Mechanism Design meets Priority Design: Redesigning the US Army's Branching Process*

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Abstract

Army cadets obtain occupations, or branches, through a centralized process. Three objectives – increasing retention, aligning talent, and enhancing trust – have guided reforms to this process since 2006. West Point's mechanism for the Class of 2020 exacerbated challenges implementing Army policy aims. We formulate these desiderata as axioms and study their implications theoretically and empirically. We show that the Army's objectives not only determine an allocation mechanism, but also a specific priority policy, a uniqueness result that integrates mechanism and priority design. These results led to a re-design of the mechanism, now adopted at both West Point and ROTC.

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1 Introduction

Each year, the US Army assigns thousands of graduating cadets from the United States Military Academy (USMA) at West Point and the Reserve Officer Training Corps (ROTC) to their first job in a military occupation, or branch, through centralized systems. The West Point and ROTC branching systems determine the branch placements for 70 percent of newly commissioned Army officers (DoD, 2020). In 2006, the US Army created a "market-based" system for branch assignments with the goal of increasing officer retention (Colarruso, Lyle, and Wardynski, 2010). The system, known as the *Branch-of-Choice* or *BRADSO program*, gives cadets heightened priority for a fraction of a branch's positions if they express a willingness to BRADSO, or extend the length of their service commitment.[1]

Since the allocation problem involves both branch assignment and length of service commitment, the Army's branching system is a natural application of the matching with contracts framework developed by Kelso and Crawford (1982) and Hatfield and Milgrom (2005). In that framework, a centralized mechanism assigns both positions and contractual terms. However, the Army's mechanism, hereafter USMA-2006, was designed while the matching with contracts model was still being developed and the original formulation in Hatfield and Milgrom (2005) did not directly apply to the Army's problem. Subsequent research by Hatfield and Kojima (2010) broadened the framework in a way that allows it to apply to the Army's problem.[2] Building on this research, Sönmez and Switzer (2013) proposed that the Army use the cumulative offer mechanism to assign cadets to branches. While this proposal had desirable theoretical properties, it required a more complex strategy space in which cadets have to rank branches and contractual terms jointly.[3] Under the USMA-2006 mechanism, cadets only rank branches and separately indicate their willingness to BRADSO for any branch. The Army considered the existing strategy space more manageable than a more complex alternative and kept the USMA-2006 mechanism in the intervening years.

In 2012, the US Army introduced Talent-Based Branching to develop a "talent market" where additional information about each cadet influences the priority a cadet receives at a branch (Colarusso, Heckel, Lyle, and Skimmyhorn, 2016). In the branch assignment process, prioritization at each branch has long been based on the order-of-merit list (OML), a composite of a cadet's academic, physical, and military performance scores. Talent-Based Branching allowed branches and cadets to better align their interests and fit for one another. Under Talent-Based Branching, branches prioritize cadets into one of three tiers: high, medium, and low. These ratings of cadets were originally a pilot initiative, but for the Class of 2020, the US Army decided to use these rat-

[1] ADSO is short for Active Duty Service Obligation. BRADSO stands for Branch of Choice Active Duty Service Obligation. BRADSO slots are 25% of total branch allocations at USMA from the Class of 2006 through 2020 and 35% for the Class of 2021, and either 50% or 60% of total branch allocations at ROTC depending on the graduating class. USMA and ROTC cadets receive branches through separate centralized branching systems.

[2] Further elaboration is provided by Echenique (2012), Schlegel (2015), and Jagadeesan (2019).

[3] In addition, Sönmez and Switzer (2013) showed that the Nash equilibrium outcome of the USMA-2006 mechanism was equivalent to the outcome of the cumulative offer mechanism if cadet preferences took a particular form, where willingness to BRADSO is secondary to rankings of branches.

ings to adjust the underlying OML-based prioritization, constructing priorities at each branch first by the tier and then by the OML within the tier.

The desire to use branching to improve talent alignment created a new objective for the Branch-of-Choice program beyond retention. Since the decision to integrate cadet ratings into the mechanism took place under an abbreviated timeline, the US Army maintained the same strategy space for the mechanism as in previous years and devised the USMA-2020 mechanism to accommodate heterogenous branch priorities. In their design, the Army created two less-than-ideal theoretical possibilities in the USMA-2020 mechanism. First, a cadet can be charged BRADSO under the USMA-2020 mechanism even if she does not need heightened priority to receive a position at that branch. While this was also possible under USMA-2006, it was nearly four times as common under USMA-2020. Second, under USMA-2020, a cadet's willingness to BRADSO for a branch can improve priorities even for base-price positions. Surveys of cadets showed that these aspects potentially undermined trust in the branching system, and led the Army to reconsider the cumulative offer mechanism, despite its more complex strategy space. At that point, the Army established a partnership with the two civilian coauthors of this paper.

This paper reports on the design of a new branching system for the Class of 2021, the *dual-price cumulative offer mechanism*, a refinement of the cumulative offer mechanism (Hatfield and Milgrom, 2005) that uses a specific choice rule for each branch that reflects the Army's objectives of retention and talent alignment. Our main formal result is that the Army's objectives, when formulated through axioms, uniquely give rise to the dual-price cumulative offer mechanism. In our setup, the cumulative offer mechanism emerges from foundational axioms in tandem with a specific choice rule, the *dual-price choice rule*, even though branches are not assumed to be endowed with choice rules in our model. In developing this result, we provide direct evidence of the relevance of these axioms in the design. To the best of our knowledge, our main result is the first joint characterization of the cumulative offer mechanism under a specific choice rule that is induced by the market designer's policy objectives.[4] Alternatively, our result is the first characterization of a refinement of the cumulative offer mechanism, where the refinement reflects the policy objectives of the central planner. Hence, we develop a model that integrates priority design with mechanism design.

While our analysis is motivated by the Army's branching process, it can be applied for any extension of a priority-based indivisible goods allocation model (such as the *school choice* model by Abdulkadiroğlu and Sönmez (2003)), where priorities of individuals can be increased with a costly action (such as paying a higher tuition) at a fraction of the units of allocated goods. It is already well-established that any such extension can be modeled as a special case of the matching with contracts model. However, the novel insight we offer is that the underlying choice rules for institutions can also be endogenously obtained in such extensions as a direct implication of natural

[4] Hirata and Kasuya (2017) and Hatfield, Kominers, and Westkamp (2021) provide characterizations of the cumulative offer mechanism for exogenously given choice rules that satisfy various technical conditions. Our main result differs from theirs in the endogeneity of the choice rule that emerges in our characterization. See Section 3.1 for an in-depth discussion of this distinction.

axioms for these extensions. These choice rules have been exogenously given in earlier literature. This distinction is the sense in which our analysis unifies mechanism design with priority design. Our model has direct applications not only for the Army's branching process, but also for the seat purchasing policies at Chinese high schools and for affirmative action with stigma, both presented in Section 3.2.

After presenting our main characterization result, we describe the evolution of the US Army's branching system. This evolution provides evidence on the importance of certain design principles that we formalize using our axioms. The Online Appendix B studies the USMA-2020 mechanism in detail and shows how this mechanism made issues related to the lack of incentive compatibility more pressing, leading the Army to abandon it. Our theoretical analysis complements the field evidence on the performance of the USMA-2020 mechanism presented in Section 4.6. Taken together, this analysis provides insight into why the Army replaced the USMA-2020 mechanism with the dual-price cumulative offer mechanism.

Finally, we provide some additional details on the adoption the dual-price cumulative offer mechanism. We first show that cadets utilized the new strategy space of the DPCO mechanism in their ranking of branch-price pairs. Second, we consider policy levers that affect the balance between talent alignment and retention in cadet assignment. The Army considered two options: increasing the number of flexible-price positions and making the price responsiveness policy more effective. Based on the tools developed in this paper, the Army decided to increase the number of flexible-price positions and to use a more effective price responsiveness policy for the Class of 2021 than it used for the Class of 2020.

The rest of this paper is organized as follows. Section 2 introduces the model and formulates Army's policy objectives as axioms. Section 3 presents the dual-price cumulative offer mechanism and our main theoretical result characterizing it as the unique mechanism that satisfies the axioms. In this section we also present two other direct applications of our characterization, which illustrate the broader scope of our theoretical contribution. Section 4 presents the evolution of the branching system in the US Army which eventually led to the design and adoption of the dual-price cumulative offer mechanism. Section 5 provides empirical evidence on how cadets utilized the richer strategy space of the dual-price cumulative offer mechanism and examines the trade-off between retention and talent alignment under the main parameters of this mechanism. Section 6 concludes the paper. All proofs, independence of the axioms in our main characterization result, and an in-depth analysis of the USMA-2020 mechanism are presented in the Online Appendix.

2 Model

A finite set of individuals I seek placement at one of a finite set of institutions B. Since our primary application is US Army branching, we refer to an individual as a **cadet** and an institution as a **branch**.[5] At any given branch $b \in B$, there are q_b identical positions. Each cadet wants at most one

[5]Section 3.2 presents other direct applications of our model outside of the US Army context.

position and can be assigned a branch under different contractual terms. Let $T = \{t^0, t^+\}$ denote the set of contractual terms or "prices." In our US Army application, the **base price** corresponds to t^0 years of mandatory service and the **increased price** corresponds to t^+ years of mandatory service through the BRADSO program.[6] For any branch $b \in B$, at most q_b^f positions can be assigned with the increased price t^+. We refer to these positions as **flexible-price** positions. Let $q_b^0 = (q_b - q_b^f)$ denote the number of **base-price** positions that can only be assigned at the price of t^0 at that branch.

2.1 Cadet Preferences and Baseline Branch Priorities

Each cadet has a strict preference relation on branch-price pairs and remaining unmatched, represented by a linear order on $(B \times T) \cup \{\emptyset\}$. We assume that at any branch $b \in B$, each cadet $i \in I$ strictly prefers a position at the base price t^0 to one at the increased price t^+. Let $\mathcal{Q}$ denote the set of linear orders on $(B \times T) \cup \{\emptyset\}$ identified by this assumption. For any $i \in I$, $\succ_i \in \mathcal{Q}$, and $b \in B$,

$$(b, t^0) \ \succ_i (b, t^+).$$

For any strict preference relation $\succ_i \in \mathcal{Q}$, let $\succeq_i$ denote the resulting weak preference relation.

Let Π denote the set of all linear orders on the set of cadets I. Each branch $b \in B$ has a strict priority order $\pi_b \in \Pi$ on the set of cadets I. We refer π_b as the **baseline priority order** at branch b. The baseline priority order represents the "baseline claims" of cadets for positions at the branch.

2.2 Price Responsiveness Policies

At any branch $b \in B$, both the baseline priority order π_b and a cadet's willingness to pay the increased price t^+ for a position affect the "overall claims" of cadets for positions at the branch. The Army must determine how much a cadet's baseline priorities should be overturned in exchange for willingness to pay the increased price. This rate of exchange is captured by the price responsiveness policy,[7] which specifies the priority advantage a cadet gains if she is willing to bear the increased price.[8]

Formally, for a given branch $b \in B$ and a baseline priority order $\pi_b \in \Pi$, a **price responsiveness policy** is a linear order ω_b^+ on $I \times T$ with the following two properties:

[6] In our US Army application, the base price corresponds to three to five years of mandatory service and the increased price corresponds to three additional years of mandatory service. USMA graduates incur a five-year service obligation upon graduation. ROTC graduates incur a three or four year service obligation upon graduation. Incurring the increased price through the BRADSO program extends the initial service obligation for USMA and ROTC cadets by three years (Army Regulation 350-100). It is rare for officers to not complete their initial service obligation, which requires them to reimburse the cost of their education to the government (Army Regulation 150-1). For West Point Graduates in the Class of 2018, this cost was $236,052 (United States Military Academy, 2019).

[7] In some of our Army related policy discussions, we also refer to the price responsiveness policy as the *BRADSO policy*.

[8] A price responsiveness policy in our model is similar to the marginal rates of substitution from price theory.

1. for any pair of cadets $i, j \in I$ and $t \in T$,

$$(i,t)\ \omega_b^+\ (j,t) \quad \Longleftrightarrow \quad i\ \pi_b\ j \quad \text{and}$$

2. for any $i \in I$,

$$(i,t^+)\ \omega_b^+\ (i,t^0).$$

A price responsiveness policy can be invoked at a branch for some or all of its positions. For the positions in which it is invoked at branch b, (i) the relative priority order of cadets with identical willingness to pay the increased price is the same as in the baseline priority order π_b and (ii) any cadet has higher claims for a position at branch b with the increased price compared to her claims for the same position at its base price. Let Ω_b^+ be the set of all linear orders on $I \times T$ which satisfy these two conditions.

How much of an advantage a price responsiveness policy grants to a cadet in securing a position at branch b due to her willingness to pay the increased price differs between distinct price responsiveness policies. Given two distinct price responsiveness policies $\omega_b^+, \nu_b^+ \in \Omega_b^+$, policy ν_b^+ **is more responsive to a price increase** than policy ω_b^+ if

$$\text{for any } i,j \in I, \qquad (i,t^+)\ \omega_b^+\ (j,t^0) \implies (i,t^+)\ \nu_b^+\ (j,t^0).$$

That is, the boost received under ν_b^+ is at least as much as the boost received under ω_b^+ for any individual when ν_b^+ is more responsive to a price increase than ω_b^+ (and strictly more for at least one individual since ω_b^+ and ν_b^+ are distinct). Therefore, policy ν_b^+ is willing to overturn more baseline priorities than policy ω_b^+.

Price Responsiveness Policies Used in Practice: Ultimate, Tiered, and Scoring-Based

We next provide three examples of price responsiveness policies, each of which is used in practice. First, given a branch $b \in B$ and a baseline priority order $\pi_b \in \Pi$, define the **ultimate price responsiveness policy** $\overline{\omega}_b^+ \in \Omega_b^+$ as one where willingness to pay the increased price overrides any differences in cadet ranking under the baseline priority order π_b at branch b. That is, for any pair of cadets $i, j \in I$,

$$(i,t^+)\ \overline{\omega}_b^+\ (j,t^0).$$

For the Classes of 2006-2019, the USMA used the ultimate price responsiveness policy. During these years, the USMA capped the positions that could be assigned at the increased price at 25 percent of total positions within each branch. At each branch $b \in B$, any cadet who was willing to pay the increased price for branch b had higher priority for the q_b^f flexible-price positions than any cadet who was not willing to pay the increased price for branch b.

To define our second example of a price responsiveness policy, given a branch $b \in B$ and a baseline priority order $\pi_b \in \Pi$, partition cadets into n tiers $I_b^1, I_b^2, \ldots, I_b^n$ so that for any two tiers

$\ell, m \in \{1, \ldots, n\}$ and pair of cadets $i, j \in I$,

$$\left.\begin{array}{l} \ell < m, \\ i \in I_b^\ell \text{ , and} \\ j \in I_b^m \end{array}\right\} \implies i \; \pi_b \; j.$$

In this partition, any cadet in tier I_b^ℓ has higher baseline priority at branch b than a cadet in tier I_b^m for $\ell < m$.

Under a **tiered price responsiveness policy** $\tau_b^+ \in \Omega_b^+$, for any tier $\ell \in \{1, \ldots, n\}$ and cadets $i, j, k \in I$,

$$\left.\begin{array}{l} i \; \pi_b \; k, \\ j \; \pi_b \; k, \text{ and} \\ i, j \in I_b^\ell \end{array}\right\} \implies \Big((k, t^+) \; \tau_b^+ \; (i, t^0) \iff (k, t^+) \; \tau_b^+ \; (j, t^0) \Big).$$

That is, given two cadets $i, j \in I$ in the same tier and a third cadet $k \in I$ with lower π_b-priority than both i and j, cadet k can gain priority over cadet i through willingness to pay the increased price if and only if cadet k can gain priority over cadet j through willingness to pay the increased price. The ultimate price responsiveness policy is a special case of the class of tiered price responsiveness policies with a single tier.

For the Classes of 2020 and 2021, the USMA used two different tiered price responsiveness policies. In both years, cadets were prioritized by each branch into one of three tiers, which we denote high, middle, and low.[9] In 2020, when a cadet expressed a willingness to pay the increased price, she had higher priority among cadets in the same tier. For example, a middle tier cadet who was willing to pay the increased price would not obtain higher priority than a high tier cadet who was unwilling to pay the increased price. Therefore, under the 2020 policy, the willingness to pay the higher price overrides any difference in cadet ranking under π_b only among cadets in the same tier.

The price responsiveness policy for the Class of 2021 granted cadets more advantage in securing a position. In 2021, when a cadet expressed a willingness to pay the increased price, she had higher priority over all other cadets if she was in the medium or high tier categories. Low tier cadets who expressed a willingness to pay, in contrast, only received higher priority among other low tier cadets. The ultimate policy is more responsive to a price increase than the 2021 policy, which is in turn more responsive to a price increase than the 2020 policy.

A third prominent example is a **scoring-based price responsiveness policy**, which is usually used when the baseline priority ranking is obtained with a scoring rule (such as a standardized exam). Under this policy, the willingness to pay the increased price increases the total score by a fixed amount, and therefore can be thought as an additional factor of the scoring rule. As we present in Section 3.2, scoring-based price responsiveness policies have been used for public high

[9]Branch rating categories are known to cadets and finalized before cadets submit their preferences for branches.

school admissions in various Chinese cities in the past, where student merit scores were increased by a fixed amount for a fraction of seats provided that they are willing to pay a higher tuition (Zhou and Wang, 2021).

2.3 Formulation through the Matching with Contracts Model

To introduce the outcome of an economy and some of the mechanisms analyzed in the paper, we use the following formulation through the *matching with contracts* model by Hatfield and Milgrom (2005).

For any $i \in I$, $b \in B$, and $t \in T$, the triple $x = (i, b, t)$ is called a **contract**. It represents a bilateral match between cadet i and branch b at the price of t. Let

$$\mathcal{X} = I \times B \times T$$

denote the set of all contracts. Given a contract $x \in \mathcal{X}$, let $\mathsf{i}(x)$ denote the cadet, $\mathsf{b}(x)$ denote the branch, and $\mathsf{t}(x)$ denote the price of the contract x. That is, $x = \big(\mathsf{i}(x), \mathsf{b}(x), \mathsf{t}(x)\big)$.

For any cadet $i \in I$, let

$$\mathcal{X}_i = \{x \in \mathcal{X} : \mathsf{i}(x) = i\}$$

denote the set of contracts that involve cadet i. Similarly, for any branch $b \in B$, let

$$\mathcal{X}_b = \{x \in \mathcal{X} : \mathsf{b}(x) = b\}$$

denote the set of contracts that involve branch b. For any cadet $i \in I$, preferences $\succ_i \in \mathcal{Q}$ defined over $B \times T \cup \{\emptyset\}$ can be redefined over $\mathcal{X}_i \cup \{\emptyset\}$ (i.e. her contracts and remaining unmatched) by simply interpreting a branch-price pair $(b, t) \in B \times T$ in the original domain as a contract between cadet i and branch b at price t in the new domain.

An **allocation** is a (possibly empty) set of contracts $X \subset \mathcal{X}$, such that

(1) for any $i \in I$, $\quad |\{x \in X : \mathsf{i}(x) = i\}| \leq 1$,
(2) for any $b \in B$, $\quad |\{x \in X : \mathsf{b}(x) = b\}| \leq q_b$, and
(3) for any $b \in B$, $\quad |\{x \in X : \mathsf{b}(x) = b \text{ and } \mathsf{t}(x) = t^+\}| \leq q_b^f$.

That is, under an allocation X, no individual can appear in more than one contract, no branch b can appear in more contracts than the number of its positions q_b, and no branch b can appear with the increased price t^+ in more contracts than the number of its flexible-price positions q_b^f. Let $\mathcal{A}$ denote the set of all allocations.

For a given allocation $X \in \mathcal{A}$ and cadet $i \in I$, the **assignment** X_i of cadet i under allocation X is defined as

$$X_i = \begin{cases} (b, t) & \text{if } (i, b, t) \in X, \\ \emptyset & \text{if } X \cap \mathcal{X}_i = \emptyset. \end{cases}$$

For the latter case, i.e. if $X_i = \emptyset$, we say that cadet i is **unmatched** under X.

For a given allocation $X \in \mathcal{A}$ and cadet $i \in I$, with a slight abuse of the notation,[10] let $\mathsf{b}(X_i)$ be defined as

$$\mathsf{b}(X_i) = \begin{cases} b & \text{if } (i,b,t) \in X, \\ \emptyset & \text{if } X \cap \mathcal{X}_i = \emptyset. \end{cases}$$

A **mechanism** is a strategy space $\mathcal{S}_i$ for each cadet $i \in I$ along with an outcome function

$$\varphi : \prod_{i \in I} \mathcal{S}_i \to \mathcal{A}$$

that selects an allocation for each strategy profile. Let $\mathcal{S} = \prod_{i \in I} \mathcal{S}_i$.

