-35	0.59	(0.12)	-0.06	(0.19)	0.02	(0.19)
Donor Age 50+	-0.83	(0.16)	-0.77	(0.21)	-0.87	(0.22)
Donor Cause of Death Anoxia	-0.04	(0.02)	-0.12	(0.06)	-0.10	(0.06)
Donor Cause of Death Stroke	0.01	(0.02)	0.02	(0.06)	0.04	(0.07)
Donor Cause of Death CNS	0.17	(0.09)	-0.16	(0.32)	-0.16	(0.36)
Donor Creatinine 0.5-1.0	-0.06	(0.03)	0.02	(0.11)	-0.01	(0.11)
Donor Creatinine 1.0-1.5	0.01	(0.03)	0.00	(0.11)	-0.04	(0.10)
Donor Creatinine >= 1.5	-0.13	(0.03)	-0.21	(0.11)	-0.23	(0.11)
Donor Pancreas Offered	0.36	(0.03)	0.54	(0.09)	0.56	(0.09)
Expanded Criteria Donor (ECD)	-0.14	(0.02)	-0.53	(0.08)	-0.53	(0.10)
Donation from Cardiac Death (DCD)	-0.10	(0.02)	-0.51	(0.06)	-0.50	(0.09)
Donor Male	0.01	(0.01)	0.05	(0.05)	0.06	(0.04)
Donor History of Hypertension	0.01	(0.02)	-0.01	(0.05)	-0.01	(0.05)
Perfect Tissue Type Match	2.33	(0.31)	2.92	(0.43)	2.89	(0.44)
2 A Mismatches	-0.08	(0.02)	0.00	(0.02)	0.00	(0.02)
2 B Mismatches	0.06	(0.02)	0.02	(0.03)	0.03	(0.03)
2 DR Mismatches	-0.06	(0.02)	-0.05	(0.02)	-0.05	(0.02)
ABO Compatible	-0.35	(0.05)	-0.40	(0.09)	-0.41	(0.09)
Regional Offer	-1.38	(0.06)	-2.90	(0.19)	-2.92	(0.19)
National Offer	-1.54	(0.04)	-3.05	(0.12)	-3.11	(0.11)
Non-NYRT Donor, NYRT Match Run	1.23	(0.02)	2.02	(0.05)	2.08	(0.05)
Patient Blood Type A	-0.17	(0.02)	-0.28	(0.07)	-0.28	(0.07)
Patient Blood Type O	-0.32	(0.02)	-0.38	(0.06)	-0.39	(0.06)
Patient on Dialysis at Registration	-0.02	(0.02)	-0.10	(0.02)	-0.09	(0.02)
Patient Age at Registration	0.04	(0.01)	0.10	(0.01)	0.10	(0.01)
Patient Age - 18 if Age >= 18	-0.05	(0.01)	-0.11	(0.01)	-0.11	(0.01)
Patient Age - 35 if Age >= 35	0.01	(0.00)	0.02	(0.01)	0.02	(0.01)
Patient Age - 50 if Age >= 50	0.00	(0.00)	0.00	(0.00)	0.00	(0.00)
Patient Age - 65 if Age >= 65	-0.01	(0.00)	0.00	(0.01)	-0.01	(0.01)
Log Waiting Time (years)					0.09	(0.06)
Log Waiting Time x 1{Over 1 Year}					-0.15	(0.07)
Log Waiting Time x 1{Over 2 Years}					-0.13	(0.12)
Log Waiting Time x 1{Over 3 Years}					0.30	(0.11)
Patient BMI at Departure					-0.07	(0.03)
Patient BMI - 18.5 if BMI >= 18.5	0.03	(0.03)	0.07	(0.04)	0.06	(0.04)
Patient BMI - 25 if BMI >= 25	0.02	(0.01)	0.02	(0.01)	0.02	(0.01)
Patient BMI - 30 if BMI >= 30	-0.01	(0.01)	-0.02	(0.01)	-0.02	(0.01)
Patient Serum Albumin	-0.02	(0.03)	-0.01	(0.03)	-0.01	(0.03)
Serum Albumin - 3.7 if >= 3.7	-0.04	(0.05)	-0.07	(0.06)	-0.06	(0.06)
Serum Albumin - 4.4 if >= 4.4	0.12	(0.05)	0.16	(0.06)	0.16	(0.06)
Log Dialysis Time at Registration (Years)	0.04	(0.00)	0.05	(0.01)	0.05	(0.01)
Log Dialysis Time at Registration x 1{Over 5 years}	0.49	(0.03)	0.44	(0.04)	0.43	(0.04)
Perfect Tissue Type Match x Prior Transplant	-0.44	(0.19)	-0.39	(0.27)	-0.29	(0.27)
Perfect Tissue Type Match x Diabetic Patient	0.03	(0.16)	0.06	(0.23)	0.06	(0.23)
Perfect Tissue Type Match x Patient Age	-0.01	(0.01)	-0.02	(0.01)	-0.02	(0.01)
Perfect Tissue Type Match x CPRA	0.85	(0.35)	1.35	(0.48)	1.53	(0.48)
Perfect Tissue Type Match x 1{CPRA above 80%}	-0.50	(0.30)	-0.30	(0.40)	-0.38	(0.41)
Perfect Tissue Type Match x ECD Donor	-0.63	(0.16)	-0.72	(0.23)	-0.72	(0.23)
Perfect Tissue Type Match x DCD Donor	-0.46	(0.33)	-1.03	(0.47)	-1.05	(0.47)
Perfect Tissue Type Match x NYRT Donor	0.44	(0.18)	-0.02	(0.26)	-0.02	(0.26)
Perfect Tissue Type Match x ABO Compatible	0.02	(0.17)	0.09	(0.24)	0.08	(0.24)
NYRT Donor x 1{2 A Mismatches}	0.16	(0.03)	0.06	(0.04)	0.05	(0.04)
NYRT Donor x 1{2 B Mismatches}	-0.02	(0.03)	-0.05	(0.04)	-0.05	(0.04)
NYRT Donor x 1{2 DR Mismatches}	-0.03	(0.03)	-0.01	(0.04)	-0.01	(0.03)
NYRT Donor x 1{Donor Age < 18}	-0.05	(0.06)	0.18	(0.22)	0.19	(0.25)
NYRT Donor x 1{Donor Age 18-35}	0.13	(0.04)	0.24	(0.15)	0.25	(0.15)

NYRT Donor x 1{Donor Age 50+}	-0.45	(0.03)	-0.69	(0.13)	-0.68	(0.12)
Patient Age x 1{Donor Age < 18}	-0.01	(0.00)	0.00	(0.00)	0.00	(0.00)
Patient Age x 1{Donor Age 18-35}	-0.02	(0.00)	0.00	(0.01)	0.00	(0.01)
Patient Age x 1{Donor Age 50+}	0.02	(0.00)	0.02	(0.01)	0.02	(0.01)
Patient Age - 35 if Age >= 35 x 1{Donor Age 18-35}	0.02	(0.01)	0.00	(0.01)	0.00	(0.01)
Patient Age - 35 if Age >= 35 x 1{Donor Age 50+}	-0.01	(0.01)	0.00	(0.01)	-0.01	(0.01)
Log Waiting Time x Prior Transplant					0.23	(0.02)
Log Waiting Time x Patient Diabetic					-0.03	(0.02)
Log Waiting Time x Patient Age					0.00	(0.00)
Log Waiting Time x CPRA					0.08	(0.05)
Log Waiting Time x 1{CPRA >= 80}					0.00	(0.05)
Log Waiting Time x Patient Serum Albumin					-0.01	(0.01)
Log Waiting Time x Patient BMI at Departure					0.00	(0.00)
Log Waiting Time x 1{Patient Blood Type A}					0.01	(0.03)
Log Waiting Time x 1{Patient Blood Type O}					-0.01	(0.03)
Patient BMI Missing					-1.27	(0.61)
Patient Serum Albumin Missing					-0.05	(0.12)
Donor Unobservable Std. Dev.			1.02	(0.03)	1.04	(0.04)
Idiosyncratic Shock Std. Dev.	1.00		1.00		1.00	
Acceptance Rate	0.140%		0.140%		0.140%	
Number of Offers	2713043		2713043		2713043	
Number of Donors	5642		5642		5642	
Number of Patients	9494		9494		9494	

Table A.4: Out-of-sample Model Validation

Relative Mean-Squared Prediction Error of CCP Estimator		
	Estimation Sample	Validation Sample
Sparse Specification	87%	88%
Baseline Specification	81%	86%
Richer Specification	77%	89%
Richest Specification	73%	136%

Note: Validation sample includes offers made between January 1, 2014 and June 30, 2014. The relative mean squared error normalizes the MSE relative to a baseline estimator that predicts a constant CCP in each period. The sparse specification reduces the interactions and knots in the piecewise linear terms included in $\chi(\cdot)$ from our baseline specification so that we estimate about one fourth of the co-efficients. The richer specification increases the number of interactions and knots in the piecewise linear terms by a factor of four from the baseline, and the last specification further increases the number of terms by another factor of three.