Given a mechanism $(\mathcal{S}, \varphi)$, the resulting **assignment function** $\varphi_i : \mathcal{S} \to B \times T \cup \{\emptyset\}$ for cadet $i \in I$ is defined as follows: For any $s \in \mathcal{S}$ and $X = \varphi(s)$,

$$\varphi_i(s) = X_i.$$

A **direct mechanism** is a mechanism where $\mathcal{S}_i = \mathcal{Q}$ for each cadet $i \in I$.

As is customary, we denote a direct mechanism with its outcome function only, thus suppressing its strategy space which is always $\mathcal{Q}^{|I|}$.

2.4 Desiderata for Allocations and Mechanisms

We next describe formal properties of allocations and mechanisms. Several of these properties are familiar ones related to efficiency, fairness, and incentives, but with some modifications for our setting. The most important deviation from more standard axioms involves how we formalize the role of the price responsiveness policy. We view each of these properties as compelling given the model. In Section 4, we also relate these properties to the evolution of the Army's branching system and show how they have emerged naturally from the Army's goals.

The first three axioms are standard in the literature for priority-based and single-unit demand resource allocation problems.

Definition 1. *An allocation $X \in \mathcal{A}$ satisfies **individual rationality** if, for any $i \in I$,*

$$X_i \succeq_i \emptyset.$$

*Likewise, a mechanism $(\mathcal{S}, \varphi)$ satisfies **individual rationality** if the allocation $\varphi(s)$ satisfies individual rationality for any strategy profile $s \in \mathcal{S}$.*

Individual rationality requires that a cadet never prefer remaining unmatched to her assignment.

[10] The abuse of notation is due to the fact that while the argument of the function $\mathsf{b}(.)$ is previously introduced as a contract, here it is an assignment. Since a cadet and an assignment uniquely defines a (possibly empty) contract, the notational abuse is innocuous.

Definition 2. *An allocation $X \in \mathcal{A}$ satisfies satisfies **non-wastefulness** if, for any $b \in B$ and $i \in I$,*

$$\left.\begin{array}{c} \big|\{x \in X : \mathsf{b}(x) = b\}\big| < q_b\,, \;\; \textit{and} \\ X_i = \emptyset \end{array}\right\} \implies \emptyset \succ_i (b, t^0).$$

*Likewise, a mechanism $(\mathcal{S}, \varphi)$ satisfies **non-wastefulness** if the allocation $\varphi(s)$ satisfies non-wastefulness for any strategy profile $s \in \mathcal{S}$.*

Non-wastefulness is a mild efficiency condition that requires that no position at a branch remains unfilled while there is an unassigned cadet who would rather receive the position at the branch at the base price t^0.

Definition 3. *An allocation $X \in \mathcal{A}$ **has no priority reversals** if, for any $i, j \in I$, and $b \in B$,*

$$\left.\begin{array}{c} \mathsf{b}(X_j) = b, \;\; \textit{and} \\ X_j \succ_i X_i \end{array}\right\} \implies j \;\pi_b\; i.$$

*Likewise, a mechanism $(\mathcal{S}, \varphi)$ **has no priority reversals** if the allocation $\varphi(s)$ has no priority reversals for any strategy profile $s \in \mathcal{S}$.*

This axiom captures the idea that once the price is fixed at $t \in T$, cadets with higher baseline priorities at any given branch b have higher claims for a position at branch b. Whenever cadet i strictly prefers another cadet j's assignment to her own, cadet j has higher baseline priority at her assigned branch than cadet i.[11] If cadet i strictly prefers cadet j's assignment even though cadet j has lower baseline priority than cadet i, then we say there is a **priority reversal**.

The previous axiom formulates the role of baseline priorities in allocations and mechanisms. In a similar vein, our next axiom formulates the role of price responsiveness policies in allocations and mechanisms. More specifically, it formulates when a cadet can have legitimate claims on positions awarded to other cadets but with a different price. The following two definitions are useful building blocks.

Definition 4. *Given an allocation $X \in \mathcal{A}$ and a cadet $i \in I$ with branch $\mathsf{b}(X_i) = b$ and $\mathsf{t}(X_i) = t^+$, a cadet $j \in I \setminus \{i\}$ has a **legitimate claim for a reduced price version** of cadet i's assignment X_i if*

$$\begin{array}{c} (b, t^0) \succ_j X_j \quad \textit{and} \\ (j, t^0) \;\omega_b^+\; (i, t^+). \end{array}$$

Cadet j's claim for a position at branch b at base price is legitimate because the price responsiveness policy ω_b^+ does not overturn her claim in favor of cadet i even when cadet i pays the increased price.

[11] This axiom is identical to the fairness axiom formulated in Sönmez and Switzer (2013). We introduce this new terminology to contrast a priority reversal with a detectable priority reversal, which is formally defined in the next section. While some priority reversals may not be observed under the strategy spaces of mechanisms the US Army used in the past, all detectable priority reversals can be observed under these mechanisms.

Definition 5. *Given an allocation $X \in \mathcal{A}$ and a cadet $i \in I$ with branch $\mathsf{b}(X_i) = b$ and $\mathsf{t}(X_i) = t^0$, a cadet $j \in I \setminus \{i\}$ has a* ***legitimate claim for an increased price version*** *of cadet i's assignment X_i if*

$$\begin{gathered}(b, t^+) \succ_j X_j\,, \\ (j, t^+)\ \omega_b^+\ (i, t^0)\,, \quad \textit{and} \\ \big(X \setminus \big\{(i, b, t^0)\big\}\big) \bigcup \big\{(j, b, t^+)\big\} \in \mathcal{A}.\end{gathered}$$

Cadet j's claim for a position at branch b at the increased price is legitimate even if cadet i has a higher baseline priority at branch b because

1. the price responsiveness policy ω_b^+ overturns the baseline priority in favor of cadet j as long as cadet j pays a higher price than cadet i, and

2. awarding the position originally given to cadet i instead to cadet j at increased price is feasible and does not result in exceeding the cap q_b^f for flexible-price positions at branch b.

Legitimate claims for reduced price and increased price versions of another cadet's assignment are conceptually similar, but they have one important difference. That difference involves an asymmetry related to feasibility, which is why the last condition in Definition 5 is absent in Definition 4. It is always feasible to replace an increased-price contract (i, b, t^+) of a cadet i with a base-price contract (j, b, t^0) of another cadet j since there is no cap on the number of positions that can be awarded at the base price t^0. However, it is not always feasible to replace a base-price contract (i, b, t^0) of a cadet i with the increased-price contract (j, b, t^+) of another cadet j because there is a cap q_b^f on the number of positions that can be awarded at the increased price.

The absence of either type of legitimate claim defines the role of the price responsiveness policy in our model, as we formulate next.

Definition 6. *An allocation $X \in \mathcal{A}$ satisfies* ***enforcement of the price responsiveness policy*** *if no cadet $j \in I$ has a legitimate claim for either a reduced price version or an increased price version of the assignment X_i of another cadet $i \in I \setminus \{j\}$.*

Likewise, a mechanism $(\mathcal{S}, \varphi)$ satisfies ***enforcement of the price responsiveness policy*** *if the allocation $\varphi(s)$ satisfies enforcement of the price responsiveness policy for any strategy profile $s \in \mathcal{S}$.*

Finally, our last axiom is the highly sought-after incentive compatibility property for direct mechanisms.

Definition 7. *A direct mechanism φ is* ***strategy-proof*** *if, for any $\succ \in \mathcal{Q}^{|I|}$, any $i \in I$, and any $\succ'_i \in \mathcal{Q}$,*

$$\varphi_i(\succ)\ \succeq_i\ \varphi_i(\succ_{-i}, \succ'_i).$$

This axiom assures that truthful preference revelation is always in the best interests of the cadets.

3 Dual-Price Cumulative Offer Mechanism and Its Characterization

The central mechanism in the matching with contracts literature is the *cumulative offer mechanism* (Hatfield and Milgrom, 2005), a direct mechanism which is based on a procedure that involves a sequence of cadet proposals and branch responses. Cadet proposals simulated under this procedure are based on their submitted preferences. Papers in the literature also assume that branch responses are determined through exogenously given choice rules that satisfy various technical conditions. The **dual-price cumulative offer (DPCO)** mechanism is a refinement of the cumulative offer mechanism, where the branch responses take a specific form given by the following choice rule $\mathcal{C}_b^{DP}$ for each branch $b \in B$.[12]

> **Dual-Price Choice Rule $\mathcal{C}_b^{DP}$**
>
> Given $b \in B$ and $X \in \mathcal{X}_b$, select (up to) q_b contracts with distinct cadets in two steps as follows:
>
> **Step 1. (Selection for the base-price positions)** Let X^1 be the set of all base-price contracts in X if there are no more than q_b^0 base-price contracts in X, and the set of base-price contracts in X with q_b^0 highest π_b-priority cadets otherwise. Pick contracts in X_1 for the base-price positions, and proceed to Step 2.
>
> **Step 2. (Selection for the flexible-price positions)** Construct the set of contracts Y from X by first removing the lower ω_b^+-priority contract of any cadet who has two contracts in X, and next removing all contracts of any cadet who has a contract already selected in X_1.
>
> Let X^2 be the set of all contracts in Y if there are no more than q_b^f contracts in Y, and the set of q_b^f highest ω_b^+-priority contracts in Y otherwise. Pick contracts in X_2 for the flexible-price positions, and terminate the procedure.
>
> The outcome of the dual-price choice rule is $\mathcal{C}_b^{DP}(X) = X_1 \cup X_2$.

Intuitively, the dual-price choice rule $\mathcal{C}_b^{DP}$ first allocates the base-price positions following the baseline priority order π_b , and next allocates the flexible-price positions following the price responsiveness policy ω_b^+.

Increased-price contracts are only selected in Step 2 of the choice rule $\mathcal{C}_b^{DP}$. This fact yields two straightforward comparative static results. First, for a given number of total positions, if the number of flexible-price positions increase, then the baseline priority order π_b is used for fewer positions and the price responsiveness policy ω_b^+ is used for more positions. As a result, the number of increased-price contracts selected by the choice rule $\mathcal{C}_b^{DP}$ weakly increases. Second, when a price responsiveness policy becomes more responsive to a price increase, increased-price contracts

[12]The DPCO mechanism is a generalization of the COSM mechanism proposed by Sönmez and Switzer (2013) for the case of the ultimate price responsiveness policy $\overline{\omega}_b^+$ for each branch $b \in B$, and a refinement of the cumulative offer mechanism for the matching with slot-specific priorities model by Kominers and Sönmez (2016).

receive weakly higher priorities. As a result, the number of increased-price contracts selected by the choice rule $\mathcal{C}_b^{DP}$ weakly increases. We collect these two observations in the following result.

Proposition 1. *Fix a branch $b \in B$, the total number of branch-b positions at q_b, and a set of branch-b contracts $X \subset \mathcal{X}_b$. Then,*

1. *the number of increased-price contracts selected under $\mathcal{C}_b^{DP}(X)$ weakly increases as the number of flexible-price positions q_b^f increases, and*
2. *the number of increased-price contracts selected under $\mathcal{C}_b^{DP}(X)$ weakly increases as the price responsiveness policy ω_b^+ is more responsive.*

We are ready to formally define the dual-price cumulative offer mechanism. Given a profile of baseline priority orders $(\pi_b)_{b \in B}$ and a profile of price responsiveness policies $(\omega_b^+)_{b \in B}$, let $\mathcal{C}^{DP} = \left(\mathcal{C}_b^{DP}\right)_{b \in B}$ denote the profile of dual-price choice rules defined above. The DPCO mechanism is a direct mechanism. the strategy space for each cadet $i \in I$ is $\mathcal{S}_i^{DPCO} = \mathcal{Q}$, where $\mathcal{Q}$ is the set of linear orders on $(B \times T) \cup \{\emptyset\}$. The outcome function ϕ^{DPCO} is given by the following procedure.

Dual-Price Cumulative Offer Mechanism

Fix a linear order of cadets $\pi \in \Pi$.[13] For a given profile of cadet preferences $\succ = (\succ_i)_{i \in I} \in \mathcal{Q}^{|I|}$, cadets propose their acceptable contracts to branches in a sequence of steps $\ell = 1, 2, \ldots$:

Step 1. Let $i_1 \in I$ be the highest π-ranked cadet who has an acceptable contract. Cadet $i_1 \in I$ proposes her most preferred contract $x_1 \in \mathcal{X}_{i_1}$ to branch $\mathsf{b}(x_1)$. Branch $\mathsf{b}(x_1)$ holds x_1 if $x_1 \in \mathcal{C}_{\mathsf{b}(x_1)}^{DP}\left(\{x_1\}\right)$ and rejects x_1 otherwise. Set $A_{\mathsf{b}(x_1)}^2 = \{x_1\}$ and set $A_{b'}^2 = \emptyset$ for each $b' \in B \setminus \{\mathsf{b}(x_1)\}$; these are the sets of contracts available to branches at the beginning of step 2.

Step $\boldsymbol{\ell}$. ($\boldsymbol{\ell \geq 2}$) Let $i_\ell \in I$ be the highest π-ranked cadet for whom no contract is currently held by any branch, and let $x_\ell \in \mathcal{X}_{i_\ell}$ be her most preferred acceptable contract that has not yet been rejected. Cadet i_ℓ proposes contract x_ℓ to branch $\mathsf{b}(x_\ell)$. Branch $\mathsf{b}(x_\ell)$ holds the contracts in $\mathcal{C}_{\mathsf{b}(x_\ell)}^{DP}\left(A_{\mathsf{b}(x_\ell)}^\ell \cup \{x_\ell\}\right)$ and rejects all other contracts in $A_{\mathsf{b}(x_\ell)}^\ell \cup \{x_\ell\}$. Set $A_{\mathsf{b}(x_\ell)}^{\ell+1} = A_{\mathsf{b}(x_\ell)}^\ell \cup \{x_\ell\}$ and set $A_{b'}^{\ell+1} = A_{b'}^\ell$ for each $b' \in B \setminus \{\mathsf{b}(x_\ell)\}$; these are the sets of contracts available to branches at the beginning of step $\ell + 1$.

The procedure terminates at a step when either no cadet remains with an acceptable contract that has not been rejected, or when no contract is rejected. All the contracts on hold in the final step are finalized as the outcome $\phi^{DPCO}(\succ)$ of the DPCO mechanism.

[13]By Kominers and Sönmez (2016), the outcome is independent of this linear order.

Our main theoretical result shows that the DPCO mechanism is the only mechanism that satisfies all our desiderata.

Theorem 1. *Fix a profile of baseline priority orders* $(\pi_b)_{b \in B} \in \Pi^{|B|}$ *and a profile of price responsiveness policies* $\left(\omega_b^+\right)_{b \in B} \in \prod_{b \in B} \Omega_b^+$. *A direct mechanism* φ

1. *satisfies individual rationality,*

2. *satisfies non-wastefulness,*

3. *has no priority reversals,*

4. *satisfies enforcement of the price responsiveness policy, and*

5. *is strategy-proof*

if and only if mechanism φ *is the DPCO mechanism* ϕ^{DPCO}.

Apart from singling out the DPCO mechanism as the unique mechanism that satisfies our desiderata, to the best of our knowledge Theorem 1 is the first joint characterization of an allocation mechanism (i.e. the cumulative offer mechanism) together with a specific choice rule (i.e. the dual-price choice rule) endogenous to the policy objectives of the central planner.

3.1 Significance of the DPCO Mechanism Characterization

Hirata and Kasuya (2017) and Hatfield, Kominers, and Westkamp (2021) present earlier axiomatic characterizations of the cumulative offer mechanism. Our characterization differs from theirs in one fundamental aspect. Starting from a given choice rule for each institution is a near universal assumption in the matching with contracts literature. In Hirata and Kasuya (2017) and Hatfield, Kominers, and Westkamp (2021) each institution is endowed with an exogenously given choice rule that satisfies various technical conditions. In our characterization, in contrast, the dual-price choice rule–one of the two pillars of the DPCO mechanism– emerges endogenously from the Army's policy objectives formulated by our desiderata. Indeed, the very concept of a choice rule is only used in our model to describe the DPCO mechanism. Our axioms do not place any structure or assume any functional form of potential branch choice rules. In fact, we do not even assume the existence of a well-defined choice rule that dictates behavior for any given branch. Instead, the dual-price choice rule in tandem with the cumulative offer mechanism is a *collective* implication of our five axioms.

It is also important to emphasize that all our axioms reflect the Army's policy objectives. None are imposed as technical conditions for the sake of obtaining an axiomatic characterization. The Army initially asked for advice on cadet branch assignment from the two civilian co-authors of this paper having observed issues with their prior mechanisms. As presented later in detail in Section 4, the Army's adjustments of its branching mechanisms to implement its price responsiveness policies (i.e. the BRADSO policies) since 2006 resulted in priority reversals along with incentive compatibility failures. Through our collaboration, we discovered that the Army's goals for cadet branch assignment could be formulated as axioms. In particular, previous efforts to accommodate the *enforcement of the price responsiveness policy* axiom resulted in an unintended failure of two other key axioms.

What is rather unusual in this literature is that our main result shows that the DPCO mechanism is the unique design that accommodates our axiomatic formulation of the Army's policy objectives. The axiom of *enforcement of the price responsiveness policy*–novel to our paper–does much of the "heavy lifting" in this uniqueness result. However, this axiom is not solely responsible for our uniqueness result. Using the terminology in the literature on axiomatic resource allocation,[14] all five axioms are *independent* in our characterization. That is, if any of the five axioms are dropped, then at least one mechanism other than the DPCO exists and satisfies the remaining four axioms.[15] From a technical perspective, this means that none of the axioms are redundant in the analysis. From a practical perspective, this means that mechanisms other than the DPCO mechanism satisfy any strict subset of the axioms, but only one mechanism, the DPCO mechanism, satisfies all five.

3.2 Applications Beyond the US Army's Branching System

The individual-proposing deferred acceptance (DA) algorithm of Gale and Shapley (1962) plays a prominent role in market design applications, in particular for priority-based resource allocation.[16] Our model is perhaps one of the most natural extensions for settings where priorities of individuals can be increased with a costly action for a subset of positions at each institution. Based on Theorem 1, we believe that the DPCO mechanism is a natural counterpart of DA for environments with this structure. Therefore, while our paper is mainly motivated by the Army's 2020 branching reform, our model in Section 2 and characterization in Theorem 1 have other direct applications.[17]

High School Seat Purchasing Policies in China. Zhou and Wang (2021) present an extension of the school choice model of Abdulkadiroğlu and Sönmez (2003) in which students gain priority for some seats at each school by paying a higher tuition through a policy they refer to as the *Ze Xiao (ZX) policy*. According to Zhou and Wang (2021), this policy was implemented in many Chinese cities from the 1990s until 2015. In many Chinese cities, a student's priority ranking at a public high school primarily depends on exam scores. Cities with the ZX policy used the scoring-based price responsiveness policy introduced in Section 2.2. For a fraction of the seats at each school, students received a certain boost to their scores in exchange for paying a higher tuition. Shanghai and Tianjin used a single level of increased tuition for the *ZX positions*, making high school admissions an exact application of our model.[18]

[14]See, for example, Moulin (1988) and Thomson (2001).

[15]See Section A.2 of the Online Appendix for examples of such mechanisms.

[16]Online Appendix Section C.2 defines the DA algorithm.

[17]The Online Appendix describes other possible applications of price-responsiveness policies for priority-based assignment markets.

[18]Zhou and Wang (2021) present the following details for Shanghai and Tianjin: "Shanghai is one of the cities that

Affirmative Action with Stigma. Aygün and Turhan (2020) consider a model where a fraction of each school's seats are reserved for disadvantaged groups of students. A student who receives these *affirmative action (AA) seats* incur a cost due to stigma. As a result, at any given school these students prefer receiving an unreserved seat to an AA seat. Their model is a special case of ours when there is a single affirmative action eligible group. Receiving a position with increased price in our model corresponds to receiving an AA seat in their model, and the price responsiveness policy that maps into their framework is the ultimate price responsiveness policy introduced in Section 2.2. Our reinterpretation of Aygün and Turhan (2020) also has some practical implications in applications where their model is relevant.