B Counterfactuals

B.1 Computation Details

B.1.1 Counterfactual Scoring Mechanisms

Our algorithm to compute steady-state equilibria for counterfactual scoring mechanisms uses a discrete time grid $t = t_0, \ldots, t_l, t_{l+1}, \ldots, T$, arbitrary initial beliefs π^0, and a sample of patients and donors as inputs. In the baseline results, the type space is given by a random sample of 300 patients and 500 donors drawn from our dataset. We discretize time into quarters for the first 15 years after registration, then every 2 years until year 25, and every 25 years thereafter. These results are not sensitive to a larger set of patient and donor types or finer time partitions. Details are provided in section C.

An equilibrium is computed by iterating through the following steps until convergence:

1. Compute the value function $V_x^k(t_l)$, given beliefs π^{k-1}, via backwards induction from $V_x^k(t_{l+1})$. This calculation also yields patient strategies $\sigma_x^k(\Gamma, t) = 1\left\{\Gamma \geq V_x^k(t)\right\}$ and departure rates $\kappa_x^k(t)$.

2. Compute the queue composition m^k given departure rates $\kappa_x^k(t)$.

3. Compute $\pi^k(t; x, z)$ using the queue composition and the accept/reject strategies $\sigma_x^k(\Gamma, t)$.

4. For step $k > 1$: Terminate if the largest change in value functions and queue length/composition between iterations – $\sup_{x,l}\left|V_x^k(t_l) - V_x^{k-1}(t_l)\right|$, $\sup_{x,l}\left|m_x^k(t_l; x) - m_x^{k-1}(t_l)\right|$, and $N^k - N^{k-1}$ – are uniformly below a tolerance level. Otherwise, repeat steps 1-4.

If this algorithm terminates, the resulting accept/reject rules yield an equilibrium (up to the threshold tolerance). Because the equilibrium we compute may not be unique, we tried different starting values for π^0. Our experiments at the estimated parameters did not find multiple equilibria. Pseudo-code is provided below.

Value Function Computation (Backwards Induction):

For a small h, the value function derived in equation (3) can be approximated as

$$(\rho + \delta_x(t))\, V_x(t) \approx \lambda \int \pi_x(t; z) \int \max\{0, \Gamma(t; x, z) + \varepsilon - V_x(t)\}\, \mathrm{d}G \mathrm{d}F + \frac{V_x(t+h) - V_x(t)}{h}$$

Because the right-hand side is monotonically decreasing in $V_x(t)$, there is a unique value of $V_x(t)$ that satisfies the equation. We will use this expression to obtain the value function by backwards induction. At iteration k, given $V_x^k(t_{l+1})$ we use the bisection method to calculate the value of v that solves:

$$(\rho + \delta_x(t_l))\, v = \lambda \int \pi_x^k(t_l; z) \int \max\{0, \Gamma(t_l; x, z) + \varepsilon - v\}\, \mathrm{d}G \mathrm{d}F + \frac{V_x^k(t_{l+1}) - v}{t_{l+1} - t_l} \tag{B.2}$$

Because this problem can be written as finding $v = f(v)$ where $f(\cdot)$ is strictly decreasing, we can take any initial guess v_0 and set the lower bound to $\min(v_0, f(v_0))$ and the upper bound to $\max(v_0, f(v_0))$. We use the initial guess $v_0 = V_x^k(t_{l+1})$.

Offer Probabilities, $\pi_{x,z}(t)$:

Section B.1.2 derives a computationally tractable approximation to offer probabilities given a scoring rule s, a large waitlist N^*, and an acceptance policy function. The expression in equation (B.5) below can be simplified and solved for analytically. We use that solution in our algorithm.

Waitlist Size/Composition, m, N:

We use $\kappa_x(t)$ and γ_x to update the queue composition. Solving the ODE in Definition 1, part 3(a), we get that for any $h > 0$,

$$m_x(t+h) = m_x(t) \exp\left(-\int_0^h \kappa_x(t+\tau)\, \mathrm{d}\tau\right),$$

where $m_x(0) = \lambda_x$. Approximating $\kappa_x(t+\tau) = \kappa_x(t+h)$ for all $\tau \in (0, h)$, we have that

$$m_x(t_{l+1}) = m_x(t_l) \exp(-\kappa_x(t_{l+1})(t_{l+1} - t_l)). \tag{B.3}$$

Finally, we scale the output so that $m_x(t_l)$ is a probability measure.

The size of the waitlist, N, is determined by part 3(b) of Definition 1.

B.1.2 Approximating Offer Probabilities

Fix a particular agent i with priority score s. Ties are broken randomly, so wlog consider each agent's tiebreaker to be drawn from a uniform distribution. Let $1-\alpha_i$ be the tie-breaker for agent i.

An offer may be the last one because it may be accepted or because the kidney may expire after the offer. This model, specified in equation (A.1), yields a probability, $p_0 = \lambda_o(z)$, the

Algorithm 1 Steady State Equilibrium

Inputs: Patient and donor characteristics, scoring rule s, parameters Γ, δ, ρ, and patient age grid $\{t_0, \ldots, t_L = T\}$. Let $t_{l_x^0}$ be the arrival time for patient of type x.
Outputs: V^*, π^*, m^*, N^*
Initialize $k = 0$ and beliefs $\pi_x^k(t)$ for all x and $t \in \{t_0, \ldots, t_L\}$
repeat
    $V^k \leftarrow$ Backwards Induction(π^k)
    $\kappa_x^k(t_l) \leftarrow \delta_x(t_l) + \lambda \sum_z \pi_{x,z}^k(t_l)\mathbb{P}(\Gamma(t_l; x, z) + \varepsilon > V_x^k(t_l))$
    $m^k, N^k \leftarrow$ Forward Simulation(κ^k) ▷ Waitlist Composition
    $\pi^k \leftarrow$ Compute Offer Probabilites(V^k, m^k, N^k) ▷ Offer Probabilities
    $k \leftarrow k + 1$
until $k > 1$, $\|V^k - V^{k-1}\|_\infty < \epsilon$, $\|m^k - m^{k-1}\|_\infty < \epsilon$, and $N^k = N^{k-1}$ ▷ Convergence
$V^* \leftarrow V^k, m^* \leftarrow m^k, N^* \leftarrow N^k, \pi^* \leftarrow \pi^k$
function BACKWARDS INDUCTION(π)
    for all x **do**
        Set $V_x(T) = 0$
        for all x and $t_l = t_{L-1}$ to $t_{l_x^0}$ **do**
            Compute $V_x(t_l)$ by solving for v in equation (B.2)
        end for
    end for
    return $V_x(t_l)$ for all x and $t_l \in \{t_{l_x^0}, \ldots, T\}$
end function
function FORWARD SIMULATION(κ)
    for all x **do**
        $m_x(t_{l_x^0}) \leftarrow \lambda_x$
        for all $t_l = t_{l_x^0+1}$ to T **do**
            $m_x(t_{l+1}) \leftarrow m_x(t_l)\exp(-\kappa_x(t_l)(t_{l+1} - t_l))$
        end for
    end for
    $N^k \leftarrow \sum_{x,t_l} m_x^k(t_l)\kappa_x^k(t_l)$ ▷ Waitlist Size: Definition 1, part 3(b)
    $m_x(t_l) \leftarrow m_x(t_l)/N^k$ for all x and t_l
    return $m_x(t_l)$ for $t_l \in \{t_{l_x^0}, \ldots, T\}$ and N^k
end function
function COMPUTE OFFER PROBABILITIES(m, V, N)
    $p^a(t_l; x, z) \leftarrow \mathbb{P}(\Gamma(t_l; x, z) + \varepsilon > V_x(t_l))$ for all x, t_l
    for all $s = \max s(t_l; x, z)$ to $\min s(t_l; x, z)$ **do**
        Compute π using equation (B.5)
    end for
    return π^k
end function

probability of a timeout before the next offer for an object of type z. For simplicity, we fix z and drop it from the notation.

An agent receives an offer if the total number of acceptances and timeouts after offers to agents with a higher priority score than agent i is strictly less than the number of copies of the object available. Consider waitlists that are composed of N agents randomly drawn from distribution m. The probability that each drawn agent is ordered above i and that the kidney is either accepted by the agent or times out is

$$p(s,\alpha) = m_H(s)\, p_H(s) + m_E(s)\, \alpha p_E(s).$$

The first term represents the case when an agent with a higher priority (group H) is drawn. The probability of the kidney becoming unavailable conditional on an agent drawn from a higher priority group is:

$$p_H(s) = p_0 + (1-p_0)\frac{1}{m_H(s)}\sum_{t,x} m(t;x)\, 1\{s(t;x) > s\}\, \mathbb{P}(\Gamma(t;x) + \varepsilon > V_x(t)).$$

The second term is the probability that an agent with priority score s is drawn. The term $p_E(s)$, representing the case when an agent in the same priority group is drawn, is defined analagously as $p_H(s)$.