Aygün and Turhan (2020) consider a model where it is public information whether an individual receives an AA seat or a regular unreserved seat. This is the reason receiving a seat via affirmative action becomes costly in their model. To reinterpreted in our framework, label the AA seats in their model as *AA-eligible seats*. These seats could either be assigned as AA seats with the stigma cost or as regular seats without the stigma cost. When the ultimate price responsiveness policy is used, this reinterpretation allows the system to give exactly the same level of affirmative action protection to the disadvantaged group, but some of the recipients of the AA-eligible seats avoid the stigma cost by receiving them as regular seats. The individuals who benefit from this reinterpretation are the members of the protected group who are do not need AA to receive a seat. This will be reflected in practical implementation as "converting" some of the AA-eligible seats to regular seats to save the stigma cost to some of the AA-eligible individuals, while providing the disadvantaged group with exactly the same AA protection level. It's as if we've reduced the number of AA-protected seats, even though we've granted the same level of AA protections. Aygün and Turhan (2020) also promote a similar concept through a class of choice rules they refer to as "dynamic reserves." With our interpretation the dual-price choice rule emerges as the single focal element of the wider class of choice rules they study.

4 The Evolution of the Army's Branching System

In this section, we describe some features of the evolution of the Army's branching system. In 2006, the system underwent a major change to improve retention. In 2020, it underwent a second major change to improve talent alignment. In recounting this evolution, we emphasize the importance of the principles formalized by our axioms. We then use administrative preference data and survey results from West Point cadets to describe how cadets used the richer strategy space of the

discontinued the ZX policy immediately after the announcement from the Ministry of Education in 2012. The total percentage of ZX students was restricted within 15% for each school in 2011, which is the percentage for ZX policy in the previous year. The ZX tuition in Shanghai was charged according to the type of school. In district-level key high schools, the basic tuition for students was 2,400 Yuan/year, whereas the ZX tuition was 6,000 Yuan/year before 2011 and 4,266 Yuan/year in 2011. For the city-level key high schools, the basic tuition was 3,000 Yuan/year, whereas the ZX tuition was 10,000 Yuan/year before 2011 and 7,000 Yuan/year in 2011. For the boarding schools, the basic tuition was 4,000 Yuan/year, whereas the ZX tuition was 13,333 Yuan/year before 2011 and 9,333 Yuan/year in 2011.

[· · ·] Tianjin cancelled its ZX policy in 2015. Before 2015, the ZX tuition was standardized across all general high schools at 8,000 Yuan/year, which was a fourfold increase in the basic tuition (2,000 Yuan/year)."

DPCO mechanism and the trade-off between talent alignment and retention.

4.1 BRADSO Program for Improved Retention

Prior to the Class of 2006, USMA cadets were assigned to Army branches using a *serial dictatorship* induced by a cadet performance ranking known as the *order of merit list (OML)*. Cadets submitted their preferences over the set of branches. The highest-OML cadet was assigned her most-preferred branch, the second highest-OML cadet was assigned her most-preferred branch among branches with remaining positions, and so on. Let us refer to this mechanism as φ^{OML}.

Declining junior officer retention rates during the late 1990s and early 2000s led to the Army to offer a menu of retention incentives to cadets at USMA and ROTC through the *Officer Career Satisfaction Program*, first implemented in 2006 (Colarruso, Lyle, and Wardynski, 2010). The most popular incentive, which involved a reform of the branching mechanism, was the *branch of choice*, or *BRADSO* program. As described above, the BRADSO program gives higher priority for a fraction of positions in each branch $b \in B$ to cadets who are willing to extend their *Active Duty Service Obligation (ADSO)* by three years if assigned to branch b. To infer which cadets were willing to serve the additional three years for any given branch, USMA required cadets to report the set of branches they were willing to serve the additional years (i.e. pay the increased price) through a new strategy space under a new mechanism, the USMA-2006 mechanism. Let $\mathcal{P}$ be the set of linear orders on $B \cup \{\emptyset\}$. Each cadet's strategy space under the USMA-2006 mechanism is $\mathcal{P} \times 2^B$.

The USMA-2006 mechanism is not a direct mechanism. Cadets only submit their preferences over branches alone and "signal" their willingness to pay the increased price at any branch rather than over branch-price pairs. The structure of this strategy space has two important implications for the axioms we introduce in Section 2.4. First, strategy-proofness is only defined for direct mechanisms and hence is not well-defined for the USMA-2006 mechanism. Second, while the remaining four axioms are all well-defined for any mechanism regardless of their strategy spaces, each depends on cadet preferences over branch-price pairs. A direct mechanism solicits this information from cadets, making it is straightforward to verify these axioms under the submitted preferences. If the direct mechanism is strategy-proof, the central planner has a formal basis to assume that the submitted preferences are truthful. In contrast, since the USMA-2006 mechanism is not direct, verification of these axioms presents a challenge. This distinction, at least partially, contributed to the Army's decision to maintain the USMA-2006 mechanism for over a decade.

4.2 Quasi-Direct Mechanisms and their Desiderata

Before formally introducing the USMA-2006 mechanism, we introduce notation to describe its strategy space and formulate axioms that are both well-defined and possible to verify under its strategy space.

A **quasi-direct mechanism** is a mechanism where the strategy space is $\mathcal{S}_i = \mathcal{P} \times 2^B$ for each cadet $i \in I$. For any strategy $s_i = (P_i, B_i) \in \mathcal{S}_i$ of cadet $i \in I$, the first component $P_i \in \mathcal{P}$ of the strategy is the cadet's preference ranking over branches (when they are awarded at the base price t^0) and remaining unmatched. The second component, $B_i \in 2^B$, is the set of branches the cadet is willing to pay the increased price t^+.

We formulate three axioms for quasi-direct mechanism, beginning with two forms of incentive-compatibility axioms. The first of these axioms states that the increased price should only be charged to cadets for whom the price responsiveness policy has been pivotal in securing a branch.

Definition 8. *A quasi-direct mechanism φ satisfies **BRADSO-incentive compatibility** (or **BRADSO-IC**) if, for any $s = \big(P_j, B_j\big)_{j\in I} \in (\mathcal{P} \times 2^B)^{|I|}$, $i \in I$, and $b \in B$,*

$$\varphi_i(s) = (b, t^+) \implies \varphi_i\big((P_i, B_i \setminus \{b\}),\ s_{-i}\big) \neq (b, t^0).$$

That is, any cadet i who receives an increased-price position at branch b under φ should not be able to profit by receiving a position at the same branch at the base price by dropping branch b from the set of branches B_i for which she's willing to pay the increased price.

Our next incentive-compatibility axiom formulates the idea that a cadet's willingness to pay the increased price at a branch should never solely enable an assignment in this branch at the base price.

Definition 9. *A quasi-direct mechanism φ is **immune to strategic BRADSO** if, for any $s = \big(P_j, B_j\big)_{j\in I} \in (\mathcal{P} \times 2^B)^{|I|}$, $i \in I$, and $b \in B$,*

$$\varphi_i(s) = (b, t^0) \implies \varphi_i\big((P_i, B_i \setminus \{b\}),\ s_{-i}\big) = (b, t^0).$$

That is, any cadet i who receives a base-price position at branch b under φ should still do so upon dropping branch b from the set of branches B_i for which she has indicated willingness to pay the increased price (in case $b \in B_i$).[19] If this axiom fails, cadet i could strategically indicate a willingness to pay the increased price at branch b and receive a base-price position (otherwise unattainable) at this branch.

Our last axiom relaxes the lack of priority reversals formulated in Definition 3 by removing dependence on cadet preference information not solicited by the mechanism.

Definition 10. *A quasi-direct mechanism φ has **no detectable priority reversals** if, for any $s = \big(P_j, B_j\big)_{j\in I} \in (\mathcal{P} \times 2^B)^{|I|}$, $b \in B$, and $i, j \in I$,*

$$\left.\begin{array}{c} \varphi_j(s) = (b, t^0), \text{ and} \\ \varphi_i(s) = (b, t^+) \quad \text{or} \quad b \; P_i \; \mathsf{b}\big(\varphi_i(s)\big) \end{array}\right\} \implies j \; \pi_b \; i.$$

This condition requires that if a cadet j is assigned a base-price position at branch b and another cadet i receives a less desired assignment by

[19] This statement holds vacuously if $b \notin B_i$.

(i) either receiving an increased-price position at the same branch or

(ii) by receiving a position at a strictly less preferred (and possibly empty) branch based on cadet i's submitted preferences P_i on $B \cup \{\emptyset\}$,

then cadet j must have higher baseline priority under branch b than cadet i.

The distinction between having no priority reversals and its weaker version, having no detectable priority reversals, is subtle. When a mechanism has priority reversals, there is a cadet i who strictly prefers the assignment of another cadet $j \in I \setminus \{i\}$ despite having higher claims for this position. Verification of this anomaly may require knowing the preferences $\succ_i \in \mathcal{Q}$ of cadet i over branch-price pairs, a challenge if the mechanism is not direct. In contrast, when a quasi-direct mechanism has detectable priority reversals, there is a cadet i who strictly prefers the assignment of another cadet $j \in I \setminus \{i\}$ no matter what cadet i's preferences $\succ_i \in \mathcal{Q}$ over branch-price pairs are provided that they are consistent with her submitted preferences $P_i \in \mathcal{P}$ over branches alone. In that sense, detectable priority reversals can be verified under a quasi-direct mechanism.

4.3 The USMA-2006 Mechanism and its Shortcomings

Sönmez and Switzer (2013) define and analyze the USMA-2006 mechanism. The USMA-2006 mechanism is a quasi-direct mechanism that extends the serial dictatorship φ^{OML} to accommodate the treatment of the flexible-price positions. When a cadet expresses willingness to pay increased price for the flexible-price positions at any branch b, the mechanism uses the ultimate price responsiveness policy $\overline{\omega}_b^+$ for allocation of these positions. If a cadet is assigned one of the q_b^0 base-price positions at branch b, she is charged the base price. If a cadet is assigned one of the q_b^f flexible-price positions, she is charged the increased price if she indicated willingness to pay the increased price at the branch and otherwise she is charged the base price. Appendix C.1 provides a formal definition of the mechanism.

Sönmez and Switzer (2013) describe shortcomings of USMA-2006, which are largely due to its non-direct strategy space. The USMA-2006 mechanism fails BRADSO-IC and has priority reversals even at its Nash equilibrium outcomes. As a remedy, Sönmez and Switzer (2013) propose using a direct mechanism where cadets rank branch-price pairs. Their proposal, known as the cadet-optimal stable mechanism, is a special case of the DPCO mechanism, with the ultimate price responsiveness policy.

The Army did not embrace this proposal for three main reasons:

1. While the USMA-2006 mechanism allows for BRADSO-IC failures and detectable priority reversals, these issues have been relatively rare in practice. Figure 1 uses administrative branching data from the USMA Classes of 2014-2019 to show that each year on average 22 cadets have been affected by BRADSO-IC failures and 20 cadets have been affected by detectable priority reversals under the USMA-2006 mechanism.[20]

[20]Our data contain information on cadet rankings, branch priorities, and capacities, covering the USMA Classes of

2. Any potential BRADSO-IC failure or detectable priority reversal can be manually corrected ex-post, since each only involves a cadet needlessly paying the increased price at her assigned branch. An ex-post manual reduction of the charged price to the base price t^0 resolves the issue.

3. Even though the USMA-2006 mechanism allows for additional priority reversals which may alter a cadet's branch assignment and consequently cannot be manually corrected ex-post, the verification of any such theoretical failure relies on cadet preferences over branch-price pairs. Since USMA-2006 is a quasi-direct mechanism, information on cadet preferences over branch-price pairs is not available.

In large part for these reasons, the Army maintained the USMA-2006 mechanism for fourteen years, even after becoming aware of Sönmez and Switzer (2013). From the Army's perspective, the costs of adopting a different strategy space were too high to justify the change. Any possible failure of the properties above could either be manually corrected ex-post or could not be verified with data solicited under the strategy space for the USMA-2006 mechanism. At this point, the introduction of a new program aimed at improved talent assignment triggered an adjustment in the mechanism, which we describe next.

4.4 Talent-Based Branching and the USMA-2020 Mechanism

The Army began piloting a new Talent-Based Branching (TBB) program in 2012 with the aim of matching cadets to branches which better fit their talents (Colarusso, Heckel, Lyle, and Skimmyhorn, 2016). A substantial component of TBB is an opportunity for branches to interview and rate cadets into three tiers. Prior to the Class of 2020, these rating categories did not influence baseline branch priorities at USMA. Ratings could only indirectly influence branch assignments either by causing some cadets to adjust their preferences for branches or by convincing the Army to make a rare, ex-post adjustment to a cadet's branch assignment after executing the USMA-2006 mechanism.

In 2019, the Army decided to incorporate branch rating categories into baseline branch priorities beginning with the USMA Class of 2020. Just as the introduction of the BRADSO program triggered a reform in the branching mechanism, the full integration of the TBB program with the branching process resulted in another adjustment. The Army replaced the USMA-2006 mechanism with another quasi-direct mechanism based on the individual-proposing deferred acceptance algorithm, where branches have heterogeneous baseline priorities over cadets according to a tiered price responsiveness policy described in Section 2.2.

2014 through 2021. In a typical year, roughly 1,000 cadets apply to about 17 or 18 different branches. From 2014-2019, an average of 231 cadets per year indicated a willingness to serve the increased price for exactly one branch, 71 cadets per year indicated a willingness to serve the increased price for exactly two branches, and 27 cadets per year indicated a willingness to serve the increased price for three or more branches. We successfully replicated the branch assignment for 99.2% of cadets in the Classes of 2014 through 2021. This and other details on the data are in Appendix Table 1 and Appendix D.1.

Under both the USMA-2006 and USMA-2020 mechanisms, the Army continued to limit the number of flexible-price positions at 25 percent of the total number of positions for each branch. However, under the USMA-2020 mechanism, the Army used the adjusted priority ranking of cadets (mainly intended for the flexible-price positions) also for the base-price positions. This change transformed branch assignment into a standard priority-based assignment problem, which in turn made it possible for the Army to use the *individual-proposing deferred acceptance algorithm* to determine the branch assignments. After branch assignment were determined, a cadet's willingness to pay increased price determined price charges. The Army charged the increased price to willing cadets in reverse-priority order, stopping when 25 percent of cadets assigned to the branch had been charged the increased-price. For example, if 100 cadets were assigned to a branch and 50 of the cadets volunteered for the increased price, the Army would charge the increased price to the 25 lowest priority cadets of the 50 willing to pay the increased price.

We next formally define the USMA-2020 mechanism. As in the case of USMA-2006 mechanism, the USMA-2020 mechanism is also a quasi-direct mechanism with strategy space $\mathcal{S}_i^{2020} = \mathcal{P} \times 2^B$ for each cadet $i \in I$. Given a strategy profile $s = (P_i, B_i)_{i \in I}$, for any branch $b \in B$, construct the following adjusted priority order $\pi_b^+ \in \Pi$ on the set of cadets I. For any $i, j \in I$,

1. $b \in B_i$ and $b \in B_j \quad \Longrightarrow \quad i \ \pi_b^+ \ j \iff i \ \pi_b \ j$,
2. $b \notin B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \ \pi_b^+ \ j \iff i \ \pi_b \ j$, and
3. $b \in B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \ \pi_b^+ \ j \iff (i, t^+) \ \omega_b^+ \ (j, t^0)$.

Under the priority order π_b^+, any two cadets are rank ordered using the baseline priority order π_b if they have indicated the same willingness to pay the increased price for branch b, and using the price responsiveness policy ω_b^+ otherwise.[21]

For any strategy profile $s = (P_i, B_i)_{i \in I}$, let μ be the outcome of the *individual-proposing deferred acceptance algorithm* for submitted cadet preferences $(P_i)_{i \in I}$ and constructed branch priorities $(\pi_b^+)_{b \in B}$.

For any strategy profile $s = (P_i, B_i)_{i \in I}$, the outcome $\varphi^{2020}(s)$ of the **USMA-2020 mechanism** is given as follows: For any cadet $i \in I$,

$$\varphi_i^{2020}(s) = \begin{cases} \emptyset & \text{if } \mu(i) = \emptyset, \\ \big(\mu(i), t^0\big) & \text{if } \mu(i) \notin B_i \text{ or } \big|\{j \in I : \mu(j) = \mu(i),\ \mu(j) \in B_j, \text{ and } i \ \pi_{\mu(i)} \ j\}\big| \geq q^f_{\mu(i)}, \\ \big(\mu(i), t^+\big) & \text{if } \mu(i) \in B_i \text{ and } \big|\{j \in I : \mu(j) = \mu(i),\ \mu(j) \in B_j, \text{ and } i \ \pi_{\mu(i)} \ j\}\big| < q^f_{\mu(i)}. \end{cases}$$

Under the USMA-2020 mechanism, each cadet $i \in I$ is asked to submit a preference relation $P_i \in \mathcal{P}$ along with a (possibly empty) set of branches $B_i \in 2^B$ for which she indicates her willing to pay the increased price t^+ to receive preferential admission. A priority order π_b^+ of cadets is

[21] When (i) the baseline priority order π_b is fixed as OML at each branch $b \in B$, and (ii) the price responsiveness policy ω_b^+ is fixed as the ultimate price responsiveness policy $\overline{\omega}_b^+$ at each branch $b \in B$, this construction gives the same adjusted priority order constructed for the USMA-2006 mechanism.

constructed for each branch b by adjusting the baseline priority order π_b using the price responsiveness policy ω_b^+ whenever a pair of cadets submitted different willingness to pay the increased price t^+ at branch b. Cadets' branch assignments are determined by the individual-proposing deferred acceptance algorithm using the submitted profile of cadet preferences $(P_i)_{i \in I}$ and the profile of adjusted priority rankings $(\pi_b^+)_{b \in B}$. A cadet pays the base price for her branch assignment if either she has not declared willingness to pay the increased price for her assigned branch or the capacity for the flexible-price positions of the branch is already filled with cadets who have lower baseline priorities. With the exception of those who remain unmatched, all other cadets pay the increased price for their branch assignments.

4.5 Shortcomings of the USMA-2020 Mechanism

We provide a detailed formal analysis of the USMA-2020 in Appendix B. The analysis illustrates the perverse incentives created by determining who is charged the increased price following the assignment stage. Example 2 in Appendix B.2 shows that the USMA-2020 mechanism fails BRADSO-IC and and admits strategic BRADSO even at equilibrium. That example also illustrates the "knife-edge" aspects of equilibrium strategies in this mechanism. Even though a minor change in the underlying economy involving a preference change in the lowest baseline priority cadet only affects the assignment of the cadet, it affects the equilibrium strategies of several higher priority cadets.[22] The fragility of our equilibrium strategies provides us intuition on the prevalence of these phenomena under the USMA-2020 mechanism, which we describe next.

After announcing the mechanism to cadets, USMA leadership recognized the possibility of detectable priority reversals under the USMA-2020 mechanism due to either failure of BRADSO-IC or presence of strategic BRADSO. In a typical year, the number of cadets willing to pay the increased price for traditionally oversubscribed branches like Military Intelligence greatly exceeds 25 percent of the branch's positions. Therefore, by volunteering to pay the increased price for an oversubscribed branch, some cadets could receive a priority upgrade even though they may not be charged for it, making detectable priority reversals possible. Moreover, unlike the detectable priority reversals under the USMA-2006 mechanism, some of these detectable priority reversals can affect cadet branch assignments, thereby making manual ex-post adjustments infeasible.

Failures of BRADSO-IC, the possibility of strategic BRADSO, or the presence of detectable priority reversals, especially when not manually corrected ex-post, could erode cadets' trust in the Army's branching process. Consider, for example, a comment from a cadet survey administered to the Class of 2020:[23]

"I believe this system fundamentally does not trust cadets to make the best choice for

[22] Example 3 in Appendix B.2 further shows that the USMA-2020 mechanism can admit detectable priority reversals even under its Bayesian Nash equilibrium outcomes.

[23] The survey was administered to the Class of 2020 immediately before they submitted their preferences for branches under the USMA-2020 mechanism. The response rate to this survey was 98%. Appendix D.2 contains specific questions and results.

themselves. It makes it so that we cannot choose what we want and have to play games to avoid force branching."

A mechanism that erodes trust is unlikely to persist in the US Army, where trust is seen as the foundation of their talent management strategy.[24] Perhaps unsurprisingly, when considering potential reforms to the USMA-2020 mechanism, the manager of the Talent-Based Branching program stated the the Army prefers a mechanism that incentivizes honest preference submissions.[25]

To address these concerns, USMA leadership executed a simulation using preliminary cadet preferences to inform cadets of the potential cutoffs for each branch.[26] The goal of this simulation was to help cadets to optimize their submitted strategies (O'Connor, 2019):

> "*We're going to tell all the cadets, we're going to show all of them, here's when the branch would have went out, here's the bucket you're in, here's the branch you would have received if this were for real. You have six days to go ahead and redo your preferences and look at if you want to BRADSO or not." Sunsdahl said. "I think it's good to be transparent. I just don't know what 21-year-olds will do with that information*."