Therefore, the number of times a kidney would become unavailable after being offered to an agent ordered above i is a binomial random variable X with parameters N and $p(s,\alpha)$. An object is available to agent i if $X < q$, where q is the number of copies of the object. Hence, the probability that i receives an offer is given by

$$\int_0^1 \mathbb{P}(X < q | s, \alpha)\, \mathrm{d}\alpha, \tag{B.4}$$

where we have integrated over the tie-breaker α, and explicit conditioning on N is subsumed for simplicity.

For large N and small $p(s,\alpha)$, the distribution of X approaches the distribution of a Poisson random variable with parameter $Np(s,\alpha)$. Therefore, the expression in equation (B.4) yields the following expression for $\pi_x(t)$:

$$\pi_x(t) = \int_0^1 \sum_{q'<q} \frac{e^{-Np(s,\alpha)}\,(Np(s,\alpha))^{q'}}{q'!}\mathrm{d}\alpha,$$

where we use the Poisson approximation to re-write $\mathbb{P}(X < q|s,\alpha)$. As a reminder, the object type z is dropped from the notation for simplicity as it is fixed, although the offer probabilities depend on it. This integral can be solved for in closed form for $q \in \{1, 2\}$:

$$\pi_x(t) = \frac{e^{-Np(s,0)} - e^{-Np(s,1)}}{N(p(s,1) - p(s,0))} + 1\{q=2\} \frac{(1+Np(s,0))e^{-Np(s,0)} - (1+Np(s,1))e^{-Np(s,1)}}{N(p(s,1) - p(s,0))}. \tag{B.5}$$

B.2 Optimal Assignments and Optimal Offer Rates

The objective functions for these two problems are identical. It is given by:

$$\sum \frac{1}{\bar{V}_x^{\mathcal{M}_0}(\lambda_0)} \left[\frac{\gamma_x}{\rho} V_x(0) + \sum_l N m_x(t_l)(t_{l+1} - t_l) V_x(t_l)\right],$$

where $\bar{V}_x^{\mathcal{M}_0}(\lambda_0)$ is defined in equation (11) and V are choice variables interpreted as in the rest of the paper. The constraints on the two problems differ and each has a separate, third choice variable. For the optimal assignment mechanism, we choose assignment policies μ. For the optimal offer mechanism, we choose offer rates π. We describe these variables and constraints below. The nonlinear problem is solved using the KNITRO optimizer interfaced with MATLAB.

B.2.1 Optimal Assignments

This allocation maximizes the objective function above by assigning an object of type z to agents currently on the list. The social planner knows the payoffs Γ_{xzt} as well as the idiosyncratic shocks ε. The planner also knows the steady state distribution of agents waiting for an assignment but not the future arrivals of objects or agents. The choice variable is the probability μ_{zxt} with which a compatible object of type z is allocated to an agent of type x who has waited for t periods. Given μ, the assignment is made to compatible agents of type x that have waited for t periods and have the highest draws of ε. Choosing μ is equivalent to choosing a cutoff $\underline{\varepsilon}_{xzt}$ such that $\mu_{xzt} = \mathbb{P}(a(\varepsilon; x, z, t) = 1) = \int 1\{\varepsilon > \underline{\varepsilon}_{xzt}\} \mathrm{d}G$, where the integral is taken with respect to ε.

There are three constraints:

1. Value Function: The agent's net present value $V_x(\cdot)$ from the expected stream of as-

signments under the policy μ_{zxt} is defined by

$$\left(1+\left(\rho+\delta_x\left(t_l\right)+\lambda \sum_z f_z \mu_{xzt_l} c_{xz}\right)\left(t_{l+1}-t_l\right)\right) V_x\left(t_l\right)=\left(t_{l+1}-t_l\right) \lambda w_x\left(t_l\right)+V_x\left(t_{l+1}\right),$$

where

$$w_x(t)=\sum_z f_z c_{xz} \int\left(\Gamma_{xzt}+\varepsilon\right) 1\left\{\varepsilon>\underline{\varepsilon}_{xzt}\right\} \mathrm{d} G,$$

f_z is the probability that the object type is z, integrals are over ε, and c_{xz} is the known (estimated) compatibility probability. These expressions for V and w are obtained by solving the value function from following the policy of accepting offers with ε above $\underline{\varepsilon}_{xzt}$, with offers made whenever an object arrives. The term $w_x(t)$ denotes the expected value to an agent of type x conditional on an object arriving.

2. Feasibility: The total mass of type z objects that are assigned upon arrival must not exceed the mass of objects that arrive. Specifically, for each z, we impose the constraint

$$\sum_{x,l} N m_x\left(t_l\right)\left(t_{l+1}-t_l\right) c_{xz} \mu_{zxt_l} \leq q_z.$$

The left hand side is the cumulative product of the (discretized) masses of each type of agent on the waitlist, $Nm_x(t_l)(t_{l+1}-t_l)$, multiplied by the assignment probabilities $c_{xz}\mu_{xzt_l}$ for each agent. This quantity cannot exceed the mass of objects that arrive, q_z.

3. Steady-State Composition: The measure of agents of type x that have waited for t periods is in steady state. This constraint is analogous to equation (B.3) above. Specifically, for each x and $l>0$, we have that

$$\begin{aligned} N m_x\left(t_{l+1}\right) & =N m_x\left(t_l\right) \exp \left(-\left(\delta_x\left(t_l\right)+\lambda \sum_z f_z c_{xz} \mu_{xzt_l}\right)\left(t_{l+1}-t_l\right)\right), \\ N m_x\left(t_0\right) & =\gamma_x . \end{aligned}$$

The term $\lambda \sum_z f_z c_{xz} \mu_{xzt_l}$ is the cumulative assignment rate across objects for an agent of type x at time t_l. This, when added to $\delta_x(t_{l+1})$, yields the total departure rate.

In addition, we impose that each μ_{xzt} belongs to unit interval.

B.2.2 Optimal Offer Rates

This problem maximizes the objective function above by choosing a probability of offering an object of type z to agents currently on the list. The social planner has full information about the payoffs Γ_{xzt}, but does not know the idiosyncratic shocks ε. She knows the steady-state distribution of agents waiting for an assignment but not the future arrivals of objects or agents. The choice variable in this problem is the probability π_{zxt} with which an arriving object of type z is offered to an agent of type x who has waited for t periods. Agents optimally choose which offers to accept given π.

As before, there are three constraints:

1. Value Function: The agent's net present value $V_x(\cdot)$ from the expected stream of assignments under the policy π_{zxt} is defined by

$$\left(1+\left(\rho+\delta_x\left(t_l\right)\right)\left(t_{l+1}-t_l\right)\right)V_x\left(t_l\right)=\left(t_{l+1}-t_l\right)\lambda w_x\left(t_l\right)+V_x\left(t_{l+1}\right),$$

   where

$$w_x\left(t\right)=\sum_z f_z\pi_{xzt}c_{xz}\int\max\left\{0,\Gamma_{xzt}+\varepsilon-V_x\left(t\right)\right\}\mathrm{d}G,$$

   f_z is the probability that the object type is z, and integrals are taken with respect to ε. As in the optimal assignment problem, $w_x(t)$ is the expected value to an agent of type x conditional on an object arriving. However, in this problem, the agent makes optimal decisions and offers do not depend on the payoff shocks. Therefore, an assignment occurs only if the agent is offered the object and the agent accepts. Acceptance occurs if the payoff shock exceeds $V_x(t)-\Gamma_{xzt}$.

2. Feasibility: The total mass of type z objects assigned must not exceed the mass of objects that arrive. Specifically, for each z, we impose the constraint

$$\sum_{x,l}\tilde{m}_x\left(t_l\right)\left(t_{l+1}-t_l\right)\pi_{zxt_l}\left[c_{xz}\int 1\left\{\Gamma_{xzt_l}+\varepsilon>V_x\left(t_l\right)\right\}\mathrm{d}G+p_{0,z}\right]\leq q_z,$$

   where the integral is over ε. This constraint is analogous to the feasibility constraint in the optimal assignment problem. The difference is that the assignment rate $c_{xz}\mu_{xzt}$ is replaced by the term

$$\pi_{zxt_l}\left[c_{xz}\int 1\left\{\Gamma_{xzt_l}+\varepsilon>V_x\left(t_l\right)\right\}\mathrm{d}G+p_{0,z}\right].$$

The term π_{zxt_l} denotes the probability that an agent of type x receives an offer for an object of type z after she has waited for t_l periods. The term in brackets is the probability that any such offer is the last offer for the object that can be made. It is the sum of the probability that object is compatible and transplanted,

$$c_{xz} \int 1\left\{\Gamma_{xzt_l} + \varepsilon > V_x\left(t_l\right)\right\} \mathrm{d}G,$$

and the probability that no more offers can be made after the current one. This term arises from the technological constraint on the number of offers that can be made for an object. The model used to determine $p_{0,z}$ is described in Appendix A.

This constraint only restricts the expected number of assignments. Therefore, the offer rates π_{xzt} may not be implementable for a specific sequence of donor and patient arrivals.

3. Steady-State Composition: The measure of agents of type x that have waited for t periods is in steady state. Specifically, for each x and $l > 0$, we have

$$\begin{aligned}
\tilde{m}_x\left(t_{l+1}\right) &= \tilde{m}_x\left(t_l\right) \exp\left(-\left(\delta_x\left(t_{l+1}\right) + \lambda \sum_z f_z \mu_{xzt_l}\right)\left(t_{l+1} - t_l\right)\right), \\
\tilde{m}_x\left(t_l\right) &= \gamma_x,
\end{aligned}$$

where

$$\mu_{xzt} = \pi_{zxt} c_{xz} \int 1\left\{\Gamma_{xzt} + \varepsilon > V_x\left(t\right)\right\} \mathrm{d}G.$$

This constraint differs from its analogue in the optimal assignment problem because here the assignment probability μ_{xzt} depends on agents' acceptance decision.

In addition, we impose that each π_{xzt} belongs to unit interval.

C Robustness and Supplementary Evidence

C.1 Robustness

Table C.5: Outcomes under Alternative Modeling Assumptions

	EV_x	Waitlist: Queue Length	Waitlist: Reduction in Discard Rate	Waitlist: Years on Waitlist	Transplanted Donors: Age	Transplanted Donors: Head Trauma	Transplanted Donors: Hyper-tensive	EV_x Decomp.: Obs.	EV_x Decomp.: Unobs.	$\Delta V_x(0)$ > -5%
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Panel A: Patient Unobserved Heterogeneity										
Pre-2014 Priorities	--	5166.5	--	2.80	44.2	15.6%	46.1%	--	--	--
Post-2014 Priorities	-1.1%	5108.3	0.3%	2.78	44.2	15.5%	46.1%	-0.3%	-0.8%	94.7%
First Come First Served	1.7%	5317.5	-1.8%	2.84	44.3	15.8%	46.1%	1.0%	0.7%	91.0%
Last Come First Served	-50.9%	3399.7	20.9%	4.28	46.9	14.4%	51.1%	-45.2%	-5.7%	14.0%
Panel B: Annual Discount Factor of 10 Percent										
Pre-2014 Priorities	--	5150.4	--	2.76	44.8	16.4%	47.0%	--	--	--
Post-2014 Priorities	-1.3%	5084.4	0.4%	2.74	44.8	16.2%	47.0%	0.0%	-1.3%	96.7%
First Come First Served	2.6%	5304.5	-2.0%	2.81	44.9	16.5%	47.0%	1.2%	1.4%	90.3%
Last Come First Served	-59.1%	3465.1	18.6%	4.13	46.9	14.9%	50.8%	-53.4%	-5.7%	9.7%
Panel C: Larger Patient and Donor Type Space										
Pre-2014 Priorities	--	4440.0	--	2.58	44.9	23.2%	43.0%	--	--	--
Post-2014 Priorities	-0.9%	4365.4	0.6%	2.54	44.9	23.2%	43.0%	0.1%	-1.0%	98.8%
First Come First Served	1.5%	4567.0	-1.7%	2.61	44.7	23.4%	42.8%	0.7%	0.8%	93.4%
Last Come First Served	-43.0%	2385.8	22.6%	2.69	47.0	20.8%	48.2%	-35.2%	-7.8%	5.0%
Panel D: No Limit on Maximum Number of Offers										
Pre-2014 Priorities	--	4597.0	--	2.55	45.0	15.8%	47.4%	--	--	--
Post-2014 Priorities	-0.9%	4553.0	0.4%	2.54	45.0	15.8%	47.3%	0.1%	-1.0%	98.3%
First Come First Served	2.8%	4607.5	-0.2%	2.56	44.7	15.9%	47.1%	1.3%	1.5%	94.3%
Last Come First Served	-28.2%	2876.8	20.8%	2.65	46.1	14.7%	49.5%	-21.1%	-7.1%	45.0%

Patient Unobserved Heterogeneity: We re-estimated the model allowing for two unobserved patient types. Specifically, we re-parametrized the CCPs as follows:

$$P_{ijt} = G\left(\alpha_i + \chi\left(x_i, z_j, t\right)\theta + \eta_j\right),$$

where $\alpha_i \in \{\alpha_1, \alpha_2\}$ with the parameters α_1 and α_2 and the share of α_1 to be estimated. This parameterization allows patients to have systematically higher or lower values of all transplants relative to their outside options. We abstract away from the initial conditions problem, setting the proportion of each patient type in our sample to the population average.

This latter assumption is appropriate for patients that registered during our sample period, but ignores selection that should arise for patients that registered prior to the sample of offers we consider.

Estimating this model requires another data augmentation step. This step draws each agent i's type given their observed decisions and the parameters α_1, α_2, and π_1. Conditional on $(\alpha_1, \alpha_2, \pi_1)$, the posterior probability that $\alpha_i = \alpha_1$ is proportional to the likelihood of observing the decisions made by agent i multiplied by π_1. This likelihood is the product of the cumulative density functions of normal distributions. The parameters α_1, α_2, and π_1 are then updated using conjugate priors. We specify diffuse normal priors for α_1 and α_2 and a Dirichlet prior for π_1 (see Section 3.4, Gelman et al., 2014). As recommended in Gelman et al. (2014), we check for re-ordering and impose the restriction that $\alpha_1 > \alpha_2$.

Table C.5 Panel A presents the results for the steady states of benchmark mechanisms considered in the main text.

Discount Factor: As discussed in Section 3, the discount faction ρ is not identified and is set to 5 percent per year. Here, we evaluate sensitivity of our results to using an annual discount rate of 10 percent. Only Steps 3 and 4 in Section 4.2 must be revised to obtain estimates with an alternative discount rate. Panel B of Table C.5 presents the counterfactual results.

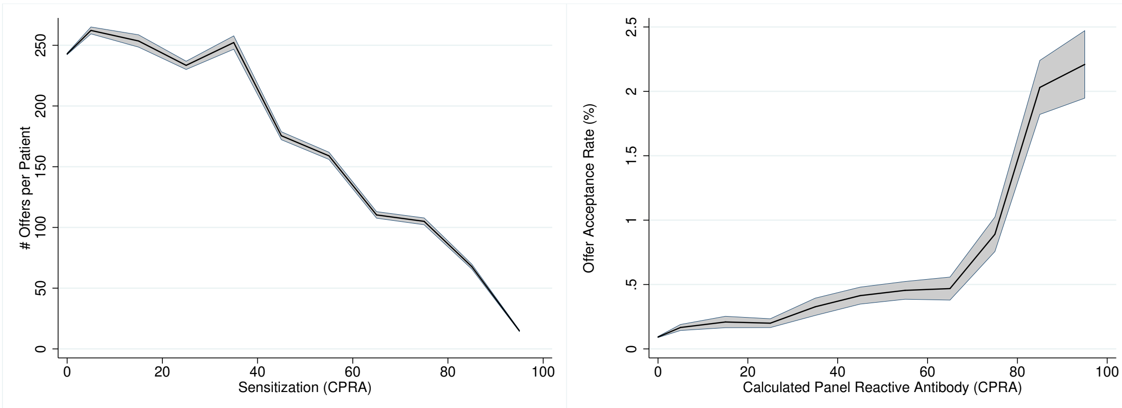
Larger Samples: The main text limits the number of types used in counterfactual calculations to 300 patient types and 500 donor types. To assess whether the results are sensitive to the specific sample and number of types, we re-calculated the counterfactuals involving scoring mechanism by drawing 1,000 patient types and 1,500 donor types. Panel C of Table C.5 presents the results.

Unlimited Offers: Our results could be sensitive to the limit on the number of offers, especially if improvements in technology that allow the OPO to make many more offers obviates the need for finding better mechanisms. Panel D of Table C.5 presents results calculated when this limit is removed.

C.2 Supplementary Evidence

Results analogous to Figure 1 and Table 3 in Agarwal et al. (2018) are presented in Figures C.1 and Table C.6 respectively.

Figure C.1: Offer and Acceptance Rate by CPRA



Note: Sample includes all offers made to NYRT patients between 2010 and 2013, including offers that did not meet pre-set donor screening criteria. Positive crossmatches are counted as acceptances. In each figure, the black line plots the mean among offers to patients in each CPRA bin, and the shaded region represents pointwise 95 percent confidence intervals.