Several open-ended survey comments from USMA cadets in the Class of 2020 mirrored USMA leadership's concern about the USMA-2020 mechanism We present three additional comments articulating concerns related to the lack of BRADSO-IC, the presence of strategic BRADSO, and the difficulty of navigating a system with both shortcomings:

1) "*Volunteering for BRADSO should only move you ahead of others if you are actually charged for BRADSO. By doing this, each branch will receive the most qualified people. Otherwise people who are lower in class rank will receive a branch over people that have a higher class rank which does not benefit the branch. Although those who BRADSO may be willing to serve longer, if they aren't charged then they can still leave after their 5 year commitment so it makes more sense to take the cadets with a higher OML.*"

2) "*I think it is still a little hard to comprehend how the branching process works. For example, I do not know if I put a BRADSO for my preferred branch that happens to be very competitive, am I at a significantly lower chance of getting my second preferred if it happens to be something like engineers? Do I have to BRADSO now if I want engineers??? Am I screwing myself over by going for this competitive branch now that every one is going to try to beat the system????*"

[24] For example, in *The Army Profession*, the US Army's Training and Doctrine Command identifies trust as an essential characteristic that defines the Army as a profession (United States Army, 2019b). The Army's People Strategy describes one of the Army's strategic outcomes as building a professional Army that retains the trust and confidence of the American people and its members (United States Army, 2019a).

[25] Lieutenant Colonel Riley Post, the Talent-Based Branching Program Manager, said "cadets should be honest when submitting preferences for branches, instead of gaming the system" in a statement in West Point's official newspaper (Garcia, 2020).

[26] Cadets in the Class of 2020 submitted preliminary preferences one month before submitting final preferences. USMA ran the USMA-2020 mechanism on these preliminary preferences to derive results for the simulation, which USMA provided to cadets 6 days prior to the deadline for submitting final preferences.

3) *"Releasing the simulation just created chaos and panicked cadets into adding a BRADSO who otherwise wouldn't have."*

4.6 Failure of Desiderata under the USMA-2006 and USMA-2020 Mechanisms

In this section, we report on the failure of BRADSO-IC, presence of strategic BRADSO, and presence of detectable priority reversals under the USMA-2006 and USMA-2020 mechanisms. We show that the USMA-2020 mechanism exacerbated challenges compared to the USMA-2006 mechanism. We use actual data submitted under these mechanisms and also simulated data generated from the DPCO mechanism for the USMA Class of 2021.

4.6.1 USMA-2006 and USMA-2020 Mechanisms in the Field

BRADSO-IC failures were much more common under USMA-2020 than under USMA-2006. Figure 1 shows that nearly four times (85 versus 22) as many cadets from the Class of 2020 (which used the USMA-2020 mechanism) were part of BRADSO-ICs than were cadets from the Classes of 2014 to 2019 (which used the USMA-2006 mechanism). Strategic BRADSOs must be more common under USMA-2020 because they are not possible under USMA-2006. For the Class of 2020, 18 cadets were part of strategic BRADSOs under the USMA-2020 mechanism. Importantly, fixing these instances ex-post would have required a change in branch assignments (rather than merely foregoing a BRADSO charge). Finally, nearly four times as many cadets were part of detectable priority reversals under the USMA-2020 mechanism than under the USMA-2006 mechanism (75 versus 20).

4.6.2 USMA-2006 and USMA-2020 Mechanisms with Simulated Data

Our comparison of prior mechanisms has so far been based on preferences submitted under those mechanisms. We can also use cadet preference data on branch-price pairs generated by the strategy-proof DPCO mechanism to simulate the outcome of USMA-2006 and USMA-2020 mechanisms under truthful strategies. This is valuable because for cadet preferences submitted under the USMA-2006 and USMA-2020 mechanisms, we could only measure detectable priority reversals (reported in Figure 1) and not all priority reversals.

To measure all priority reversals, we use preferences over branch-price pairs under the DPCO mechanism to construct a truthful strategy, denoted $s_i = (P_i, B_i)$, under a quasi-direct mechanism by using the branch rank ordering for P_i and assuming that if a cadet ever expresses a willingness to pay the increased price at a branch, then the cadet is willing to pay the increased price under B_i. Taking this constructed strategy as input, we then simulate the USMA-2006 and USMA-2020 mechanism using the branch capacities and priorities from the Class of 2021. Under the USMA-2006 mechanism simulation, there are 29 priority reversals and 20 are detectable priority reversals. Under the USMA-2020 mechanism simulation, there are 204 priority reversals and 197 are detectable priority reversals. This suggests that, in practice, detectable priority reversals likely constitute the majority of priority reversals among the Classes of 2014-2019, which used the USMA-2006 mechanism, and the Class of 2020, which used the USMA-2020 mechanism.

Using truthful strategies to evaluate the USMA-2006 and USMA-2020 mechanism, Figure 2 shows that there are nearly seven times as many BRADSO-IC failures under the USMA-2020 mechanism compared to the USMA-2006 mechanism (146 vs. 21) and seven times as many priority reversals under the USMA-2020 mechanism compared to the USMA-2006 mechanism (204 vs. 29). This pattern of behavior suggests that the comparison reported in Figure 1 potentially understates the dramatic increase in BRADSO-IC failures and priority reversals stemming from the adoption of the USMA-2020 mechanism because the Figure 1 comparison is based on strategies submitted under the strategy space of quasi-direct mechanisms and not underlying cadet preferences.

One reason the comparison between USMA-2006 and USMA-2020 in Figure 1 is not as striking as the comparison in Figure 2 is that, as we have presented in Section 4.5, many cadets were well-aware of the necessity to strategically make their increased price willingness choices under the USMA-2020 mechanism. Our analysis in Appendix B illustrates the perverse incentives in the USMA-2020 mechanism. For the Class of 2020, a dry-run of the mechanism where cadets submitted indicative rankings of branches and learned about their assignment took place. After observing their dry-run assignment, cadets were allowed to submit a final set of rankings under USMA-2020, and therefore had the opportunity to revise their strategies in response to this feedback. Figure 3 tabulates strategic BRADSOs, BRADSO-IC failures, and detectable priority reversals under indicative and final preferences. Final preferences result in fewer strategic BRADSOs, BRADSO-IC failures, and detectable priority reversals. This pattern is consistent with some cadets responding to the dry-run by ranking branch choices in response to these issues.

In general, cadets form their preferences over branches over time as they acquire more information about branches and their own tastes. Therefore, the change documented in Figure 3 may simply reflect general preference formation from acquiring information about branches, and not revisions to preferences in response to the specific mechanism. We briefly investigate this possibility by looking at the presence of strategic BRADSOs, BRADSO-IC failures, and priority reversals using data on the indicative and final preferences from the Class of 2021. This class participated in the strategy-proof DPCO mechanism. We take indicative and final cadet preferences under DPCO mechanism and construct truthful strategies, following the approach described above, for the USMA-2020 mechanism. Figure 4 shows that with preferences constructed from a strategy-proof mechanism, there are only modest differences in strategic BRADSOs, BRADSO-IC failures, and priority reversals between the indicative and final rounds. This comparison supports our claim that revisions of rank order lists in response to a dry-run of the USMA-2020 mechanism might understate the issues this mechanism created, and why these issues became so pronounced with the USMA-2020 mechanism relative to the USMA-2006 mechanism.

5 U.S. Army's 2020 Branching Reform and Adoption of the DPCO Mechanism

To resolve the problems with the USMA-2020 mechanism, most notably its failure of BRADSO-IC, the possibility of strategic BRADSO, and the resulting detectable priority reversals, the Army established a partnership with the two civilian co-authors of this paper to redesign their branching mechanism. Critical to achieving these objectives was the Army's decision to permit cadets in the Class of 2021 to submit preferences over branch-price pairs, thus paving the way to adopt a direct mechanism. This decision was aided by evidence from a cadet survey that mitigated concerns that rating branch-price pairs would be overly complex or unnecessary. Indeed, some of the cadets indicated the need for a system that would allow them to rank order branch-price pairs. One cadet wrote:

> "[...] *I believe that DMI (Department of Military Instruction) could elicit a new type of ranking list. Within my proposed system, people could add to the list of 17 branches BRADSO slots and rank them within that list. For example: AV (Aviation) > IN (Infantry) > AV:B (Aviation with BRADSO).*"

More generally, the survey revealed that more than twice as many cadets prefer a mechanism that allows them to submit preferences over branch-price pairs relative to a mechanism that requires them to submit preferences over branches and then separately indicate their willingness to pay an increased price for each branch as in the USMA-2006 and USMA-2020 mechanisms.[27]

5.1 Cadet Utilization of the Richer Strategy Space of the DPCO Mechanism

Preference data from the Class of 2021 confirm that cadets used the flexibility to express preferences over branch-price pairs. Figure 5 provides details on the extent to which cadets did not rank a branch with increased price immediately after the branch at base price. For each of 994 cadet first branch choices, 272 cadets rank that branch with increased price as their second choice and 36 cadets rank that branch with increased price as their third choice or lower. These 36 cadets would not have been able to express this preference under the strategy space of a quasi-direct mechanism like the USMA-2006 mechanism or the USMA-2020 mechanism. When we consider the next branch on a cadet's rank order list, cadets also value the flexibility of the new mechanism. For the branch that appears next on the rank order list, 78 cadets rank that branch with increased price as their immediate next highest choice and 24 cadets rank that branch with increased price two or more places below on their rank order list. These 24 cadets also would not have been able to express this preference under a quasi-direct mechanism.

[27] A question on the survey asked cadets whether they prefer a mechanism that allows them to submit preferences over branch-price pairs, like the DPCO mechanism, or a mechanism that requires them to submit preferences over branches alone while separately indicating willingness to pay an increased price, or BRADSO, for each branch. Appendix D.2 shows that 50 percent of respondents preferred the mechanism that permitted ranking branch-price pairs, 21 percent preferred the mechanism without the option to rank branch-price pairs, 24 percent were indifferent, and 5 percent did not understand.

5.2 Effects of Price Responsiveness Policies Under DPCO Mechanism

After adopting the USMA-2020 mechanism, Army and USMA leadership had several discussions about the potential price responsiveness policy for the Class of 2021 and possibly increasing the share of flexible-price positions. As described in the excerpt below from a news article describing an interview with the Talent-Based Branching Program Manager, selecting these parameters presented the Army with a trade-off between retention and talent alignment (Garcia, 2020):

> *"A key question the Army considered when designing this year's mechanism was how much influence to give cadets who are willing to BRADSO. If every cadet who volunteers to BRADSO can gain priority, or "jump" above, every cadet who did not volunteer to BRADSO, then that could improve Army retention through more cadets serving an additional three years, but it could also result in more cadets being assigned to branches that do not prefer them."*

The comparative static results in Proposition 1 in Section 3 motivate our empirical analysis of different price responsiveness policies and different levels of flexible-price positions. While the results on the BRADSO collected (i.e. the flexible-price positions awarded at the increased price) given in Proposition 1 hold for a given branch, in theory they may not hold in aggregate across all branches under the DPCO mechanism.[28] However, as we show next, the comparative static properties do hold in our simulations with the Class of 2021 data for several price responsiveness policies.

The Army considered three price responsiveness policies: the ultimate price responsiveness policy and two tiered price responsiveness policies. Under the BRADSO-2020 price responsiveness policy, a cadet who expressed a willingness to sign a BRADSO contract only obtained priority over other cadets who had the same categorical branch rating. Under the BRADSO-2021 price responsiveness policy, a cadet who expressed a willingness to sign a BRADSO contract obtained higher priority over all other cadets if she was in the medium or high category. To illustrate the trade-off between talent alignment and retention, Figure 6 uses preferences from the Class of 2021 and re-runs the DPCO mechanism under these three price responsiveness policies for different levels of flexible-price positions q_b^f, where q_b^f is expressed as a percentage of the total number of positions for branch b.

To measure the effects of price responsiveness policies on BRADSOs collected, Figure 6 shows how the number of BRADSOs charged increases with q_b^f and with the "closeness" of the price responsiveness policy to the ultimate price responsiveness policy. That is, for a given q_b^f, the BRADSO-2021 policy results in more BRADSOs charged than the BRADSO-2020 policy, but fewer BRADSOs charged than the ultimate price responsiveness policy. When the fraction of the flexible-price positions is small, there is relatively little difference between price responsiveness policies.

[28] The fact that a global comparative static result does not hold in matching models with slot-specific priorities has been explored in other work, including Dur, Kominers, Pathak, and Sönmez (2018) and Dur, Pathak, and Sönmez (2020). Both papers contains examples showing that how a comparative static across all branches need not hold. However, the two papers also show empirically that these theoretical cases do not apply in their applications.

For example, when the fraction of the flexible-price positions is 15% of all positions, 55 BRADSOs are charged under the ultimate BRADSO policy, 47 BRADSOs are charged under BRADSO-2021, and 38 BRADSOs are charged under BRADSO-2020. When the fraction of the flexible-price positions is larger, the price responsiveness policy has a larger effect on BRADSOs collected. When the fraction of the flexible-price positions is 65%, 118 BRADSOs are charged under the ultimate BRADSO policy, 95 BRADSOs are charged under BRADSO-2021, and 65 BRADSOs are charged under BRADSO-2020.

The ability to run this analysis on the effects of price responsiveness policies is a side-benefit of a strategy-proof mechanism, like the DPCO mechanism. At the request of the Army, we conducted a similar analysis using data from the Class of 2020, but this analysis required stronger assumptions on cadet preferences.[29] As a result of this analysis, the Army decided to adopt the BRADSO-2021 policy and increase the fraction of the flexible-price positions from 25 to 35 percent. These are both policies that increase the power of BRADSO. However, USMA decided against adopting the ultimate price responsiveness policy because branches remained opposed to giving more BRADSO power to low-tier cadets.

6 Conclusion

In July 2019, the US Army implemented sweeping changes to their Talent-Based Branching Program by adopting the USMA-2020 mechanism for the West Point, or USMA, Class of 2020. The impetus for this change was to give Army branches greater influence and to ultimately assign cadets to better-fitting branches. However, the USMA-2020 mechanism retained the same restricted strategy space as the previous USMA-2006 mechanism. The performance of the USMA-2020 mechanism made several underlying issues more apparent.

Our paper describes these reforms and shows how they facilitated the adoption of the DPCO mechanism for the Class of 2021. Our main result is that the DPCO mechanism is the only mechanism that satisfies intuitive criteria, all formulating the Army's objectives. We also present an empirical analysis of the USMA-2020 mechanism in Section 4, complemented with an in depth theoretical analysis of this mechanism in Appendix B. That investigation provides insights into the perverse incentives in this mechanism and why these challenges became difficult to ignore for the Class of 2020.

When it was first formulated in Sönmez and Switzer (2013), cadet-branch matching became the first real-life application of the matching with contracts framework with a non-trivial role for the contractual terms. Our work builds on foundational theory by Kelso and Crawford (1982), Hatfield and Milgrom (2005), and Hatfield and Kojima (2010) and applied theory papers by Sönmez and Switzer (2013) and Sönmez (2013). This sequence of papers opened the door to influence mechanisms deployed in the field, and eventually led to the redesign of USMA's mechanism. In

[29]Because cadets in the Class of 2020 did not submit preferences over branch-price pairs, we assumed that all BRADSOs are consecutive, and also considered different assumptions on the prevalence of non-consecutive BRADSOs. These assumptions are not needed when cadets can rank branch-price pairs in a strategy-proof mechanism.

this respect, we contribute to a market design literature where abstract theoretical models, which are often not contemplated in terms of particular applications, go on to have practical applications and ultimately influence the design of real-world mechanisms.

While the Army initially resisted reforms to the USMA branching process, the challenges due to failures of certain key principles formalized by our axioms led the Army to partner with the civilian coauthors of this paper to fix these challenges. The Army sought a mechanism that promoted retention and talent alignment as USMA-2020 did, but that was also incentive compatible. The desire for incentive compatibility was partly to build cadets' trust in Army labor markets (Garcia, 2020) and partly to obtain truthful information on cadet preferences. The latter objective is particularly important for Army efforts to address the lack of minority representation in branches like Infantry and Armor, which produce a disproportionate share of Army generals (Briscoe, 2013; Kofoed and mcGovney, 2019). In that sense, the reform shows the relevance and power of the matching with contracts framework, and the importance of building mechanisms with straightforward incentives to engender trust between organizations and their employees.

After seeing the value of DPCO, the Army made two decisions to expand its utilization and to help achieve other policy objectives. First, although the Army originally planned to use the USMA-2020 mechanism to branch more than 3,000 ROTC cadets graduating in 2021, after observing the shortcomings of the USMA-2020 mechanism, the Army changed course and decided to adopt the DPCO mechanism for ROTC instead. The decision to use the DPCO mechanism for ROTC was in part due to concerns that ROTC's previous branching mechanism generated dead zones that made priority reversals particularly visible, as discussed in Sönmez (2013). Second, for the West Point and ROTC Classes of 2022, the Army has asked us to modify the DPCO mechanism to help address shortages of cadets willing to volunteer for the Army's *branch-detail program*.[30] We hope to report on these developments in future work.

[30]Officers who participate in the branch-detail program serve the first three to four years of their Army career in a "detail" branch before transitioning to the branch they received through their commissioning source's branching process. The Army faces an imbalance in branch strengths when there is a shortage of cadets willing to branch-detail.

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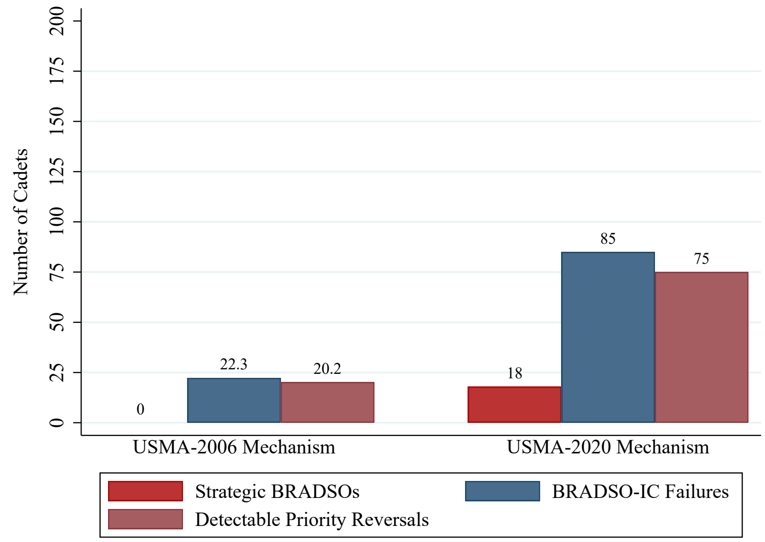
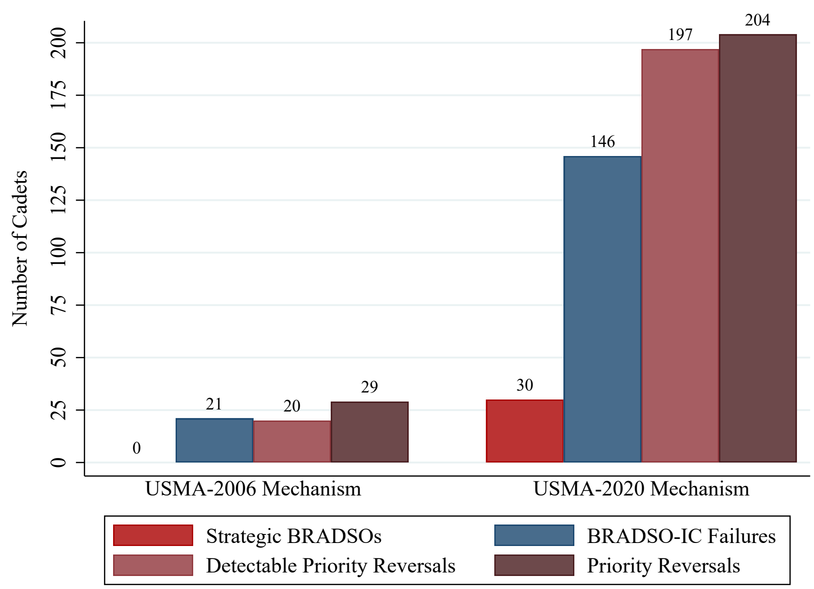
Figure 1: Comparison of Outcomes of the USMA-2006 and USMA-2020 Mechanisms



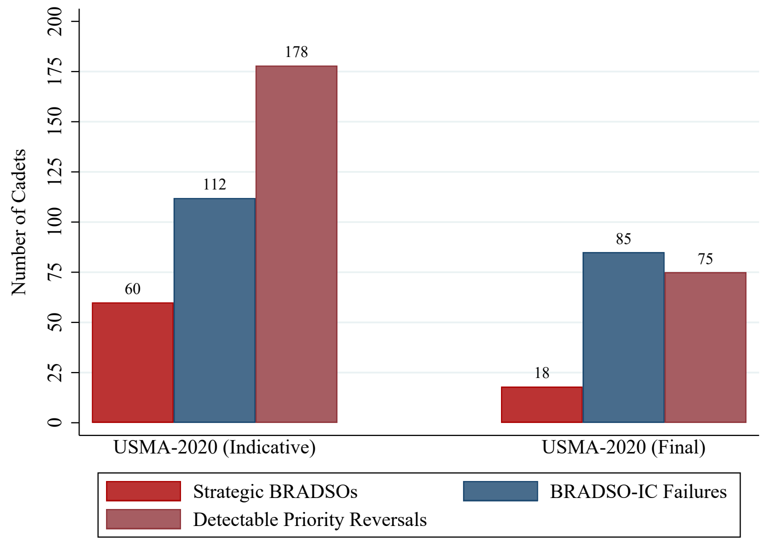
Notes. This figure reports Strategic BRADSOs, BRADSO-IC Failures, and Detectable Priority Reversals under the USMA-2006 and USMA-2020 Mechanisms. The first three columns correspond to outcomes under USMA-2006 Mechanism averaged over classes from 2014-2019. The last three columns correspond to outcomes under USMA-2020 Mechanism for the Class of 2020.