Table C.6: Evidence of Response to Dynamic Incentives

	Dependent Variable: Offer Accepted				
	(1)	(2)	(3)	(4)	(5)
Calculated Panel Reactive Antibodies (CPRA)	0.0148	0.00869	0.00822	0.00757	0.00770
	(0.000764)	(0.000889)	(0.000879)	(0.000831)	(0.000833)
Variables Affecting Priority		X	X	X	X
Patient Characteristics			X	X	X
Donor and Match Characteristics				X	X
Interaction between CPRA and # HLA Mismatches					X
Mean Acceptance Rate	0.142%	0.142%	0.142%	0.142%	0.142%
Observations	2713043	2713043	2713043	2713043	2713043
R-squared	0.003	0.006	0.009	0.104	0.104

Notes: Estimates from a linear probability model of offer acceptance on patient Calculated Panel Reactive Antibodies (CPRA). The sample is offers made to NYRT patients between 2010 and 2013, including offers that did not meet pre-set screening criteria. CPRA is measured on a [0,1] scale at the time of the offer. Column 1 controls for a CPRA=0 indicator. Column 2 adds controls affecting patient priority: indicators for CPRA>=0.2, CPRA>=0.8, and age<18, as well as waiting time indicators and linear controls for 1-3, 3-5, and >5 years. Column 3 adds other patient characteristics. Column 4 adds controls for donor and match characteristics. Column 5 adds interactions between CPRA indicators and # HLA mismatches. Patient characteristics are indicators for age 18-35, 35-50, and 50-65; indicators for diabetes, blood type, and the patient's transplant center; linear controls and indicators for dialysis time 1-3, 3-5, 5-10, and >10 years; and an indicator for health status at listing. Donor controls are linear age; linear creatinine with indicators for 0.6-1.8 and >1.8; the donor's mean acceptance rate; and indicators for diabetes, donation after cardiac death (DCD), and expanded criteria donor (ECD). Match characteristics are linear # HLA mismatches; indicators for zero HLA mismatch, 0 and 1 DR mismatch, identical blood type, offer year, and local donor; linear controls for (+) and (-) age difference; and interactions between local and zero-HLA mismatch, and local and donor age, donor over 40 and pediatric patient, donor over 55 and patient age 18-35, and donor over 60 and patient age 35-50 and over 50. Standard errors, clustered by donor, are in parentheses.

D Additional Theoretical Results

D.1 Existence of Steady-State Equilibria

This section proves that a steady-state equilibrium exists for sequential offer mechanisms that use a scoring rule. We make the following assumptions:

Assumption 1. *(i) The exogenous arrival rates λ and γ_x are finite*

(ii) The exogenous departure rate $\delta(\tau; x)$ is bounded below by $\underline{\delta} > 0$ and bounded above by $\bar{\delta}$, unifomly for $t \in [0, T)$ and all $x \in \chi$

(iii) The conditional probability density function $f_{\Gamma|t,x,z}$ exists, and is uniformly bounded

(iv) The conditional moment,$\mathbb{E}[|\Gamma| \,|\tau, x, z] = \int |\Gamma| \, \mathrm{d}F_{\Gamma|\tau,x,z}$ where $\Gamma = \Gamma(x, z, \tau) + \varepsilon$, is uniformly bounded in t, x, z

(v) The family of functions $g\left(t; x, z, \bar{\Gamma}\right) = F_{\Gamma|t,x,z}\left(\bar{\Gamma}\right)$ indexed by $\bar{\Gamma}$, x, z is Lipschitz continuous in t with a common constant

(vi) The object arrival rate λ is strictly less than the total agent arrival rate $\sum \gamma_x$

(vii) The set of scores $S = \{s(t; x, z) : (t, x, z) \in [0, T] \times \chi \times \zeta\}$ is finite.

Most empirical models will satisfy the continuity and boundedness assumptions above. The two substantive assumptions are parts (vi) and (vii). Part (vi) assumes that the objects that need to be assigned are scarce in order to ensure that the queue is unlikely to be empty. Part (vii) restricts the mechanisms for which we prove existence. The assumption is used to ensure that the set of all functions $\pi_{xz}(t)$ is sufficiently small (more precisely, compact). Other assumptions that yield this conclusion would also suffice.

Our main result proves existence of a steady-state equilibrium.

Theorem 1. *Suppose Assumption 1 is satisfied. Then a steady-state equilibrium for a sequential offer mechanism with a scoring rule exists.*

Proof. The proof proceeds by applying the Brower-Schauder-Tychonoff Fixed Point Theorem (Corollary 17.56, Aliprantis and Border, 2006). The proof proceeds in three parts.

Part 1, Definition of Ω: The equilibrium objects are defined by five types of functions:

1. The conditional choice probabilities, given t and the agent and object characteristics x and z. We consider these choice probabilities as a function $p_\sigma : [0, T] \times \chi \times \zeta \to [0, 1]$.

2. The value function $V : \chi \times [0,T] \to \mathbb{R}_+$. It is convenient to define this function, although it is somewhat redundant with the choice probabilities above.

3. The offer probabilities $\pi : [0,T] \times \chi \times \zeta \to [0,1]$ where $\pi(t;x,z) = H_z(s_{xz}(t)) \times \mathbb{P}(c_{ij} = 1|x,z)$.

4. The distribution of agent types $m : \chi \times [0,T] \to \mathbb{R}_+$.

5. The queue length $N \in \mathbb{R}$.

We denote the tuple of these objects by $\omega = (p_\sigma, V, \pi, m, N)$. We endow each of the functions in the first four objects with the supremum norm over its domain. The norm for ω is denoted $\|\omega\| = \|p_\sigma\| + \|V\| + \|\pi\| + \|m\| + |N|$. Therefore, ω is an element of a Banach space.

We further restrict ω to belong to a subset Ω of this Banach space. Specifically, we restrict its components as follows:

1. The functions $V_x(t)$ are uniformly bounded by $\lambda T \sup_{\tau,x,z} \int |\Gamma| \, \mathrm{d}F_{\Gamma|\tau,x,z}$and are Lipschitz continuous with a common constant $\left(1+\rho+\bar{\delta}\right)\lambda \sup_{\tau,x,z} \int |\Gamma| \, \mathrm{d}F_{\Gamma|\tau,x,z}$.

   Note that the optimal value of $V_x(t)$ satisfied this property. To see this, observe that $\frac{\mathrm{d}}{\mathrm{d}t}V_x(t) = -\lambda \exp(-\rho(\tau-t))\, p(\tau|t,x)L(t) + \lambda \int_t^T (-\rho - \delta_x(t))\, L(\tau) d\tau$ where $L(\tau) = \int \pi_{ij}(\tau) \int \max\{0, \Gamma_{ij}(\tau) - V_i(\tau)\} dGdF$. The result follows since $L(\tau)$ is bounded by $\sup_{\tau,x,z} \int |\Gamma| \, \mathrm{d}F_{\Gamma|\tau,x,z}$.

2. The functions $p_\sigma(t;x,z)$ are uniformly bounded by 1 and Lipschitz continuous with a common constant K, where

$$K = \left(1+\rho+\bar{\delta}\right)\lambda \sup_{\tau,x,z}\left(\int |\Gamma| \, \mathrm{d}F_{\Gamma|\tau,x,z} \sup_{\Gamma} f_{\Gamma|\tau,x,z}(\Gamma)\right) + \sup_{\bar{\Gamma},x,z,t,t'} \left|F_{\Gamma|t,x,z}\left(\bar{\Gamma}\right) - F_{\Gamma|t',x,z}\left(\bar{\Gamma}\right)\right| / |t-t'|.$$

   Note that Assumption 1 implies that K is finite. That the equilibrium value satisfies this assumption can be seen from part 1.

3. The functions $\pi_{x,z}(t)$ such that $\pi_{x,z}(t) = \pi_{x,z}(t')$ if $s_{xz}(t) = s_{xz}(t')$ with range $[0,1]$.

4. The term $N \in \left[\underline{N}, \bar{N}\right]$, where $\underline{N} = \left(\sum_x \gamma_x - \lambda\right)/\bar{\delta}$ and $\bar{N} = \frac{\sum_x \gamma_x}{\underline{\delta}}$. These bounds are obtained by considering the extremal cases in which no agent is assigned and when every kidney is assigned. Note that $\underline{N} > 0$ because Assumption 1 requires that $\sum_x \gamma_x > \lambda$ and $\delta(\tau;x)$ is uniformly bounded above.