Figure 2: USMA-2006 and USMA-2020 Mechanism Performance under Truthful Strategies Simulated from Preference Data from Class of 2021



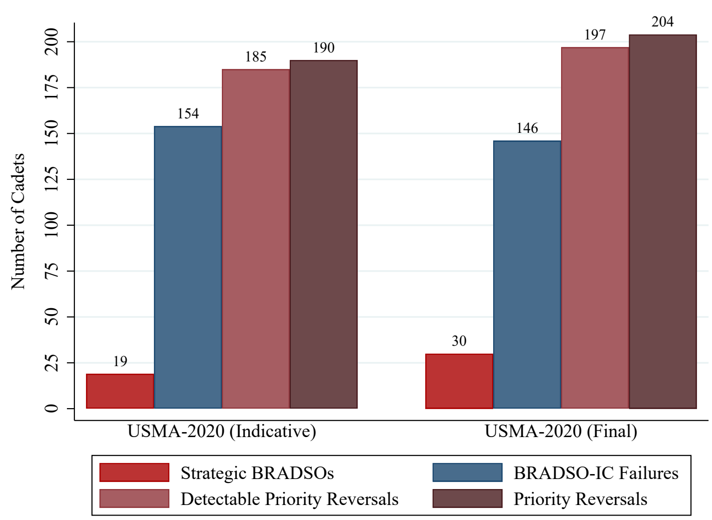
Notes. This figure uses data from the Class of 2021 to simulate the outcomes of the mechanisms USMA-2006 and USMA-2020. We use preferences over branch-price pairs under the DPCO mechanism to construct truthful strategies for USMA-2006 and USMA-2020 by assuming that willingness to BRADSO at a branch means the cadet's strategy under the USMA-2006 and USMA-2020 mechanisms has her willing to BRADSO. To compute Priority Reversals, we compare a cadet's outcome in either the USMA-2006 or USMA-2020 mechanism to a cadet's preference submitted under the DPCO mechanism. If a cadet prefers a higher ranked choice and has higher priority over a cadet who is assigned that choice, then the cadet is part of a Priority Reversal.

Figure 3: USMA-2020 Mechanism Performance Under Indicative and Final Strategies



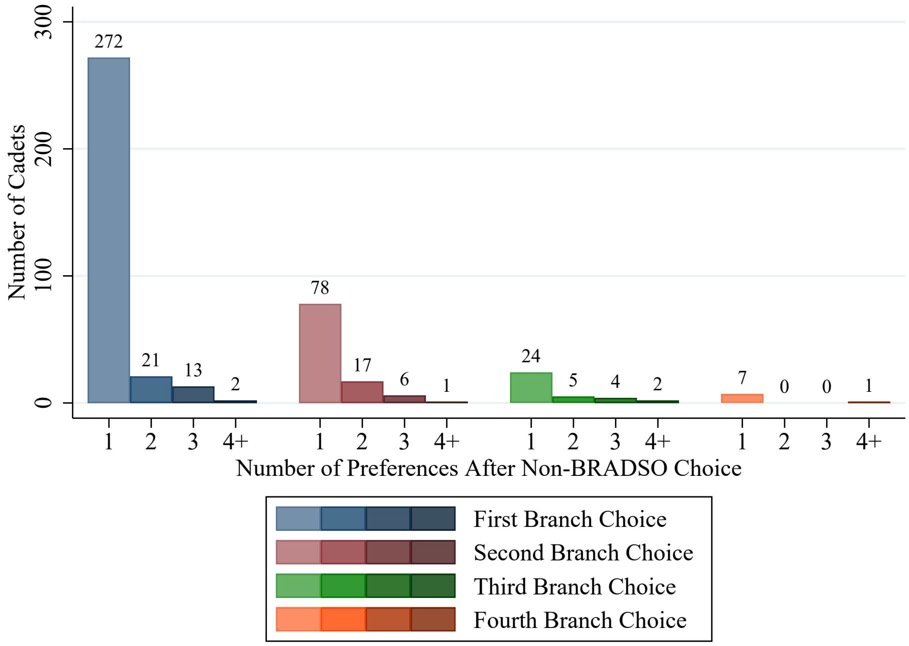
Notes. This figure reports on the number of Strategic BRADSOs, BRADSO-IC failures, and Detectable Priority Reversals under indicative strategies submitted in a dry-run of the USMA-2020 mechanism and final strategies of the USMA-2020 mechanism for the Class of 2020.

Figure 4: USMA-2020 Mechanism Performance under Truthful Strategies Simulated from Indicative and Final Preference Data from Class of 2021



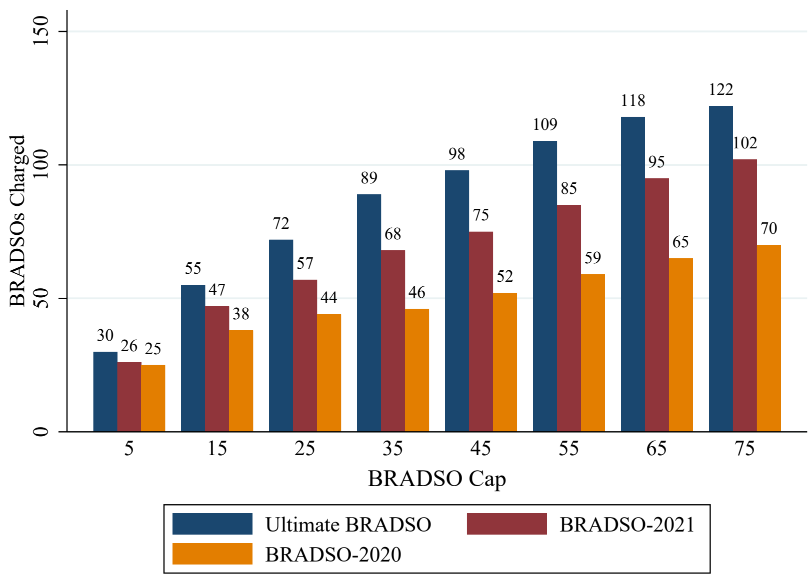
Notes. USMA used the strategy-proof DPCO mechanism for the Class of 2021. This figure uses data from the indicative and final rounds from the Class of 2021 on cadet preferences, branch priorities, and branch capacities to simulate the outcome of the USMA-2020 mechanism. Since the strategy space of the mechanism USMA-2020 differs from that of the mechanism DPCO, cadet strategies that correspond to truthful branch-preferences and BRADSO willingness are are simulated from cadet preferences over branch-price pairs under the DPCO mechanism. Truthful strategies are constructed from Class of 2021 preferences by assuming that a preference indicating willingness to BRADSO at a branch means the cadet's strategy under the USMA-2006 and USMA-2020 mechanisms has her willing to BRADSO. USMA-2020 (Indicative) reports outcomes using strategies constructed from preferences submitted in the dry-run of the DPCO mechanism. USMA-2020 (Final) reports outcomes using strategies constructed from preferences submitted in the final run of the DPCO mechanism.

Figure 5: BRADSO Ranking Relative to Non-BRADSO Ranking by Class of 2021



Notes. This figure reports where in the preference list a branch is ranked with BRADSO relative to where it is ranked without BRADSO. A value of 1 (2 or 3) indicates that the branch is ranked with BRADSO immediately after (two places or three places after, respectively) the branch is ranked at base price. 4+ means that the a branch is ranked with BRADSO four or more choices after the branch is ranked at base price.

Figure 6: Number of BRADSOs Charged Across price responsiveness policies and Cap Sizes



Notes. This figure reports on the number of BRADSOs charged for three price responsiveness policies: Ultimate BRADSO, BRADSO-2020, and BRADSO-2021 using data from the Class of 2021. The BRADSO cap ranges from 5% to 75% of slots at each branch. Each outcome is computed by running DPCO mechanism given stated cadet preferences under different price responsiveness policies and cap sizes.

A Online Appendix: Main Characterization Result

A.1 Proof of Theorem 1

Proof of Theorem 1: Fix $(\pi_b)_{b\in B} \in \Pi^{|B|}$ and $(\omega_b^+)_{b\in B} \in \prod_{b\in B} \Omega_b^+$.

We first show that the mechanism ϕ^{DPCO} satisfies the five axioms. For the proofs of *individual rationality, non-wastefulness, lack of priority reversals*, and *enforcement of the price responsiveness policy*, also fix $\succ \in \mathcal{Q}^{|I|}$.

Individual rationality: No cadet $i \in I$ ever makes a proposal to a branch b at the increased price t^+ under the DPCO mechanism, unless her preferences are such that $(b, t^+) \succ_i \emptyset$. Hence the DPCO mechanism satisfies *individual rationality*.

Non-wastefulness: For any branch $b \in B$, unless there are already q contracts with distinct cadets on hold, it is not possible for all contracts of any given cadet to be rejected at any step of the DPCO mechanism. Hence the DPCO mechanism satisfies *non-wastefulness*.

Lack of priority reversals: Suppose that $\phi_j^{DPCO}(\succ) \succ_i \phi_i^{DPCO}(\succ)$ for a pair of cadets $i, j \in I$. Since the DPCO mechanism is *individually rational*, $\phi_j^{DPCO}(\succ) \neq \emptyset$. Let branch $b \in B$ and price $t \in \{t^0, t^+\}$ be such that $\phi_j^{DPCO}(\succ) = (b, t)$. Let k be the final step of the DPCO mechanism. Since $\phi_j^{DPCO}(\succ) \succ_i \phi_i^{DPCO}(\succ)$, cadet i has proposed the contract (i, b, t) to branch b at some step of the cumulative offer process, which is rejected by branch b (strictly speaking for the first time) either immediately or at a later step. Since the proposed contracts remain available until the termination of the procedure under the DPCO mechanism, the contract (i, b, t) is also rejected by branch b at the final Step k of the DPCO mechanism. In contrast, since $\phi_j^{DPCO}(\succ) = (b, t)$, contract (j, b, t) is chosen by branch b at the final step k of the DPCO mechanism. If the contract (j, b, t) is accepted as one of the first q_b^0 positions under the choice rule $\mathcal{C}_b^{DP}$, then $j \; \pi_b \; i$. Otherwise, if the contract (j, b, t) is accepted as one of the last q_b^f positions under the choice rule $\mathcal{C}_b^{DP}$, then $(j, b, t) \; \omega_b^+ \; (i, b, t)$. In either case we have $j \; \pi_b \; i$, proving that the DPCO mechanism *has no priority reversals*.

Enforcement of the price responsiveness policy: First suppose that cadets $i, j \in I$ are such that $\phi_i^{DPCO}(\succ) = (b, t^+)$ and $(b, t^0) \succ_j \phi_j^{DPCO}(\succ)$. The relation $(b, t^0) \succ_j \phi_j^{DPCO}(\succ)$ implies that cadet j has proposed the contract (j, b, t^0) to the branch b at some step of the DPCO mechanism, which is rejected by branch b either immediately or at a later step. Let k be the final step of the DPCO mechanism. Since the proposed contracts remain available until the termination of the procedure under the DPCO mechanism, the contract (j, b, t^0) is also rejected by branch b at the final Step k of the DPCO mechanism. More specifically, it is rejected by the choice rule $\mathcal{C}_b^{DP}$ at the final Step k both for the first q_b^0 positions using the baseline priority order π_b and for the last q_b^f positions using the price responsiveness policy ω_b^+. In contrast, contract (i, b, t^+) is chosen by branch b at the final Step k of the DPCO mechanism using the price responsiveness policy ω_b^+. Therefore,

$$\left.\begin{array}{l} \phi_i^{DPCO}(\succ) = (b, t^+), \text{ and} \\ (b, t^0) \succ_j \phi_j^{DPCO}(\succ) \end{array}\right\} \quad \Longrightarrow \quad (i, t^+) \; \omega_b^+ \; (j, t^0).$$

That is, cadet j does not have a legitimate claim for a lower-price version of $X_i = (b, t^+)$.

Next suppose that cadets $i, j \in I$ are such that $\phi_i^{DPCO}(\succ) = (b, t^0)$, $(b, t^+) \succ_j \phi_j^{DPCO}(\succ)$, $(j, t^+)\ \omega_b^+\ (i, t^0)$, and moreover, let cadet i be the lowest π_b-priority cadet with an assignment of $\phi_i^{DPCO}(\succ) = (b, t^0)$. The relation $(b, t^+) \succ_j \phi_j^{DPCO}(\succ)$ implies that cadet j has proposed the contract (j, b, t^+) to the branch b at some step of the DPCO mechanism, which is rejected by branch b either immediately or at a later step. Let k be the final step of the DPCO mechanism. Since the proposed contracts remain available until the termination of the procedure under the DPCO mechanism, the contract (j, b, t^+) is also rejected by branch b at the final Step k of the DPCO mechanism. More specifically, it is rejected by the choice rule $\mathcal{C}_b^{DP}$ at the final Step k even for the last q_b^f positions using the price responsiveness policy ω_b^+. Therefore, since by assumption we have $(j, t^+)\ \omega_b^+\ (i, t^0)$, cadet i must have received one of the first q^0 positions using the baseline priority ranking π_b. But since cadet i is the lowest π_b-priority cadet with an assignment of $\phi_j^{DPCO}(\succ) = (b, t^0)$, that means no cadet has received any of the last q_b^f positions at the base price of t^0. Therefore, since ϕ^{DPCO} satisfies *non-wastefulness*,

$$\left.\begin{array}{l} \phi_i^{DPCO}(\succ) = (b, t^0), \\ (b, t^+) \succ_j \phi_j^{DPCO}(\succ), \text{ and} \\ (j, t^+)\ \omega_b^+\ (i, t^0) \end{array}\right\} \implies |\{i' \in I : \phi_{i'}^{DPCO}(\succ) = (b, t^+)\}| = q_b^f.$$

Hence,

$$\left(\phi^{DPCO} \setminus \{(i, b, t^0)\}\right) \bigcup \{(j, b, t^+)\} \notin \mathcal{A},$$

showing that cadet j does not have a legitimate claim for a price-increased version of $\phi_i^{DPCO}(\succ) = (b, t^0)$.

Since no cadet has a legitimate claim for either a priced-reduced or a priced-increased version of another cadet's assignment, the DPCO mechanism satisfies *enforcement of the price responsiveness policy*.

Strategy-proofness: The DPCO mechanism is a special case of the cumulative offer mechanism for *matching problems with slot-specific priorities* formulated in Kominers and Sönmez (2016). Hence *strategy-poofness* of the DPCO mechanism is a direct corollary of their Theorem 3, which proves *strategy-proofness* of the cumulative offer mechanism more broadly for matching problems with slot-specific priorities.

Uniqueness: We prove uniqueness via two lemmata.

Lemma 1. *Let $X, Y \in \mathcal{A}$ be two distinct allocations that satisfy individual rationality, non-wastefulness, enforcement of the price responsiveness policy, and have no priority reversals. Then there exists a cadet $i \in I$ who receives non-empty and distinct assignments under X and Y.*

Proof of Lemma 1: The proof is by contradiction. Fix $\succ \in \mathcal{Q}^{|I|}$. Let $X, Y \in \mathcal{A}$ be two distinct allocations that satisfy *individual rationality, non-wastefulness, enforcement of the price responsiveness policy*, and *have no priority reversals*. To derive the desired contradiction, suppose that, for any cadet

$i \in I$,
$$X_i \neq Y_i \quad \Longrightarrow \quad X_i = \emptyset \text{ or } Y_i = \emptyset. \tag{1}$$

Pick any branch $b \in B$ such that $X_b \neq Y_b$. Let $j \in I$ be the highest π_b-priority cadet who is assigned to branch b either under X or under Y but not both. W.l.o.g., let cadet j be assigned to branch b under allocation X but not under allocation Y. By relation (1),

$$Y_j = \emptyset. \tag{2}$$

Since allocation Y satisfies *non-wastefulness*, there exists a cadet $k \in I$ who is assigned to branch b under allocation Y but not under allocation X. By relation (1),

$$X_k = \emptyset, \tag{3}$$

and therefore, by the choice of cadet j, we have

$$j \; \pi_b \; k. \tag{4}$$

Moreover, since allocation Y *has no priority reversals*, by relations (2) and (4) we have

$$Y_k = (b, t^+) \tag{5}$$

and
$$\underbrace{\emptyset}_{=Y_j} \succ_j (b, t^+), \tag{6}$$

which in turn implies
$$X_j = (b, t^0) \tag{7}$$

by *individual rationality* of allocation X.

Since allocation Y satisfies *enforcement of the price responsiveness policy*, relations (4) and (5) imply

$$(k, t^+) \; \omega_b^+ \; (j, t^0), \tag{8}$$

for otherwise cadet j would have a legitimate claim for a lower-price version of cadet k's assignment $Y_k = (b, t^+)$.

Relation (5) and *individual rationality* of allocation Y imply

$$(b, t^+) \succ_k \emptyset. \tag{9}$$

Define
$$I^* \equiv \{i \in I \, : \, X_i = (b, t^+)\}.$$

Since allocation X satisfies *enforcement of the price responsiveness policy*, relations (3), (7), (8), and

(9) imply

$$|I^*| = q_b^f, \tag{10}$$

for otherwise cadet k would have a legitimate claim for a price-increased version of cadet j's assignment $X_j = (b, t^0)$.

Since allocation X *has no priority reversals* and

$$(b, t^+) \succ_k \underbrace{X_k}_{=\emptyset}$$

by relations (3) and (9), for any $i \in I^*$ we have,

$$i \; \pi_b \; k. \tag{11}$$

But since $Y_k = (b, t^+)$ by relation (5) and $|I^*| = q_b^f$ by relation (10), there exists a cadet $\ell \in I^*$ with $Y_\ell \neq (b, t^+) = X_\ell$, and therefore by relation (1) we have,

$$Y_\ell = \emptyset. \tag{12}$$

Since X satisfies *individual rationality* and $\ell \in I^*$, we have

$$(b, t^+) \;\; \succ_\ell \;\; \emptyset,$$

and therefore relations (5), (11), and (12) imply allocation Y *has a priority reversal*, giving us the desired contradiction and completing the proof of Lemma (1). ♢

Lemma 2. *There can be at most one direct mechanism that satisfies individual rationality, non-wastefulness, enforcement of the price responsiveness policy, strategy-proofness, and has no priority reversals.*

Proof of Lemma 2: The proof of this lemma is inspired by a technique introduced by Hirata and Kasuya (2017). Towards a contradiction, suppose there exists two distinct direct mechanisms φ and ψ that satisfy *individual rationality, non-wastefulness, enforcement of the price responsiveness policy, strategy-proofness*, and *have no priority reversals*. Let the preference profile $\succ^* \in \mathcal{Q}^{|I|}$ be such that,

1. $\varphi(\succ^*) \neq \psi(\succ^*)$, and

2. the aggregate number of acceptable contracts between all cadets is minimized among all preference profiles $\tilde{\succ} \in \mathcal{Q}^{|I|}$ such that $\varphi(\tilde{\succ}) \neq \psi(\tilde{\succ})$.

Let $X = \varphi(\succ^*)$ and $Y = \psi(\succ^*)$. By Lemma 1, there exists a cadet $i \in I$ such that

1. $X_i \neq \emptyset$,

2. $Y_i \neq \emptyset$, and

3. $X_i \neq Y_i$.