5. The functions $m_x(t)$ are uniformly bounded by $\frac{\sup_x \gamma_x}{\underline{N}}$ and are Lipschitz continuous with a common constant $\frac{\sup_x \gamma_x}{\underline{N}}\left(\bar{\delta}+\lambda\right)$. The steady-state value satisfies this requirement since $T\gamma_x$ is the maximum mass of agents of type x, and because

$$|\dot{m}_x(t)| = m_x(t)\kappa_x(t) \leq m_x(t)\left(\bar{\delta}+\lambda\right).$$

Part 2, definition of $A:\Omega\to\Omega$: Denote $A_V[\omega]$ as the V component of $A[\omega]$, where $\omega\in\Omega$. Likewise, define A_π, A_{p_σ}, A_m and A_N. This map is defined as follows:

$$\begin{aligned}
A_V[\omega](x,t) &= \int_t^T \exp\left(-\rho(\tau-t)\right)p(\tau|t;x)\left(\lambda\int\pi(\tau;x,Z)\int\max\left\{0,\Gamma-V(\tau;x)\right\}\mathrm{d}F_{\Gamma|\tau,x,Z}\mathrm{d}F_Z\right)\mathrm{d}\tau\\
A_{p_\sigma}[\omega](x,z,t) &= \int 1\left\{\Gamma\geq A_V[\omega](x,t)\right\}\mathrm{d}F_{\Gamma|x,z,t}\\
A_m[\omega](x,t) &= \gamma_x\exp\left(-\int_0^t\delta(\tau;x)+\lambda\int\pi(\tau;x,Z)\,p_\sigma(\tau;x,z)\,\mathrm{d}F_Z\mathrm{d}\tau\right)/N\\
A_N[\omega] &= \max\left\{\underline{N},\min\left\{\frac{\sum_x\gamma_x}{\sum_x\int_0^T m_x(t)\,\kappa_x(t)\,\mathrm{d}t},\bar{N}\right\}\right\}\\
A_\pi[\omega](x,z,t) &= H_z\left(s_{xz}(t);A_{p_\sigma}[\omega],A_m[\omega],A_N[\omega]\right)\times\mathbb{P}\left(c_{ij}=1|x,z\right),
\end{aligned}$$

where

$$p(\tau|t;x) = \exp\left(-\int_t^\tau\delta(\tau';x)\,\mathrm{d}\tau'\right)$$

is the probability that an agent of type x departs the list prior to τ conditional on being on the list at t. To ensure that the image is a subset of Ω, we need to show that $A[\omega]\in\Omega$ for all $\omega\in\Omega$. We do this for each of the components separately:

1. A_V: Since $\exp(-\rho(\tau-t))$, $p(\tau|t;x)$ and $\pi(\tau;x,Z)$ are in $[0,1]$, and

$$\int\max\left\{0,\Gamma-V(\tau;x)\right\}\mathrm{d}F_{\Gamma|\tau,x,Z}\leq\int|\Gamma|\,\mathrm{d}F_{\Gamma|\tau,x,Z},$$

we have that $A_V[\omega]$ is uniformly bounded by $\lambda T\sup_{\tau,x,z}\int|\Gamma|\,\mathrm{d}F_{\Gamma|\tau,x,z}$. Further, for any t, $t'\in[0,T]$, with $t<t'$, we have that

$$\begin{aligned}
&|A_V[\omega](t)-A_V[\omega](t')|\\
&=\left|\int_t^{t'}\exp\left(-\rho(\tau-t)\right)p(\tau|t;x)\left(\lambda\int\pi(\tau;x,Z)\int\max\left\{0,\Gamma-V(\tau;x)\right\}\mathrm{d}F_{\Gamma|\tau,x,Z}\mathrm{d}F_Z\right)\mathrm{d}\tau\right|\\
&\leq\lambda|t'-t|\left(1+\rho+\bar{\delta}\right)\sup_{\tau,x,z}\int|\Gamma|\,\mathrm{d}F_{\Gamma|\tau,x,z}.
\end{aligned}$$

Therefore, $A_V[\omega]$ satisfies the necessary restrictions.

2. A_{p_σ} : Note that $A_{p_\sigma}[\omega]$ is uniformly bounded by 1. Moreover, for any x and z, and $t,\, t' \in [0,T]$, we have that

$$
\begin{aligned}
&\left|A_{p_\sigma}[\omega](t,x,z) - A_{p_\sigma}[\omega](t',x,z)\right| \\
=&\left|\int 1\{\Gamma \geq A_V[\omega](x,t)\}\,\mathrm{d}F_{\Gamma|x,z,t} - \int 1\{\Gamma \geq A_V[\omega](x,t')\}\,\mathrm{d}F_{\Gamma|x,z,t'}\right| \\
=&\left|\int \left(1\{\Gamma \geq A_V[\omega](x,t)\} - 1\{\Gamma \geq A_V[\omega](x,t')\}\right)\mathrm{d}F_{\Gamma|x,z,t}\right| \\
&+\left|\int 1\{\Gamma \geq A_V[\omega](x,t')\}\,\mathrm{d}\left(F_{\Gamma|x,z,t} - F_{\Gamma|x,z,t'}\right)\right| \\
\leq&\left|\int_{\min\{A_V[\omega](x,t),A_V[\omega](x,t')\}}^{\max\{A_V[\omega](x,t),A_V[\omega](x,t')\}} 1\mathrm{d}F_{\Gamma|x,z,t}\right| \\
&+\left|F_{\Gamma|x,z,t'}\left(A_V[\omega](x,t')\right) - F_{\Gamma|x,z,t}\left(A_V[\omega](x,t')\right)\right| \\
\leq&\lambda\left(1+\rho+\bar{\delta}\right)|t'-t|\sup_{\tau,x,z}\left(\int|\Gamma|\,\mathrm{d}F_{\Gamma|\tau,x,z}\sup_{\Gamma} f_{\Gamma|\tau,x,z}(\Gamma)\right) + \sup_{\bar{\Gamma},x,z}\left(\left|F_{\Gamma|t,x,z}\left(\bar{\Gamma}\right) - F_{\Gamma|t',x,z}\left(\bar{\Gamma}\right)\right|/\left|t-t'\right|\right) \\
\leq&\left[\lambda\left(1+\rho+\bar{\delta}\right)\sup_{\tau,x,z}\left(\int|\Gamma|\,\mathrm{d}F_{\Gamma|\tau,x,z}\sup_{\Gamma} f_{\Gamma|\tau,x,z}(\Gamma)\right) + \sup_{\bar{\Gamma},x,z,t,t'}\left(\left|F_{\Gamma|t,x,z}\left(\bar{\Gamma}\right) - F_{\Gamma|t',x,z}\left(\bar{\Gamma}\right)\right|/\left|t-t'\right|\right)\right]|t
\end{aligned}
$$

Therefore, $A_{p_\sigma}[\omega]$ satisfies the necessary restrictions.

3. A_π : Observe that $A_\pi[\omega](x,z,t) \in [0,1]$ and $A_\pi[\omega](x,z,t) = A_\pi[\omega](x,z,t')$ if $s_{xz}(t) = s_{xz}(t')$ by construction.

4. A_m : Since $\exp\left(-\int_0^t \delta(\tau;x) + \lambda\int\pi(\tau;x,Z)\,p_\sigma(\tau;x,z)\,\mathrm{d}F_{\Gamma|\tau,x,Z}\mathrm{d}\tau\right) \leq 1$ and $\underline{N} > 0$, we have that $A_m[\omega]$ is uniformly bounded by $\frac{\sup_x \gamma_x}{\underline{N}}$. Further, the derivative at t of $A_m[\omega](t)$ is equal to

$$\left(-\delta(t;x) - \lambda\int\pi(t;x,Z)\,p_\sigma(t;x,z)\,\mathrm{d}F_Z\right)A_m[\omega](t)\,.$$

This derivative is bounded in absolute value by $\left(\bar{\delta}+\lambda\right)\frac{\sup_x \gamma_x}{\underline{N}}$.

5. A_N: By construction, $A_N[\omega]$ belongs to $\left[\underline{N},\bar{N}\right]$, satisfying the necessary restrictions.

Part 3, existence of equilibria: It is straightforward to verify that Ω is convex. Lemma 1 implies that the components Ω_V, Ω_m, and Ω_{p_σ} are compact sets. Lemma 2 shows that Ω_π is compact. Assumption 1 (i), (ii), and (vi) imply that $\underline{N} > 0$ and $\bar{N}$ is finite, implying that Ω_N is compact. Therefore, Ω is compact. Lemma 3 shows that A is a continous map. Therefore, the Brouwer-Schauder-Tychonoff Theorem (Corollary 17.56, Aliprantis and Border, 2006) implies that there exists $\omega^* \in \Omega$ such that $A[\omega^*] = \omega^*$.

To complete the proof, we show that any fixed point $\omega^* = (p_\sigma^*, V^*, \pi^*, m^*, N^*)$ corresponds to a steady state equilibrium. Observe that for each x,

$$V^*(t;x) = \int_t^T \exp(-\rho(\tau - t))\, p(\tau|t;x) \left(\lambda \int \pi^*(\tau;x,Z) \int \max\{0, \Gamma - V^*(\tau;x)\}\, \mathrm{d}F_{\Gamma|\tau,x,Z} \mathrm{d}F_Z\right) \mathrm{d}\tau.$$

Therefore, $V^*(t;x)$ is the value of declining an offer and following the optimal strategy given the offer rate π^*. Therefore,

$$p_\sigma^*(x,z,t) = A_{p_\sigma}[\omega^*](x,z,t) = \int 1\{\Gamma \geq V^*(t;x)\}\, \mathrm{d}F_{\Gamma|x,z,t}.$$

For each (x,z,t), $F^{-1}_{\Gamma|x,z,t}(p_\sigma^*(x,z,t)) = V^*(t;x)$. Therefore, $\sigma^*(\Gamma,t) = 1\left\{\Gamma \geq F^{-1}_{\Gamma|x,z,t}(p_\sigma^*(x,z,t))\right\}$ is an optimal strategy, satisfying requirement 1 in Definition 1.