Since both allocations X and Y satisfy *individual rationality*,

$$X_i \succ_i^* \emptyset \quad \text{and} \quad Y_i \succ_i^* \emptyset.$$

W.l.o.g., assume

$$X_i \succ_i^* Y_i \succ_i^* \emptyset.$$

Construct the preference relation $\succ_i' \in \mathcal{Q}$ as follows:

If $X_i = (b, t^0)$ for some $b \in B$, then

$$(b, t^0) \succ_i' \emptyset \succ_i' (b', t') \qquad \text{for any } (b', t') \in B \times T \setminus \{(b, t^0)\}.$$

Otherwise, if $X_i = (b, t^+)$ for some $b \in B$, then

$$(b, t^0) \succ_i' (b, t^+) \succ_i' \emptyset \succ_i' (b', t') \qquad \text{for any } (b', t') \in B \times T \setminus \{(b, t^0), (b, t^+)\}.$$

Since $X_i \succ_i^* Y_i \succ_i^* \emptyset$ and $(b, t^0) \succ_i^* (b, t^+)$, the preference relation $\succ_i'$ has strictly fewer acceptable contracts for cadet i than the preference relation $\succ_i^*$.

By *strategy-proofness* of the mechanism ψ, we have

$$\underbrace{\psi_i(\succ_i^*, \succ_{-i}^*)}_{=Y_i} \succeq_i^* \psi_i(\succ_i', \succ_{-i}^*),$$

and since no branch-price pair $(b', t') \in B \times T$ with $Y_i \succeq_i' (b', t')$ is acceptable under $\succ_i'$, by *individual rationality* of the mechanism ψ we have

$$\psi_i(\succ_i', \succ_{-i}^*) = \emptyset. \tag{13}$$

Similarly, by *strategy-proofness* of the mechanism φ, we have

$$\varphi_i(\succ_i', \succ_{-i}^*) \succeq_i' \underbrace{\varphi_i(\succ_i^*, \succ_{-i}^*)}_{=X_i},$$

which in turn implies

$$\varphi_i(\succ_i', \succ_{-i}^*) \neq \emptyset. \tag{14}$$

But then, by relations (13) and (14) we have

$$\varphi(\succ_i', \succ_{-i}^*) \neq \psi(\succ_i', \succ_{-i}^*),$$

giving us the desired contradiction, since between all cadets the preference profile $(\succ_i', \succ_{-i}^*)$ has strictly fewer acceptable contracts than the preference profile $\succ^*$. This completes the proof of

Lemma 2. $\diamondsuit$

Since we have already shown that the DPCO mechanism satisfies all five axioms, Lemma 2 establishes the uniqueness, concluding the proof of Theorem 1. $\square$

A.2 Independence of Axioms in Theorem 1

We establish the independence of the axioms in Theorem 1 by presenting five direct mechanisms. Each fails one of our five axioms and satisfies the other four. Our result shows that none of the axioms are redundant in Theorem 1 and each is important for the characterization of DPCO mechanism.

A.2.1 A mechanism that satisfies all axioms except individual rationality

Given any preference profile $\succ \in \mathcal{Q}^{|I|}$ and individual $i \in I$, let $\succ_i^0 \in \mathcal{Q}$ be the preference relation where the relative preference ranking of all branch-price pairs in $B \times T$ is the same as in $\succ_i$, and remaining unmatched (i.e. $\emptyset$) is the last choice. Define the direct mechanism ϕ^0 as, for any preference profile $\succ \in \mathcal{Q}^{|I|}$,

$$\phi^0(\succ) = \phi^{DPCO}(\succ^0).$$

Mechanism ϕ^0 satisfies all axioms except *individual rationality*.

A.2.2 A mechanism that satisfies all axioms except non-wastefulness

Define the direct mechanism $\phi^\emptyset$ as, for any preference profile $\succ \in \mathcal{Q}^{|I|}$,

$$\phi^\emptyset(\succ) = \emptyset.$$

Mechanism $\phi^\emptyset$ satisfies all axioms except *non-wastefulness*.

A.2.3 A mechanism that satisfies all axioms except enforcement of the price responsiveness policy

The individual-proposing deferred acceptance mechanism given in Online Appendix C.2 satisfies all axioms except *enforcement of the price responsiveness policy*.[31]

A.2.4 A mechanism that satisfies all axioms except lack of priority reversals

The outcome of the mechanism ψ is derived from the outcome of the DPCO mechanism as follows.

Fix a branch $b \in B$. Given any preference profile $\succ \in \mathcal{Q}^{|I|}$, let $i \in I$ be the lowest π_b-priority individual with $\mathsf{b}\big(\phi_i^{DPCO}(\succ)\big) = b$. Let the preference relation $\succ_i^{-b} \in \mathcal{Q}$ be constructed from $\succ_i$ by

[31] More broadly the DPCO mechanism when implemented with a different profile of price responsiveness policies than the underlying one also satisfies all axioms except *enforcement of the price responsiveness policy*.

making branch-price pairs (b, t^0) and (b, t^+) unacceptable, but otherwise keeping the rest of the preference order same as in $\succ_i$. Let the outcome of the mechanism ψ be given as

- $\psi(\succ) = \phi^{DPCO}(\succ_{-i}, \succ_i^{-b})$ if all q_b^f flexible-price positions at branch b are awarded at the increased price t^+ under both $\phi^{DPCO}(\succ_{-i}, \succ_i^{-b})$ and $\phi^{DPCO}(\succ)$, and
- $\psi(\succ) = \phi^{DPCO}(\succ)$ otherwise.

For any given branch $b \in B$, mechanism ψ derives its outcome mostly using the DPCO mechanism, except it "ignores" the lowest π_b-priority individual who receives a position at branch b under the DPCO mechanism provided that all flexible-price positions at branch b are awarded at the increased price t^+ under the DPCO mechanism whether the lowest π_b-priority individual is being ignored or not. If in either scenario some of the q_b^f flexible-price positions are awarded at the base price t^0 or remain idle, then the outcome of the mechanism ψ is the same as the outcome of the DPCO mechanism.

Mechanism ψ satisfies all axioms except the *lack of priority reversals*. The detailed construction above assures that it does not also lose *enforcement of the price responsiveness policy* or *strategy-proofness* due to the modification.

A.2.5 A mechanism that satisfies all axioms except strategy-proofness

The outcome of the mechanism φ is derived from the outcome of the DPCO mechanism as follows.

Fix a branch $b \in B$. Given any preference profile $\succ \in \mathcal{Q}^{|I|}$, let $i \in I$ be the lowest π_b-priority individual with $\mathsf{b}\big(\phi_i^{DPCO}(\succ)\big) = b$. If

1. $\phi_i^{DPCO}(\succ) = (b, t^0)$,
2. $(b, t^+) \succ_i \emptyset$, and
3. $\Big(\phi^{DPCO}(\succ) \setminus \big\{(i, b, t^0)\big\}\Big) \cup \big\{(i, b, t^+)\big\} \in \mathcal{A}$,

then let $\varphi(\succ) = \Big(\phi^{DPCO}(\succ) \setminus \big\{(i, b, t^0)\big\}\Big) \cup \big\{(i, b, t^+)\big\}$. Otherwise, i.e. if any of the three conditions fail, then let $\varphi(\succ) = \phi^{DPCO}(\succ)$.

Compared to the outcome of the DPCO mechanism, the mechanism φ simply increases the charged price for the lowest π_b-priority individual who receive a position at branch b under the DPCO mechanism, if doing so is *feasible* and does not violate *individual rationality*.

Mechanism φ satisfies all axioms except *strategy-proofness*. The affected individual can profit by declaring the branch-price pair (b, t^+) as unacceptable under the mechanism φ. The detailed construction above assures that the mechanism does not also lose *individual rationality*, *lack of priority reversals*, or *enforcement of the price responsiveness policy* due to the modification.

B Formal Analysis of USMA-2020 Mechanism

In Sections 4.5 and 4.6.1, we presented the shortcomings of the USMA-2020 mechanism. In this section of the Online Appendix, we present a more in-depth analysis of the USMA-2020 mechanism to offer additional insight on why it resulted in a much more complex branching system than its predecessor USMA-2006 mechanism.

As with the USMA-2006 mechanism, truthful revelation of branch preferences is not a dominant strategy under the USMA-2020 mechanism, thereby making its formal analysis challenging. Fortunately, focusing on a simpler version of the model with a single branch is sufficient to illustrate and analyze the main challenges of the USMA-2020 mechanism.

Suppose we consider a single branch $b \in B$. When there is a single branch $b \in B$, there are only two preferences for any cadet $i \in I$. The base price contract (i, b, t^0) is by assumption preferred by cadet i to both its increased price version (i, b, t^+) and also to remaining unmatched. Therefore, the only variation in cadet i's preferences depends on whether the increased price contact (i, b, t^+) is preferred to remaining unmatched. For any cadet $i \in I$, $|\mathcal{Q}| = 2$ when there is a single branch $b \in B$, since

- indicating willingness to pay the increased price t^+ under a quasi-direct mechanism can be naturally mapped to the preference relation where the increased price contact (i, b, t^+) is acceptable, whereas
- not doing so can be naturally mapped to the preference relation where the increased price contact (i, b, t^+) is unacceptable,

any quasi-direct mechanism can be interpreted as a direct mechanism. Therefore, unlike the general version of the model, the axioms of BRADSO-IC and elimination of strategic BRADSO are also well-defined for direct mechanisms when there is a single branch, and moreover they are both implied by strategy-proofness.[32]

B.1 Single-Branch Mechanism ϕ^{DP} and Its Characterization

We next introduce a single-branch direct mechanism that is key for our analysis of the USMA-2020 mechanism. The main feature of this mechanism is its iterative subroutine (in Step 2), which determines how many flexible-price positions are assigned at the increased price and which cadets receive these positions.

Mechanism ϕ^{DP}

For any given profile of cadet preferences $\succ = (\succ_i)_{i \in I} \in \mathcal{Q}^{|I|}$, construct the allocation $\phi^{DP}(\succ)$ as follows:

[32] BRADSO-IC and elimination of strategic BRADSO together are equivalent to strategy-proofness when there is a single branch. Strategy-proofness of a single branch, called non-manipulability via contractual terms also plays an important role in the analysis of Hatfield, Kominers, and Westkamp (2021).

Step 0. Let $I^0 \subset I$ be the set of q_b^0 highest π_b-priority cadets in I. For each cadet $i \in I^0$, finalize the assignment of cadet i as $\phi_i^{DP}(\succ) = (b, t^0)$.

Step 1. Let $I^1 \subset I \setminus I^0$ be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. *Tentatively* assign each cadet in I^1 a position at the base price t^0. Relabel the set of cadets in I^1 so that cadet $i^1 \in I^1$ has the lowest π_b-priority in I^1, cadet $i^2 \in I^1$ has the second-lowest π_b-priority in I^1, ..., and cadet $i^{q_b^f} \in I^1$ has the highest π_b-priority in I^1. Also relabel the lowest π_b-priority cadet in I^0 as $i^{q_b^f+1}$.

Step 2. This step determines how many positions are assigned at the increased price t^+.

Step 2.0. Let $J^0 \subset I \setminus (I^0 \cup I^1)$ be the set of cadets in $I \setminus (I^0 \cup I^1)$ who declared the position at the increased price t^+ as acceptable:

$$J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^+) \ \succ_j \ \emptyset\}.$$

If

$$\big|\{j \in J^0 : (j, t^+) \ \omega_b^+ \ (i^1, t^0)\}\big| = 0,$$

then finalize Step 2 and proceed to Step 3. In this case no position will be assigned at the increased price t^+.

Otherwise, if

$$\big|\{j \in J^0 : (j, t^+) \ \omega_b^+ \ (i^1, t^0)\}\big| \geq 1,$$

then proceed to Step 2.1.

Step 2.ℓ. ($\ell = 1, \ldots, q_b^f$) Let

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^+) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^+) \succ_{i^\ell} \emptyset. \end{cases}$$

If

$$\big|\{j \in J^\ell : (j, t^+) \ \omega_b^+ \ (i^{\ell+1}, t^0)\}\big| = \ell,$$

then finalize Step 2 and proceed to Step 3.[33] In this case ℓ positions will be assigned at the increased price t^+.

Otherwise, if

$$\big|\{j \in J^\ell : (j, t^+) \ \omega_b^+ \ (i^{\ell+1}, t^0)\}\big| \geq \ell + 1,$$

then proceed to Step 2.$(\ell + 1)$, unless $\ell = q_b^f$, in which case finalize Step 2 and proceed to Step 3.

[33] Since $J^\ell \supseteq J^{\ell-1}$ by construction, the fact that the procedure has reached Step 2.ℓ implies that the inequality $\big|\{j \in J^\ell : (j, t^+) \ \omega_b^+ \ (i^{\ell+1}, t^0)\}\big| \geq \ell$ must hold.

Step 3. Let Step 2.n be the final sub-step of Step 2 leading to Step 3. $\{i^1, \ldots, i^n\} \subset I^1$ is the set of cadets in I^1 who each lose their tentative assignment (b, t^0). For each cadet $i \in I^1 \setminus \{i^1, \ldots, i^n\}$, finalize the assignment of cadet i as $\phi_i^{DP}(\succ) = (b, t^0)$.

For each cadet $i \in J^n$ with one of the n highest π_b-priorities in J^n, finalize the assignment of cadet i as $\phi_i^{DP}(\succ) = (b, t^+)$. Finalize the assignment of any remaining cadet as $\emptyset$.

The key step in the procedure is Step 2 where it is determined how many of the q_b^f flexible-price positions are to be awarded at the increased price t^+. To determine this number, the price responsiveness policy ω_b^+ is used to check

(1) whether there is at least one cadet with a lower baseline priority π_b than cadet i^1, who is willing to pay the increased price t^+ and whose increased price contract has higher priority under the price effectiveness policy ω_b^+ than the base price contract of cadet i^1;

(2) whether there are at least two cadets each with a lower baseline priority π_b than cadet i^2, who are each willing to pay the increased price t^+ and whose increased price contracts have higher priority under the price effectiveness policy ω_b^+ than the base price contract of cadet i^2;

$\vdots$

(q_b^f) whether there are at least q_b^f cadets each with a lower baseline priority π_b than cadet $i^{q_b^f}$, who are each willing to pay the increased price t^+ and whose increased price contracts have higher priority under the price effectiveness policy ω_b^+ than the base price contract of cadet $i^{q_b^f}$.

Once the number of positions awarded through increased price t^+ contracts is determined in this way, all other positions are assigned to the highest baseline priority cadets as base price contracts. The increased price contracts are awarded to the remaining highest baseline priority cadets who are willing to pay the increased price t^+.

Example 1. (Mechanics of Mechanism ϕ^{DP}) There is a single branch b with $q_b^0 = 3$ and $q_b^f = 3$. There are eight cadets, with their set given as $I = \{i^1, i^2, i^3, i^4, i^5, i^6, j^1, j^2\}$. The baseline priority order π_b is given as

$$i^6 \; \pi_b \; i^5 \; \pi_b \; i^4 \; \pi_b \; i^3 \; \pi_b \; i^2 \; \pi_b \; i^1 \; \pi_b \; j^1 \; \pi_b \; j^2,$$

and the price responsiveness policy is the ultimate price responsiveness policy $\overline{\omega}_b^+$. Cadet preferences are given as

$$(b, t^0) \succ_i (b, t^+) \succ_i \emptyset \qquad \text{for any } i \in \{i^1, i^3, i^5, j^1\}, \text{ and}$$
$$(b, t^0) \succ_i \emptyset \succ_i (b, t^+) \qquad \text{for any } i \in \{i^2, i^4, i^6, j^2\}.$$

We next run the procedure for the mechanism ϕ^{DP}.

Step 0: There are three base-price positions. The three highest π_b-priority cadets in the set I are i^6, i^5, and i^4. Let $I^0 = \{i^4, i^5, i^6\}$, and finalize the assignments of cadets in I^0 as $\phi^{DP}_{i^6}(\succ) = \phi^{DP}_{i^5}(\succ) = \phi^{DP}_{i^4}(\succ) = (b, t^0)$.

Step 1: There are three flexible-price positions. Three highest π_b-priority cadets in the set $I \setminus I^0$ are i^3, i^2, and i^1. Let $I^1 = \{i^1, i^2, i^3\}$, and the tentative assignment of each cadet in I^1 is (b, t^0). There is no need to relabel the cadets since cadet i^1 is already the lowest π_b-priority cadet in I^1, cadet i^2 is the second lowest π_b-priority cadet in I^1, and cadet i^3 is the highest π_b-priority cadet in I^1.

Step 2.0: The set of cadets in $I \setminus (I^0 \cup I^1) = \{j^1, j^2\}$ for whom the assignment (b, t^+) is acceptable is $J^0 = \{j^1\}$. Since

$$\underbrace{\left|\{j \in J^0 : (j, t^+) \; \overline{\omega}_b^+ \; (i^1, t^0)\}\right|}_{=|J^0|=|\{j^1\}|=1} \geq 1,$$

we proceed to Step 2.1.

Step 2.1: Since $(b, t^+) \succ_{i^1} \emptyset$, we have $J^1 = J^0 \cup \{i^1\} = \{i^1, j^1\}$. Since

$$\underbrace{\left|\{j \in J^1 : (j, t^+) \; \overline{\omega}_b^+ \; (i^2, t^0)\}\right|}_{=|J^1|=|\{i^1,j^1\}|=2} \geq 2,$$

we proceed to Step 2.2.

Step 2.2: Since $\emptyset \succ_{i^2} (b, t^+)$, we have $J^2 = J^1 = \{i^1, j^1\}$. Since

$$\underbrace{\left|\{j \in J^2 : (j, t^+) \; \overline{\omega}_b^+ \; (i^3, t^0)\}\right|}_{=|J^2|=|\{i^1,j^1\}|=2} = 2,$$

we finalize Step 2 and proceed to Step 2.3.

Step 3: Step 2.2 is the last sub-step of Step 2. Therefore two lowest π_b-priority cadets in I^1, i.e cadets i^1 and i^2, lose their tentative assignments of (b, t^0). In contrast, the only remaining cadet in the set $I^1 \setminus \{i^1, i^2\}$, i.e cadet i^3 maintains her tentative assignment, which is finalized as $\phi^{DP}_{i^3}(\succ) = (b, t^0)$.

The two highest priority cadets in J^2 are i^1 and j^1. Their assignments are finalized as $\phi^{DP}_{i^1}(\succ) = \phi^{DP}_{j^1}(\succ) = (b, t^+)$. Assignments of the remaining cadets i^2 and j^2 are finalized as $\emptyset$. The final allocation is:

$$\phi^{DP}(\succ) = \begin{pmatrix} i^1 & i^2 & i^3 & i^4 & i^5 & i^6 & j^1 & j^2 \\ (b, t^+) & \emptyset & (b, t^0) & (b, t^0) & (b, t^0) & (b, t^0) & (b, t^+) & \emptyset \end{pmatrix}.$$

■

Our next result is the following characterization of the the single-branch direct mechanism ϕ^{DP}.

Proposition 2. *Suppose there is a single branch b. Fix a baseline priority order $\pi_b \in \Pi$ and a price responsiveness policy $\omega_b^+ \in \Omega_b^+$. A direct mechanism φ satisfies*

1. *individual rationality,*
2. *non-wastefulness,*
3. *enforcement of the price responsiveness policy,*
4. *BRADSO-IC, and*
5. *has no priority reversals,*

if and only if $\varphi = \phi^{DP}$.

Since (i) a quasi-direct mechanism becomes a direct mechanism when there is a single branch, and (ii) strategy-proofness implies BRADSO-IC in this environment, Theorem 1 and Proposition 2 immediately imply the following result.

Corollary 1. *Suppose there is a single branch b. Fix a baseline priority order $\pi_b \in \Pi$ and a price responsiveness policy $\omega_b^+ \in \Omega_b^+$. Then, for any preference profile $\succ \in \mathcal{Q}^{|I|}$,*

$$\phi^{DP}(\succ) = \phi^{DPCO}(\succ).$$

The mechanism ϕ^{DP} is merely an alternative formulation of the DPCO mechanism that does not rely on the cumulative offer procedure when there is a single branch. This formulation is helpful for the single-branch equilibrium analysis of the USMA-2020 mechanism we present next.

B.2 Equilibrium Outcomes under the USMA-2020 Mechanism

While the USMA-2020 mechanism is not a direct mechanism in general, when there is a single branch it can be interpreted a direct mechanism. In this case, for any cadet $i \in I$ the first part of the strategy space $\mathcal{S}_i = \mathcal{P} \times 2^B$ becomes redundant, and the second part simply solicits whether branch b is acceptable by cadet i or not (analogous to a direct mechanism).

Our next result shows that when there is a single branch the truthful outcome of the direct mechanism ϕ^{DP} is the same as the unique Nash equilibrium outcome of the mechanism φ^{2020}.