By construction, $\pi^*(x,z,t) = A_\pi[\omega^*](x,z,t) = H_z(s_{xz}(t); p_\sigma^*, m^*, N^*) \times \mathbb{P}(c_{ij} = 1|x,z)$ satisfies requirement 2 of Definition 1 because p_σ^* equals the acceptance probability of a type z object by an agent of type x at time t.

Finally, $m^* = A_m[\omega^*]$ and $N^* = A_N[\omega^*]$ together satisfy requirement 3 in Definition 1. The restriction of $A_N[\omega^*]$ to $\left[\underline{N}, \bar{N}\right]$ cannot strictly bind because $\underline{N}$ and $\bar{N}$ denote the smallest and largest possible queue lengths given the exogenous arrival and departure rates.

□

D.2 Lemmata

Lemma 1. *Suppose $X \subset C([a,b])$ is the set of all functions on the bounded interval $[a,b]$ that are uniformly bounded by K_1 and have a common Lipschitz constant K_2. Then X is compact.*

Proof. Note that the set of functions X is uniformly equicontinuous. By the Arzela-Ascoli theorem, any sequence of functions $x_n \in X$ has a uniformly convergent subsequence x_{n_k}. Denote the limit of this sequence by x^*, i.e. for each t, $x^*(t) = \lim_{k\to\infty} x_{n_k}(t)$. Therefore, $\sup_t |x^*(t)| \leq \lim_{k\to\infty} \sup_t |x_{n_k}(t)| \leq K_1$. Similarly, $|x^*(t) - x^*(t')| = \lim_{k\to\infty} |x_{n_k}(t) - x_{n_k}(t')| \leq K_2 |t - t'|$. Hence, $x^* \in X$. Consquently, we have that X is sequentially compact, which is equivalent to X being compact. □

Lemma 2. *Assumption 1(vii) implies that the set Ω_π consisting of functions $\pi : [0, T] \times \chi \times \zeta \to [0, 1]$ endowed with the supremum norm such that $\pi_{xz}(t) = \pi_{xz}(t')$ if $s_{xz}(t) = s_{xz}(t')$ is compact.*

Proof. Assumption 1(vii) and finiteness of χ and ζ imply that the set of scores $s_{xz}(t)$ over all χ, ζ, and $t \in [0, T]$ is finite. Therefore, π is an element of a finite dimensional Euclidean space. Further, Ω_π is closed and bounded by definition. By the Heine-Borel theorem, Ω_π is compact. □

Lemma 3. *Suppose Assumption 1 is satisfied. Then the map $A : \Omega \to \Omega$ is continuous.*

Proof. We do this for each component of A separately.

A_V: Let Ω_0 be an arbitrary subset of Ω. Consider $\omega \in \bar{\Omega}_0$, where $\bar{\Omega}_0$ is the closure of Ω_0. Since $\omega \in \bar{\Omega}_0$, there exists a sequence $\omega_n \in \Omega_0$ such that $\|\omega_n - \omega\| = \varepsilon_n \to 0$. Denote $\tilde{V}_n = A_V[\omega_n]$ and drop x from the notation as it belongs to a finite set. Now, consider

$$
\begin{aligned}
&\left|\tilde{V}_n(t) - \tilde{V}(t)\right| \\
=&\left|\int_t^T \exp(-\rho(\tau - t))\, p(\tau|t)\, \lambda \left(\int \pi_n(\tau; Z) \int \max\{0, \Gamma - V_n(\tau)\}\, \mathrm{d}F_{\Gamma|\tau,Z} \mathrm{d}F_Z\right) \mathrm{d}\tau\right. \\
&\left. - \int_t^T \exp(-\rho(\tau - t))\, p(\tau|t)\, \lambda \left(\int \pi(\tau; Z) \int \max\{0, \Gamma - V(\tau)\}\, \mathrm{d}F_{\Gamma|\tau,Z} \mathrm{d}F_Z\right) \mathrm{d}\tau\right| \\
\leq& T\lambda \sup_{t,z} \left|\pi_n(t; z) \int \max\{0, \Gamma - V_n(t)\}\, \mathrm{d}F_{\Gamma|t,z} - \pi(t; z) \int \max\{0, \Gamma - V(t)\}\, \mathrm{d}F_{\Gamma|t,z}\right| \\
\leq& T\lambda \sup_{t,z} \left|\pi_n(t; z) \int |\max\{0, \Gamma - V_n(t)\} - \max\{0, \Gamma - V(t)\}|\, \mathrm{d}F_{\Gamma|t,z}\right| \\
&+ T\lambda \sup_{t,z} \left||\pi_n(t; z) - \pi(\tau; z)| \int \max\{0, \Gamma - V(t)\}\, \mathrm{d}F_{\Gamma|t,z}\right| \\
\leq& T\lambda \sup_{t,z} |V_n(t) - V(t)| + T\lambda \sup_{t,z} \int |\Gamma|\, \mathrm{d}F_{\Gamma|t,z} \sup_{t,z} |\pi_n(t; z) - \pi(t; z)| \\
\leq& T\lambda \left(1 + \sup_{t,z} \int |\Gamma|\, \mathrm{d}F_{\Gamma|t,z}\right) \varepsilon_n.
\end{aligned}
$$

Since $\varepsilon_n \to 0$, Assumption 1(i) and (iv) imply that the right hand side converges to zero.

A_{p_σ}: Continuity follows by noting that A_V is continuous in the sup-norm and $F_{\Gamma|t,x,z}$ is absolutely continuous with respect to Lebesgue measure for each t, x, z (Assumption 1(iii)).

A_m : It is sufficient to fix x because χ is a finite set. Lemma 4 implies that the map defined by $A_\kappa[\omega](t) = \delta(t; x) + \lambda \int \pi(t; x, Z)\, p_\sigma(t; x)\, \mathrm{d}F_Z$ is continuous. Moreover, $\sup_t A_\kappa[\omega](t)$ is bounded above (Assumption 1(i)). Therefore, $A_{\kappa^*}[\omega](t) = -\int_0^t \delta(\tau;x)+\lambda\int \pi(\tau;x,Z)\,p_\sigma(t;x)\,\mathrm{d}F_Z \mathrm{d}\tau$ defines a continuous map from Ω to $C([0,T])$. Since a composition of continuous functions is continuous, and $g(a) = \gamma_x \exp(a)/N$ is continuous for all $N > 0$, A_m is continuous.

A_N: First we show that $A_N[\omega_n]$ is continuous. Lemma 4 implies that the map $A_\kappa[\omega](t) = \delta(t;x) + \lambda\int \pi(t;x,Z)\,p_{\sigma_n}(t;x,Z)\,\mathrm{d}F_Z$ is continuous for each x. A similar argument implies that $A_{\bar\kappa}[\omega] = \sum_x \int_0^T m_x(t)\,\kappa_x(t)\,\mathrm{d}t$ is continuous because $m_x(t)$ is bounded by γ_x. Further, $A_{\bar\kappa}[\omega] \in [\underline{\delta},\infty]$ since $\delta(t;x)$ is uniformly bounded below by $\underline{\delta}$ (Assumption 1(ii)). Since a composition of real-valued continuous functions is continuous, and the reciprocal function is continuous for all arguments other than 0, A_N is a continuous map.

A_π: Denote $\tilde{A}[\omega] = (A_{p_\sigma}[\omega], A_m[\omega], A_N[\omega])$. We have shown that $\tilde{A}$ is continuous and compact. Note that for any sequence ω_n,

$$\sup_{x,z,t} |A_\pi[\omega_n](x,z,t)| \le \sup_{x,z,t} \left| H_z\left(s_{xz}(t);\tilde{A}[\omega_n]\right)\right| \le \sup_{z,s}\left| H_z\left(s;\tilde{A}[\omega_n]\right)\right|,$$

where the first inequality follows from the fact that $\mathbb{P}(c_{ij}=1|x,z) \in [0,1]$ and the second inequality follows from set inclusion. Therefore, Lemma 5 and continuity of $\tilde{A}$ imply that for each z, $\sup_s \left| H_z\left(s;\tilde{A}[\omega_n]\right) - H_z\left(s;\tilde{A}[\omega]\right)\right| \to 0$ if ω_n converges to ω. Since z belongs to a finite set, we therefore have that $\sup_{x,z,t}|A_\pi[\omega_n](x,z,t) - A_\pi[\omega](x,z,t)| \to 0$. Hence, A_π is a continuous map. $\square$

Lemma 4. *Fix x. The map $A_\kappa : \Omega \to L_\infty([0,T])$, where $A_\kappa[\omega](t) = \delta(t;x)+\lambda\int \pi(t;x,Z)\,p_\sigma(\tau;x,Z)\,\mathrm{d}F_Z$ is continuous if λ is finite, and π and p_σ are uniformly bounded by* 1.