Proposition 3. *Suppose there is a single branch b. Fix a baseline priority order $\pi_b \in \Pi$, a price responsiveness policy $\omega_b^+ \in \Omega_b^+$, and a preference profile $\succ \in \mathcal{Q}^{|I|}$. Then the strategic-form game induced by the mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$ has a unique Nash equilibrium outcome that is equal to the allocation $\phi^{DP}(\succ)$.*[34]

Caution is needed when interpreting Proposition 3; if interpreted literally, this result can be misleading. What is more consequential for Proposition 3 is not the result itself, but rather its

[34]Using the terminology of the *implementation theory*, this result can be alternatively stated as follows: When there is a single branch, the mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$ implements the allocation rule ϕ^{DP} in Nash equilibrium.

proof which constructs the equilibrium strategies of cadets. The proof provides insight into why the failure of BRADSO-IC, the presence of strategic BRADSO, and the presence of detectable priority reversals are all common phenomena under the real-life implementation of the USMA-2020 mechanism (despite the outcome equivalence suggested by Proposition 3).

Given the byzantine structure of the Nash equilibrium strategies even with a single branch, it is perhaps not surprising that reaching such a well-behaved Nash equilibrium is highly unlikely to be observed under the USMA-2020 mechanism. The following example illustrates the knife-edge structure of the Nash equilibrium strategies under the USMA-2020 mechanism.

Example 2. (Knife-Edge Nash Equilibrium Strategies)

To illustrate how challenging it is for the cadets to figure out their best responses under the USMA-2020 mechanism, we present two scenarios. The scenarios differ from each other minimally, but cadet best responses differ dramatically. Our first scenario is same as the one we presented in Example 1.

Scenario 1: There is a single branch b with $q_b^0 = 3$ and $q_b^f = 3$. There are eight cadets, $I = \{i^1, i^2, i^3, i^4, i^5, i^6, j^1, j^2\}$. The baseline priority order π_b is given as

$$i^6 \; \pi_b \; i^5 \; \pi_b \; i^4 \; \pi_b \; i^3 \; \pi_b \; i^2 \; \pi_b \; i^1 \; \pi_b \; j^1 \; \pi_b \; j^2 \quad \text{and}$$

and the price responsiveness policy is the ultimate price responsiveness policy $\overline{\omega}_b^+$. Cadet preferences are

$$\begin{aligned} &(b,t^0) \succ_i (b,t^+) \succ_i \emptyset \qquad && \text{for any } i \in \{i^1, i^3, i^5, j^1\}, \text{ and} \\ &(b,t^0) \succ_i \emptyset \succ_i (b,t^+) \qquad && \text{for any } i \in \{i^2, i^4, i^6, j^2\}. \end{aligned}$$

Let s^* be a Nash equilibrium strategy for Scenario 1 under the USMA-2020 mechanism. Recall that when there is a single branch b, the strategy space for each cadet $i \in I$ is simply $\mathcal{S}_i = \{b, \emptyset\}$. We construct the Nash equilibrium strategies in several phases.

Phase 1: Consider cadets i^1 and j^1, each of whom prefers the increased price assignment (b, t^+) to remaining unmatched. Since there are six positions altogether and there are five higher π_b-priority cadets than either of these two cadets, at most one of them can receive a position (at any cost) unless each of them submit a strategy of b. And if one of them submits a strategy of $\emptyset$, the other one has a best response strategy of b assuring a position at the increased price rather than remaining unmatched. Hence, $s^*_{i^1} = s^*_{j^1} = b$ at any Nash equilibrium.

Phase 2: Consider cadet j^2 who prefers remaining unmatched to the increased price assignment (b, t^+). Since she is the lowest π_b-priority cadet, she cannot receive an assignment of (b, t^0) regardless of her strategy. In contrast, she can guarantee remaining unmatched with a strategy of $s_{j^2} = \emptyset$. While this does not at this point rule out a strategy of $s_{j^2} = b$ at Nash equilibrium (just yet), it means $\varphi^{2020}_{j^2}(s^*) = \emptyset$.

Phase 3: Consider cadet i^2 who prefers remaining unmatched to the increased price assignment

(b, t^+). She is the fifth highest π_b-priority cadet, so she secures a position if she submits a strategy of $s_{i^2} = b$, but the position will have to be at the increased price t^+, since the lowest π_b-priority cadet j^2 is remaining unmatched from Phase 2, and therefore there cannot be three cadets with lower π_b-priority who receive an assignment of (b, t^+). But since cadet j^2 prefers remaining unmatched to the increased price assignment (b, t^+), she cannot receive an assignment of (b, t^+) at Nash equilibria. Hence, cadet i^2's Nash equilibrium strategy is $s^*_{i^2} = \emptyset$, and her Nash equilibrium assignment is $\varphi^{2020}_{i^2}(s^*) = \emptyset$.

Phase 4: Consider the remaining cadets i^3, i^4, i^5 and i^6. Since cadets i^2 and j^2 have to remain unmatched (from Phases 2 and 3) at Nash equilibria, they each receive a position at Nash equilibrium. Since only the two cadets i^1 and j^1 from Phases 1-3 have Nash equilibrium strategies of b, the lowest π_b-priority cadet of the four cadets i^3, i^4, i^5, i^6 who submit a strategy of b receives an assignment of (b, t^+). But this cannot happen at Nash equilibria since that particular cadet can instead submit a strategy of $\emptyset$ receiving a more preferred assignment of (b, t^0). Hence, $s^*_i = \emptyset$ and $\varphi^{2020}_i(s^*) = (b, t^0)$ for any $i \in \{i^3, i^4, i^5, i^6\}$.

The unique Nash equilibrium strategy s^* and its Nash equilibrium outcome $\varphi^{2020}(s^*)$ for Scenario 1 are given as:

Cadet	i^1	i^2	i^3	i^4	i^5	i^6	j^1	j^2
Nash equilibrium strategy	b	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	b	$\emptyset$
Nash equilibrium assignment	(b, t^+)	$\emptyset$	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^+)	$\emptyset$

Scenario 1 involves *BRADSO-IC* failures for cadets i^3 and i^5 whose Nash equilibrium strategies force them into hiding their willingness to pay the increased price t^+. Any deviation from her Nash equilibrium strategy by truthfully declaring her willingness to pay the increased price t^+ will result in an detectable priority reversal for cadet i^5.

Scenario 2: This scenario differs from Scenario 1 in only the preferences of the lowest π_b-priority cadet j^2 and nothing else. Thus, cadet preferences for this scenario are given as:

$$(b, t^0) \succ'_i (b, t^+) \succ'_i \emptyset \qquad \text{for any } i \in \{i^1, i^3, i^5, j^1, j^2\}, \text{ and}$$
$$(b, t^0) \succ'_i \emptyset \succ'_i (b, t^+) \qquad \text{for any } i \in \{i^2, i^4, i^6\}.$$

Let s' be a Nash equilibrium strategy for Scenario 2 under the USMA-2020 mechanism.

Phase 1: Identical to Phase 1 for Scenario 1, and thus $s'_{i^1} = s'_{j^1} = b$ at any Nash equilibrium.

Phase 2: Consider cadet i^2 who prefers remaining unmatched to the increased price assignment (b, t^+), and cadets i^3 and j^2, each of whom prefers the increased price assignment (b, t^+) to remaining unmatched. Since (i) there are six positions altogether, (ii) three cadets with higher π_b-priority than each one of i^2, i^3, and j^2, and (iii) $s'_{i^1} = s'_{j^1} = b$ from Phase 1, at most one of the cadets i^2, i^3, j^2 can receive an assignment of (b, t^0) if any. Therefore, submitting a strategy of $s_{i^3} = \emptyset$ is a best response for cadet i^3 only if both cadets i^2 and j^2 also submit a strategy of $\emptyset$ each. But this cannot happen in Nash equilibria, since it gives cadet j^2 a profitable deviation by submitting a strategy of $s_{j^2} = b$ and jumping ahead of cadets i^2 and i^3 securing her a position. Hence $s'_{i^3} = b$ and $\varphi^{2020}_{i^3}(s') = (b, t^+)$. When cadet i^3 joins the two cadets from Phase 1 each also submitting a strategy of b, this assures that exactly three positions will be assigned at the increased price t^+. Therefore a strategy of f $s_{i^2} = b$ assures assures cadet i^2 an assignment of (b, t^+), which cannot happen at Nash equilibrium. Therefore, $s'_{i^2} = \emptyset$ and $\varphi^{2020}_{i^2}(s') = \emptyset$. This not only assures that $\varphi^{2020}_{i^3}(s') = \varphi^{2020}_{i^1}(s') = \varphi^{2020}_{j^1}(s') = (b, t^+)$, but it also means that $s'_{j^2} = b$ at Nash equilibrium, for otherwise with two lower π_b-priority cadets with strategies of $\emptyset$, cadet i^3 would have an incentive to deviate himself and receiving the position at the base price rather than the increased price.

Phase 3: Consider the remaining cadets i^4, i^5 and i^6. Of all lower π_b-priority cadets, only the cadet i^2 and has Nash equilibrium strategies of $\emptyset$ from Phases 1 and 2. Therefore the lowest π_b-priority cadet of the three cadets i^4, i^5, i^6 who submit a strategy of $\emptyset$ receives an assignment of $\emptyset$. But this cannot happen at Nash equilibria since that particular cadet can instead submit a strategy of b and receive a more preferred assignment of (b, t^0) since three lower π_b-priority cadets already receive an assignment of (b, t^+) each from Phase 2. Therefore, regardless of their preferences $s'_{i^4} = s'_{i^5} = s'_{i^6} = b$, and $\varphi^{2020}_{i^4}(s') = \varphi^{2020}_{i^5}(s') = \varphi^{2020}_{i^6}(s')(b, t^0)$.

The unique Nash equilibrium strategy s' and its Nash equilibrium outcome $\varphi^{2020}(s')$ for Scenario 2 are given as:

Cadet	i^1	i^2	i^3	i^4	i^5	i^6	j^1	j^2
Nash equilibrium strategy	b	$\emptyset$	b	b	b	b	b	b
Nash equilibrium assignment	(b, t^+)	$\emptyset$	(b, t^+)	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^+)	$\emptyset$

Not only does the Nash equilibrium strategies of cadets i^4 and i^6 involve strategic BRADSO in Scenario 2 and they have to declare willingness to pay the increased price t^+ even though under their true preferences they do not, but any deviation from this Nash equilibrium strategy by declaring their unwillingness to pay the increased price t^+ will result in detectable priority reversals for both cadets.

Another key insight from this example is the dramatic difference between the Nash equilibrium strategies due to one minor change in the underlying economy, a preference change in the lowest base priority cadet. This minor change only affects the assignment of cadet i^3 by changing it from (b, t^0) to (b, t^+). It also changes the Nash equilibrium strategy of not only cadet i^3, and also all other higher π_b-priority cadets i^4, i^5, and i^6. Moreover, in addition to BRADSO-IC failures and the presence of strategic BRADSO under Nash equilibria, any deviation from these strategies result in detectable priority reversals. The fragility of our equilibrium strategies provides us intuition on the prevalence of these phenomena under the USMA-2020 mechanism. ■

Example 2 shows that while the failure of BRADSO-IC and the presence of strategic BRADSO can be observed at Nash equilibria of the USMA-2020 mechanism, the presence of detectable priority reversals is out-of-equilibrium behavior under complete information when there is a single branch. Our next example shows that if the complete information assumption is relaxed there can also be detectable priority reversals in the Bayesian equilibria of the USMA-2020 mechanism.

Example 3. (Detectable Priority Reversals at Bayesian Equilibria)

Suppose there is a single branch b with $q^0_b = q^f_b = 1$ and three cadets i_1, i_2, and i_3. The baseline priority order π_b is such that

$$i_1 \; \pi_b \; i_2 \; \pi_b \; i_3,$$

and the price responsiveness policy ω^+_b is the ultimate price responsiveness policy $\overline{\omega}^+_b$.

Each cadet has a utility function that is drawn from a distribution with the following two elements, u and v, where:

$$u(b,t^0) = 10, \; u(\emptyset) = 8, \; u(b,t^+) = 0, \quad \text{and} \quad v(b,t^0) = 10, \; v(b,t^+) = 8, \; v(\emptyset) = 0.$$

Let us refer to cadets with a utility function $u(.)$ as type 1 and cadets with a utility function $v(.)$ as type 2. All cadets have a utility of 10 for their first choice assignment of (b,t^0), a utility of 8 for their second choice assignment, and a utility of 0 for their last choice assignment. For type 1 cadets, the second choice is remaining unmatched whereas for type 2 cadets the second choice is receiving a position at the increased price t^+. Suppose each cadet can be of the either type with a probability of 50 percent, and they are all expected utility maximizers.

The unique Bayesian Nash equilibrium s^* under the incomplete information game induced by the USMA-2020 mechanism is, for any cadet $i \in \{i_1, i_2, i_3\}$,

$$s^*_i = \begin{cases} \emptyset & \text{if cadet } i \text{ is of type 1, and} \\ b & \text{if cadet } i \text{ is of type 2.} \end{cases}$$

That is, truth-telling is the unique Bayesian Nash equilibrium strategy for each cadet. However, this unique Bayesian Nash equilibrium strategy results in detectable priority reversals whenever either

1. cadet i_1 is of type 1 and cadets i_2, i_3 are of type 2, or
2. cadet i_1 is of type 2 and cadets i_2, i_3 and are of type 1.

While cadet i_2 receives a position at the base price t^0 in both cases, the highest baseline priority cadet i_1 remains unassigned in the first case and receives a position at the increased price t^+ in the second case. ■

B.3 Proofs for Results in Online Appendix Section B

Proof of Proposition 2: Suppose there is only one branch $b \in B$, and fix a profile of cadet preferences $\succ \in \mathcal{Q}^{|I|}$. We first show that the direct mechanism ϕ^{DP} satisfies the five axioms.

Individual rationality: This axiom holds immediately under ϕ^{DP}, since no cadet $i \in I$ is considered for a position at the increased price t^+ unless her submitted preferences is such that $(b,t^+) \succ_i \emptyset$.

Non-wastefulness: Since there is only one branch and we already established *individual rationality*, we can focus on cadets who consider a position at the base price to be acceptable. With this observation, *non-wastefulness* also holds immediately under ϕ^{DP}, since all positions are allocated at Steps 0 and 1 at the base price t^0 either as a final assignment or a tentative one. Tentative assignments from Step 1 may be altered later on by increasing their price to t^+ and possibly changing their recipients, but not by leaving the position unassigned, hence assuring non-wastefulness.

Lack of priority reversals: Under the mechanism ϕ^{DP}, each of the q_b^0 highest π_b-priority cadets are assigned a position at the base price t^0 at Step 0, and each of the next q_b^f highest π_b-priority cadets are tentatively assigned a position at the base price t^0 at Step 1. Tentative positions are lost in Step 2 only if there is excess demand from qualified cadets who are willing to pay the increased price t^+, and starting with the lowest π_b priority cadets with tentative assignments. That assures that, for any $i, j \in I$,

$$\phi_j^{DP}(\succ) = (b, t^0) \succ_i \phi_i^{DP}(\succ) \quad \Longrightarrow \quad j \; \pi_b \; i. \tag{15}$$

Moreover positions at the increased price t^+ are offered to cadets with highest π_b priorities among those (i) who fail to receive a position at the base price t^0 and (ii) who declare the expensive assignment (b, t^+) as acceptable. Therefore, for any $i, j \in I$,

$$\phi_j^{DP}(\succ) = (b, t^+) \succ_i \phi_i^{DP}(\succ) = \emptyset \quad \Longrightarrow \quad j \; \pi_b \; i. \tag{16}$$

Relations 15 and 16 imply that mechanism ϕ^{DP} *has no priority reversals*.

BRADSO-IC: Fix a cadet $i \in I$. For a given profile of preferences for all cadets except cadet i, whether cadet $i \in I$ receives an assignment of (b, t^0) under the mechanism ϕ^{DP} is independent of cadet i's preferences under the mechanism ϕ^{DP}: Cadets who are among the q_b^0 highest π_b-priority cadets in I always receive an assignment at the base price t^0; cadets who are not among the q highest π_b-priority cadets in I never receive an assignment at the base price t^0; and for any cadet i who has one of the highest q but not one of the highest q_b^0 priorities, whether she receives an assignment at the base price t^0 depends on how many lower π_b-priority cadets are both willing to pay the increased price t^+ and also able to "jump ahead of" cadet i through the price responsiveness policy. Hence if a cadet receives a position under ϕ^{DP} at the increased price t^+, changing her reported preferences can only result in losing the position altogether. Therefore mechanism ϕ^{DP} satisfies *BRADSO-IC*.

Enforcement of the price responsiveness policy: The procedure for the mechanism ϕ^{DP} initially assigns all positions to the q_b highest π_b-priority cadets at the base price t^0, although the assignments of the q_b^f-lowest π_b-priority cadets among these awardees are only tentative. Step 2 of the procedure for mechanism ϕ^{DP} ensures that, if any cadet $j \in I$ loses her tentative assignment (b, t^0) from Step 1, then any cadet $i \in I$ who receives an assignment of (b, t^+) is such that

$(i, t^+)\ \omega_b^+\ (j, t^0)$. Therefore,

$$\left.\begin{array}{l} \phi_i^{DP}(\succ) = (b, t^+), \text{ and} \\ (b, t^0) \succ_j \phi_j^{DP}(\succ) \end{array}\right\} \quad \Longrightarrow \quad (i, t^+)\ \omega_b^+\ (j, t^0). \tag{17}$$

Moreover, Step 2 of the same procedure also ensures that, for any $\ell \in \{1, \ldots, q_b^f\}$, the ℓ^{th} lowest π_b-priority cadet i^ℓ with a tentative assignment of (b, t^0) cannot maintain this tentative assignment, for as long as there are at least ℓ lower π_b-priority cadets who are both willing to pay the increased price t^+ and also able to "jump ahead of" the cadet i^ℓ through the price responsiveness policy. Therefore,

$$\left.\begin{array}{l} \phi_i^{DP}(\succ) = (b, t^0), \\ (b, t^+) \succ_j \phi_j^{DP}(\succ), \text{ and} \\ (j, t^+)\ \omega_b^+\ (i, t^0) \end{array}\right\} \quad \Longrightarrow \quad \left|\left\{i' \in I : \phi_{i'}^{DP}(\succ) = (b, t^+)\right\}\right| = q_b^f$$

$$\Longrightarrow \quad \left(\phi^{DP} \setminus \left\{(i, b, t^0)\right\}\right) \bigcup \left\{(j, b, t^+)\right\} \notin \mathcal{A}. \tag{18}$$

Relations (17) and (18) imply that mechanism ϕ^{DP} satisfies *enforcement of the price responsiveness policy*.

Uniqueness: We next show that mechanism ϕ^{DP} is the only mechanism that satisfies all five axioms.

Let the direct mechanism φ satisfy *individual rationality, non-wastefulness, BRADSO-IC, enforcement of the price responsiveness policy*, and *has no priority reversals*. We want to show that $\varphi(\succ) = \phi^{DP}(\succ)$.

If there are less than or equal to q cadets for whom the assignment (b, t^0) is acceptable under the preference profile $\succ$, all such cadets must receive an assignment of (b, t^0) by *individual rationality, non-wastefulness*, and *BRADSO-IC*. Since this is also the case under the allocation $\phi^{DP}(\succ)$, the result holds immediately for this case.

Therefore, w.l.o.g assume that there are strictly more than q cadets for whom the assignment (b, t^0) is acceptable under the preference profile $\succ$. Let I^0 be the set of q_b^0 highest π_b-priority cadets in I. By *non-wastefulness*, all positions are assigned under $\varphi(\succ)$. Since at most q_b^f positions can be awarded at the increased price t^+, at least q_b^0 positions has to be allocated at the base price t^0. Therefore,

$$\text{for any } i \in I^0, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{DP}(\succ) \tag{19}$$

by *lack of priority reversals*.

Let I^1 be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. Relabel the cadets in the set I^1 so that for any $\ell \in \{1, \ldots, q_b^f\}$, cadet i^ℓ is the ℓ^{th}-lowest π_b-priority cadet in I^1. Let

$$J^0 = \left\{j \in I \setminus (I^0 \cup I^1) : (b, t^+) \succ_j \emptyset\right\}.$$

By *individual rationality* and the *lack of priority reversals*,

$$\text{for any } i \in I \setminus (I^0 \cup I^1 \cup J^0), \qquad \varphi_i(\succ) = \emptyset = \phi_i^{DP}(\succ). \tag{20}$$

By relations (19) and (20), the only set of cadets whose assignments are yet to be determined under $\varphi(\succ)$ are cadets in $I^1 \cup J^0$. Moreover, by the *lack of priority reversals*, cadets in J^0 can only receive a position at the increased price t^+. That is,

$$\text{for any } j \in J^0, \qquad \varphi_j(\succ) \neq (b, t^0). \tag{21}$$

For the next phase of our proof, we will rely on the sequence of individuals $i^1, \ldots, i^{q_b^f}$ and the sequence of sets $J^0, J^1, \ldots$ that are constructed for the Step 2 of the mechanism ϕ^{DP}. Here individual i^1 is the q^{th} highest π_b-priority cadet in set I, cadet i^2 is the $(q-1)^{\text{th}}$ highest π_b-priority cadet in set I, and so on. The starting element of the second sequence is $J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^+) \succ_j \emptyset\}$. Assuming Step 2.$n$ is the last sub-step of Step 2, the remaining elements of the latter sequence for $n \geq 1$ is given as follows: For any $\ell \in \{1, \ldots, n\}$,

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^+) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^+) \succ_{i^\ell} \emptyset \end{cases}$$

We have three cases to consider.