Proof. Let Ω_0 be an arbitrary subset of Ω. Consider $\omega \in \bar\Omega_0$. Since $\omega \in \bar\Omega_0$, there exists a sequence $\omega_n \in \Omega_0$ such that $\|\omega_n - \omega\| = \varepsilon_n \to 0$. Now, consider $A_\kappa[\omega_n](t) = \lambda\int \pi_n(t;x,Z)\,p_{n,\sigma}(\tau;x,Z)\,\mathrm{d}F_{\Gamma|\tau,x,Z}$.

$$\begin{aligned}
\|A_\kappa[\omega_n] - A_\kappa[\omega]\| &= \lambda\left\| \int \pi_n(t;x,Z)\,p_{n,\sigma}(t;x,Z)\,\mathrm{d}F_Z - \int \pi(t;x,Z)\,p_\sigma(t;x,Z)\,\mathrm{d}F_Z\right\| \\
&\le \lambda \sup_{z,t} |\pi_n(t;x,z)\,p_{n,\sigma}(t;x,z) - \pi(t;x,z)\,p_\sigma(t;x,z)| \\
&\le \lambda \sup_{z,t} |\pi_n(t;x,z)\,(p_{n,\sigma}(t;x,z) - p_\sigma(t;x,z))| \\
&+ \lambda \sup_{z,t} |(\pi_n(t;x,z) - \pi(t;x,z))\,p_\sigma(t;x,z)| \\
&\le \lambda \sup_{z,t} |p_{n,\sigma}(t;x,z) - p_\sigma(t;x,z)| + \lambda \sup_{z,t} |\pi_n(t;x,z) - \pi(t;x,z)| \le 2\lambda\varepsilon_n.
\end{aligned}$$

Therefore, $A_\kappa\left[\bar{\Omega}_0\right] \subset \overline{A_\kappa\left[\Omega_0\right]}$, implying that A_κ is continuous (Theorem 2.27, Aliprantis and Border, 2006). □

Lemma 5. *Fix z. The map $A_H : \Omega \to L_\infty(\mathbb{R})$ defined by $A_H[\omega](s) = H_z(s; p_\sigma, m, N)$ is continuous.*

Proof. We omit z from the notation for simplicity as it is fixed. Equation (B.5) derives the following expression for A_H:

$$A_H[\omega](t,x,z) = \int_0^1 \sum_{q'<q} \frac{e^{-Np(s,\alpha)}\left(Np(s,\alpha)\right)^{q'}}{q'!} \mathrm{d}\alpha,$$

where $p(s,\alpha)$, $p_H(s)$, and $p_E(s)$ are defined in Section B.1.2. We have $\mathbb{P}(\Gamma(t;x,z)+\varepsilon > V_x(t))$ with the acceptance probabilities $p_\sigma(t;x,z)$. Recall that $m_H(s) = \sum_{t,x} m(t;x)\, 1\{s(t;x) > s\}$ and $m_E(s) = \sum_{t,x} m(t;x)\, 1\{s(t;x) = s\}$. We prove continuity of A_H by first proving continuity of the components m_H, m_E, p_H and p_E.

Continuity of m_H and m_E: Consider a sequence m_n that converges in sup norm on x,t to m:

$$\begin{aligned}
\left|m_{n,H}(s) - m_H(s)\right| &\le \sum_x \int_0^T \left|m_n(t;x) - m(t;x)\right| 1\{s(t;x) > s\}\, \mathrm{d}t \\
&\le |\chi|\, T \sup_{x,t} \left|m_n(t;x) - m(t;x)\right|.
\end{aligned}$$

Because this bound is independent of s, $\sup_s \left|m_{n,H}(s) - m_H(s)\right|$ converges to zero. Therefore, $A_{m_H} : \Omega \to L_\infty(\mathbb{R})$ defined by $A_{m_H}[\omega](s) = m_H(s)$ is a continuous map because $A_{m_H}\left(\bar{\Omega}_0\right) = \overline{A_{m_H}(\Omega_0)}$ for any $\Omega_0 \subseteq \Omega$ (Theorem 2.27, Aliprantis and Border, 2006). An identical argument shows that $A_{m_E} : \Omega \to L_\infty(\mathbb{R})$ defined by $A_{m_E}[\omega](s) = m_E(s)$ is continuous.

Continuity of p_H and p_E: We show the argument only for p_H because the argument for p_E is identical. Consider a sequence of ω_n that converges to ω, and the map $A_{p_H} : \Omega \to L_\infty(\mathbb{R})$ defined by $A_{p_H}[\omega](s) = p_0 + (1-p_0)\dfrac{1}{m_H(s)} \sum_{t,x} m(t;x)\, 1\{s(t;x) > s\}\, p_\sigma(t;x)$. Since p_0 is fixed, we need to show continuity of the map from ω to $\dfrac{1}{m_H(s)} \sum_{t,x} m(t;x)\, 1\{s(t;x) > s\}\, p_\sigma(t;x)$.

For each s,

$$\begin{aligned}
&\left|\frac{1}{m_{n,H}(s)}\sum_{t,x} m_n(t;x)\,1\{s(t;x)>s\}\,p_{n,\sigma}(t;x,z) - \frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\,p_\sigma(t;x)\right| \\
&\leq \left|\frac{1}{m_{n,H}(s)}\sum_{t,x} m_n(t;x)\,1\{s(t;x)>s\}\,p_{n,\sigma}(t;x) - \frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\,p_{n,\sigma}(t;x)\right| \\
&+ \left|\frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\,p_{n,\sigma}(t;x) - \frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\,p_\sigma(t;x)\right| \\
&\leq \left|\frac{1}{m_{n,H}(s)}\sum_{t,x} m_n(t;x)\,1\{s(t;x)>s\} - \frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\right| |p_{n,\sigma}(t;x)| \\
&+ \left|\frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\right| |p_{n,\sigma}(t;x)-p_\sigma(t;x)| \\
\leq &\left|\frac{1}{m_{n,H}(s)}\sum_{t,x} m_n(t;x)\,1\{s(t;x)>s\} - \frac{1}{m_H(s)}\sum_{t,x} m(t;x)\,1\{s(t;x)>s\}\right| + |p_{n,\sigma}(t;x)-p_\sigma(t;x)| \\
&= |p_{n,\sigma}(t;x)-p_\sigma(t;x)|
\end{aligned}$$

The first inequality follows from the triangle inequality. The second follows from the fact that $|p_{n,\sigma}(t;x)-p_\sigma(t;x)|$ is bounded by 1 and $m_H(s) = \sum_{t,x} m(t;x)\,1\{s(t;x)>s\}$ by definition. The third follows from $m_{n,H}(s) = \sum_{t,x} m_n(t;x)\,1\{s(t;x)>s\}$ and $m_H(s) = \sum_{t,x} m(t;x)\,1\{s(t;x)>s\}$ for all s. If ω_n converges to ω, then $\sup_{t,x}|p_{n,\sigma}(t;x)-p_\sigma(t;x)|$ conveges to zero. Therefore, $\sup_s |A_{p_H}[\omega_n](s) - A_{p_H}[\omega](s)|$ converges to zero. Hence, A_{p_H} is continuous because $A_{p_H}\left(\bar{\Omega}_0\right) = \overline{A_{p_H}(\Omega_0)}$ for any $\Omega_0 \subseteq \Omega$ (Theorem 2.27, Aliprantis and Border, 2006).

Continuity of $p(s,\alpha)$: The map $A_{p^H} : \Omega \to L_\infty(\mathbb{R}\times[0,1])$ defined by $A_p[\omega](s,\alpha) = m_H(s)\,p_H(s) + m_E(s)\,\alpha p_E(s)$ is continuous because α is bounded by 1, the maps from ω to $m_H(s)$, $p_H(s)$, $m_E(s)$, $p_E(s)$ are continuous.

Continuity of A_H: The map from Ω to $\sum_{q'<q} \frac{e^{-Np(s,\alpha)}(Np(s,\alpha))^{q'}}{q'!}$ is continuous because the components are continuous. This term is bounded by 1. Therefore, $\int_0^1 \sum_{q'<q} \frac{e^{-Np(s,\alpha)}(Np(s,\alpha))^{q'}}{q'!}\mathrm{d}\alpha$ defines a continous map from Ω to the $L_\infty([0,T])$ for each x. $\square$

References

Aliprantis, Charalambos D. and Kim C. Border, *Infinite dimensional analysis : a* *hitchhiker's guide*, 3rd ed., Springer, 2006.

Gelman, Andrew, John B Carlin, Hal S Stern, and Donald B Rubin, *Bayesian data analysis*, third ed., Chapman & Hall/CRC Boca Raton, FL, USA, 2014.