Case 1. $n = 0$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{DP}, we have

$$\left|\{j \in J^0 : (j, t^+) \; \omega_b^+ \; (i^1, t^0)\}\right| = 0. \tag{22}$$

Therefore, by relations 20, 21, and condition (1) of the axiom *enforcement of the price responsiveness policy*,

$$\text{for any } i \in I \setminus (I^0 \cup I^1), \qquad \varphi_i(\succ) = \emptyset = \phi_i^{DP}(\succ). \tag{23}$$

Hence by *non-wastefulness*,

$$\text{for any } i \in I^1, \qquad \varphi_i(\succ) \in \{(b, t^0), (b, t^+)\}. \tag{24}$$

But since φ satisfies *individual rationality*, relation (24) implies that $\varphi_i(\succ) = (b, t^0)$ for any $i \in I^1$ with $\emptyset \succ_i (b, t^+)$. Furthermore for any $i \in I^1$ with $(b, t^+) \succ_i \emptyset$, instead reporting the fake preference relation $\succ_i' \in \mathcal{Q}$ with $\emptyset \succ_i' (b, t^+)$ would guarantee cadet i an assignment of $\varphi_i(\succ_{-i}, \succ_i') = (b, t^0)$ due to the same arguments applied for the economy $(\succ_{-i}, \succ_i')$, and therefore by *BRADSO-IC* these cadets too must receive an assignment of (b, t^0) each. Hence

$$\text{for any } i \in I^1, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{DP}(\succ). \tag{25}$$

Relations (19), and (25) imply $\varphi(\succ) = \phi^{DP}(\succ)$, completing the proof for Case 1.■

Case 2. $n \in \{1, \ldots, q_b^f - 1\}$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{DP}, we have

$$\text{for any } \ell \in \{1, \ldots, n\}, \qquad \big|\{j \in J^{\ell-1} : (j, t^+)\ \omega_b^+\ (i^\ell, t^0)\}\big| \geq \ell, \tag{26}$$

and

$$\big|\{j \in J^n : (j, t^+)\ \omega_b^+\ (i^{n+1}, t^0)\}\big| = n. \tag{27}$$

Since mechanism φ satisfies condition (2) of the axiom *enforcement of the price responsiveness policy*, the *lack of priority reversals* and relation 26 imply

$$\text{for any } i \in \{i^1, \ldots, i^n\}, \qquad \varphi_i(\succ) \neq (b, t^0). \tag{28}$$

Therefore, by *non-wastefulness* and relations (19), (20), (21), and (28), at least n positions must be assigned at the increased price t^+.

Moreover, since mechanism φ satisfies *non-wastefulness, lack of priority reversals*, and condition (1) of the axiom *enforcement of the price responsiveness policy*, relation (27) implies

$$\text{for any } i \in \{i^{n+1}, \ldots, i^{q_b^f}\}, \qquad \varphi_i(\succ) \in \{(b, t^0), (b, t^+)\}. \tag{29}$$

But since φ satisfies *individual rationality*, relation (29) implies that $\varphi_i(\succ) = (b, t^0)$ for any $i \in \{i^{n+1}, \ldots, i^{q_b^f}\}$ with $\emptyset \succ_i (b, t^+)$. Furthermore for any $i \in \{i^{n+1}, \ldots, i^{q_b^f}\}$ with $(b, t^+) \succ_i \emptyset$, instead reporting the fake preference relation $\succ'_i \in \mathcal{Q}$ with $\emptyset \succ'_i (b, t^+)$ would guarantee cadet i an assignment of $\varphi_i(\succ_{-i}, \succ'_i) = (b, t^0)$ due to the same arguments applied for the economy $(\succ_{-i}, \succ'_i)$, and therefore by *BRADSO-IC* these cadets must also receive an assignment of (b, t^0) each. Hence

$$\text{for any } i \in \{i^{n+1}, \ldots, i^{q_b^f}\}, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{DP}(\succ). \tag{30}$$

Since we have already shown that at least n positions must be assigned at an increased price of t^+, relation (30) implies that exactly n positions must be assigned this cost, and therefore for any cadet $j \in J^n$ who is one of the n highest π_b-priority cadets in J^n,

$$\varphi_j(\succ) = (b, t^+) = \phi_i^{DP}(\succ) \tag{31}$$

by the *lack of priority reversals*.

Relations (19), (30), and (31) imply $\varphi(\succ) = \phi^{DP}(\succ)$, completing the proof for Case 2. ■

Case 3. $n = q_b^f$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{DP}, we have

$$\text{for any } \ell \in \{1, \ldots, q_b^f\}, \qquad \big|\{j \in J^{\ell-1} : (j, t^+)\ \omega_b^+\ (i^\ell, t^0)\}\big| \geq \ell. \tag{32}$$

Since mechanism φ satisfies condition (2) of the axiom *enforcement of the price responsiveness policy*, relation 32 implies

$$\text{for any } i \in \underbrace{\{i^1, \ldots, i^{q_b^f}\}}_{=I^1}, \qquad \varphi_i(\succ) \neq (b, t^0). \tag{33}$$

Therefore, by *non-wastefulness* and the *lack of priority reversals*, exactly q_b^f positions must be assigned at the increased price t^+. Hence for any cadet $j \in J^{q_b^f}$ who is one of the q_b^f highest π_b-priority cadets in $J^{q_b^f}$,

$$\varphi_j(\succ) = (b, t^+) = \phi_i^{DP}(\succ) \tag{34}$$

by *elimination of priority reversals*.

Relations (19) and (34) imply $\varphi(\succ) = \phi^{DP}(\succ)$, completing the proof for Case 3, thus finalizing the proof of the theorem. ■ □

Proof of Proposition 3: Suppose that there is only one branch $b \in B$. Fixing the profile of cadet preferences $\succ \in \mathcal{Q}$, the baseline priority order π_b, and the price responsiveness policy ω_b^+, consider the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. When there is only one branch, the first part of the strategy space becomes redundant and the second part contains only the two elements b and $\emptyset$. Hence, for any cadet $i \in I$, the strategy space of cadet $i \in I$ under the USMA-2020 mechanism is $\mathcal{S}_i^{2020} = \{\emptyset, b\}$.

For a given strategy profile $s \in \mathcal{S}^{2020}$, construct the priority order $\pi_b^+(s)$ as follows: For any $i, j \in I$,

1. $s_i = s_j \quad \Longrightarrow \quad i \ \pi_b^+(s)\ j \iff i\ \pi_b\ j,$
2. $s_i = b$ and $s_j = \emptyset \quad \Longrightarrow \quad i\ \pi_b^+(s)\ j \iff (i, t^+)\ \omega_b^+\ (j, t^0).$

Let $I^+(s)$ be the set of q_b highest $\pi_b^+(s)$-priority cadets in I.

For any cadet $i \in I$, the outcome of the USMA-2020 mechanism is given as,

$$\varphi_i^{2020}(s) = \begin{cases} \emptyset & \text{if } \ i \notin I^+(s), \\ (b, t^0) & \text{if } \ i \in I^+(s) \text{ and } s_i = \emptyset, \\ (b, t^0) & \text{if } \ i \in I^+(s) \text{ and } s_i = b \ \text{ and } \ \big|\{j \in I^+(s) : s_j = b \text{ and } i\ \pi_b\ j\}\big| \geq q_b^f, \\ (b, t^+) & \text{if } \ i \in I^+(s) \text{ and } s_i = b \ \text{ and } \ \big|\{j \in I^+(s) : s_j = b \text{ and } i\ \pi_b\ j\}\big| < q_b^f. \end{cases}$$

We first prove a lemma on the structure of Nash equilibrium strategies of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$.

Lemma 3. *Let s^* be a Nash equilibrium of the strategic-form game induced by the mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. Then, for any $i, j \in I$,*

$$\varphi_j^{2020}(s^*) \succ_i \varphi_i^{2020}(s^*) \quad \Longrightarrow \quad j\ \pi_b\ i.$$

Proof of Lemma 3: Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. Contrary to the claim suppose that, there exists $i, j \in I$ such that

$$\varphi_j^{2020}(s^*) \succ_i \varphi_i^{2020}(s^*) \quad \text{and} \quad i \; \pi_b \; j.$$

There are three possible cases, where in each case we reach a contradiction by showing that cadet i has a profitable deviation by mimicking the strategy of cadet j:

***Case 1*:** $\varphi_j^{2020}(s^*) = (b, t^0)$ and $\varphi_i^{2020}(s^*) = (b, t^+)$.

Since by assumption $\varphi_i^{2020}(s^*) = (b, t^+)$,

$$s_i^* = b.$$

Moreover the assumptions $\varphi_j^{2020}(s^*) = (b, t^0),\; \varphi_i^{2020}(s^*) \neq (b, t^0),$ and $i \; \pi_b \; j$ imply

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = \emptyset. \tag{35}$$

But then, relation (35) and the assumption $i \; \pi_b \; j$ imply that, for the alternative strategy $\hat{s}_i = \emptyset$ for cadet i,

$$i \in I^+(s_{-i}^*, \hat{s}_i),$$

and thus

$$\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0) \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy. This completes the proof for Case 1. ■

***Case 2*:** $\varphi_j^{2020}(s^*) = (b, t^0)$ and $\varphi_i^{2020}(s^*) = \emptyset$.

Since by assumption $\varphi_j^{2020}(s^*) = (b, t^0),\; \varphi_i^{2020}(s^*) = \emptyset,$ and $i \; \pi_b \; j$, we must have

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = b \quad \text{and} \quad \big|\{k \in I^+(s^*) : s_k^* = b \text{ and } j \; \pi_b \; k\}\big| \geq q_b^f, \tag{36}$$

and

$$s_i^* = \emptyset.$$

But then, relation (36) and the assumption $i \; \pi_b \; j$ imply that, for the alternative strategy $\hat{s}_i = b$ for cadet i,

$$i \in I^+(s_{-i}^*, \hat{s}_i) \quad \text{and} \quad \hat{s}_i = b \quad \text{and} \quad \big|\{k \in I^+(s_{-i}^*, \hat{s}_i) : s_k^* = b \text{ and } i \; \pi_b \; k\}\big| \geq q_b^f,$$

and thus

$$\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0) \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy. This completes the proof for Case 2. ■

***Case 3*:** $\varphi_j^{2020}(s^*) = (b, t^+)$ and $\varphi_i^{2020}(s^*) = \emptyset$.

Since by assumption $\varphi_j^{2020}(s^*) = (b, t^+)$,

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = b. \tag{37}$$

Moreover, since $\varphi_i^{2020}(s^*) = \emptyset$ by assumption,

$$i \notin I^+(s^*).$$

Therefore, since $i \ \pi_b \ j$ by assumption,

$$j \in I^+(s^*) \ \text{ and } \ i \notin I^+(s^*) \qquad \Longrightarrow \qquad s_i^* = \emptyset.$$

But then, again thanks to assumption $i \ \pi_b \ j$, the relation (37) implies that, for the alternative strategy $\hat{s}_i = b$ for cadet i,

$$i \in I^+(s^*_{-i}, \hat{s}_i),$$

and thus

$$\underbrace{\varphi_i^{2020}(s^*_{-i}, \hat{s}_i)}_{\in\{(b,t^0),(b,t^+)\}} \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy,[35] completing the proof for Case 3, and concluding the proof of Lemma 3. ■ ♢

For the next phase of our proof, we rely on the construction in the Step 2 of the mechanism ϕ^{DP}: Let I^0 be the set of q_b^0 highest π_b-priority cadets in I, and I^1 be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. Relabel the set of cadets in I^1, so that i^1 is the lowest π_b-priority cadet in I^1, i^2 is the second lowest π_b-priority cadet in I^1,..., and $i^{q_b^f}$ is the highest π_b-priority cadet in I^1. Note that, cadet i^1 is the q^{th} highest π_b-priority cadet in set I, cadet i^2 is the $(q-1)^{\text{th}}$ highest π_b-priority cadet in set I, and so on. Let $J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^+) \succ_j \emptyset\}$. Assuming Step 2.$n$ is the last sub-step of Step 2 of the mechanism ϕ^{DP}, for any $\ell \in \{1, \ldots, n\}$, let

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^+) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^+) \succ_{i^\ell} \emptyset \end{cases}$$

Recall that, under the mechanism ϕ^{DP}, exactly n cadets receive an assignment of (b, t^+). We will show that, the same is also the case under the Nash equilibria of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$.

Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. We have three cases to consider:

Case 1: $n = 0$

[35] Unlike the first two cases, in this case cadet i may even get a better assignment than cadet j (i.e. cadet i may receive an assignment of (b, t^0)) by mimicking cadet j's strategy.

Since by assumption $n = 0$ in this case,

$$\{j \in J^0 : (j, t^+) \ \omega_b^+ \ (i^1, t^0)\} = \emptyset. \tag{38}$$

Towards a contradiction, suppose there exists a cadet $i \in I \setminus (I^0 \cup I^1)$ such that $i \in I^+(s^*)$. Since cadet i^1 is the q^{th} highest π_b-priority cadet in I, the assumption $i \in I^+(s^*)$ and relation (38) imply

$$i \notin J^0 \implies \emptyset \succ_i (b, t^+). \tag{39}$$

Moreover, since cadet i is not one of the q highest π_b-priority cadets in I,

$$i \in I^+(s^*) \implies s_i^* = b. \tag{40}$$

But this means cadet i can instead submit an alternative strategy $\hat{s}_i = \emptyset$, assuring that she remains unmatched, contradicting s^* is a Nash equilibrium. Therefore,

$$\text{for any } i \in I \setminus (I^0 \cup I^1), \qquad (i, t^+) \ \omega_b^+ \ (i^1, t^0) \implies s_i^* = \emptyset, \tag{41}$$

which in turn implies

$$I^+(s^*) = I^0 \cup I^1. \tag{42}$$

Hence all cadets in $I^0 \cup I^1$ receive a position under $\varphi^{2020}(s^*)$. Next consider the lowest π_b-priority cadet $i \in I^0 \cup I^1$ such that $\varphi_i^{2020}(s^*) = (b, t^+)$. This can only happen if $s_i^* = b$. But this means cadet i can instead submit an alternative strategy $\hat{s}_i = \emptyset$, assuring that $\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0)$ by relation (41), contradicting s^* is a Nash equilibrium. Hence

$$\text{for any } i \in I^0 \cup I^1, \qquad \varphi_i^{2020}(s^*) = (b, t^0) = \phi_i^{DP}(\succ), \tag{43}$$

and therefore $\varphi^{2020}(s^*) = \phi^{DP}(\succ)$.

Finally observe that the strategy profile s' where $s_i' = \emptyset$ for any cadet $i \in I$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{DP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 1. ■

For any $\ell \in \{1, \ldots, n\}$, let $\overline{J^\ell}$ be the set of ℓ highest π_b-priority cadets in the set J^ℓ:

$$\overline{J^\ell} = \left\{ j \in J^\ell : \ \left| \{ i \in J^\ell : \ i \ \pi_b \ j \} \right| < \ell \right\}$$

Before proceeding with the next two cases, we prove the following lemma that will be helpful for both cases.

Lemma 4. *Suppose there are $n > 0$ positions allocated at the increased price t^+ under the allocation $\phi^{DP}(\succ)$. Then, for any Nash equilibrium s^* of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$ and $\ell \in \{1, \ldots, n\}$,*

1. $\varphi^{2020}_{i^\ell}(s^*) = (b, t^+) \iff (b, t^+) \succ_{i^\ell} \emptyset$, *and*

2. $\varphi^{2020}_{i}(s^*) = (b, t^+)$ *for any* $i \in \overline{J^\ell}$.

Proof of Lemma 4: Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. First recall that,

$$\text{for any } j \in I \setminus (I^0 \cup I^1), \qquad \varphi^{2020}_j(s^*) \in \{(b, t^+), \emptyset\},$$

and therefore, since any cadet $j \in I \setminus (I^0 \cup I^1 \cup J^0)$ prefers remaining unmatched to receiving a position at the increased price t^+ and she can assure remaining unmatched by submitting the strategy $s_j = \emptyset$,

$$\text{for any } j \in I \setminus (I^0 \cup I^1 \cup J^0), \qquad \varphi^{2020}_j(s^*) = \emptyset. \tag{44}$$

Also, by the mechanics of the Step 2 of the mechanism ϕ^{DP},

$$\text{for any } \ell \in \{1, \ldots, n\}, \qquad \left|\{j \in J^{\ell-1} : (j, t^+) \ \omega^+_b \ (i^\ell, t^0)\}\right| \geq \ell. \tag{45}$$

The proof of the lemma is by induction on ℓ. We first prove the result for $\ell = 1$.

Consider the highest π_b-priority cadet j in the set $\{j \in J^0 : (j, t^+) \ \omega^+_b \ (i^1, t^0)\}$. By relation 45, such a cadet exists.

First assume that $(b, t^+) \succ_{i^1} \emptyset$. In this case, $J^1 = J^0 \cup \{i^1\}$ and cadet i^1 is the highest π_b-priority cadet in J^1. Hence $\overline{J^1} = \{i^1\}$ in this case. Consider the Nash equilibrium strategies of cadet i^1 and cadet j. If $s^*_{i^1} = \emptyset$, then by relation (44) her competitor cadet j can secure himself an assignment of (b, t^+) by reporting a strategy of $s_j = b$, which would mean cadet i^1 has to remain unassigned, since by Lemma 3 no cadet in $I^0 \cup I^1$ can envy the assignment of cadet i^1 at Nash equilibria. In contrast, reporting a strategy of $s_{i^1} = b$ assures that cadet i^1 receives a position, which is preferred at any price to remaining unmatched by assumption $(b, t^+) \succ_{i^1} \emptyset$. Therefore, $s^*_{i^1} = b$, and hence

$$(b, t^+) \succ_{i^1} \emptyset \quad \Longrightarrow \quad \begin{cases} \varphi^{2020}_{i^1}(s^*) = (b, t^+), \text{ and} \\ \varphi^{2020}_{i}(s^*) = (b, t^+) \quad \text{for any } i \in \overline{J^1} = \{i^1\}. \end{cases} \tag{46}$$

Next assume that $\emptyset \succ_{i^1} (b, t^+)$. In this case $J^1 = J^0$ and cadet j is the highest π_b-priority cadet in J^1. Hence $\overline{J^1} = \{j\}$ in this case. By Lemma 3, no cadet in $(I^0 \cup I^1) \setminus \{i^1\}$ can envy the assignment of cadet i^1 at Nash equilibria. Therefore, a strategy of $s_{i^1} = b$ means that cadet i receives an assignment of (b, t^+), which is inferior to remaining unmatched by assumption. Therefore $s^*_{i^1} = \emptyset$. Moreover reporting a strategy of $s_j = \emptyset$ means that cadet j remains unmatched, whereas reporting a strategy of $s_j = b$ assures that she receives an assignment of (b, t^+), which is preferred to remaining unmatched since $j \in J^0$. Therefore, $s^*_{i^1} = \emptyset$, and hence

$$\emptyset \succ_{i^1} (b, t^+) \quad \Longrightarrow \quad \begin{cases} \varphi^{2020}_{i^1}(s^*) = \emptyset, \text{ and} \\ \varphi^{2020}_{i}(s^*) = (b, t^+) \quad \text{for any } i \in \overline{J^1} = \{j\}. \end{cases} \tag{47}$$

Relations (46) and (47) complete the proof for $\ell = 1$.

Next assume that the inductive hypothesis holds for $\ell = k < n$. We want to show that the result holds for $\ell = (k+1)$ as well.

By the inductive hypothesis,

$$\text{for any } i \in \overline{J^k}, \quad \varphi_i^{2020}(s^*) = (b, t^+). \tag{48}$$

By relation 45, there are at least $k+1$ cadets in the set $\{j \in J^k : (j, t^+) \ \omega_b^+ \ (i^{k+1}, t^0)\}$. Therefore, since there are k cadets in the set $\overline{J^k}$, there is at least one cadet in the set

$$\{j \in J^k : (j, t^+) \ \omega_b^+ \ (i^{k+1}, t^0)\} \setminus \overline{J^k}.$$

Let j be the highest π_b-priority cadet in this set.

First assume that $(b, t^+) \succ_{i^{k+1}} \emptyset$. In this case $J^{k+1} = J^k \cup \{i^{k+1}\}$ and cadet i^{k+1} is the highest π_b-priority cadet in J^{k+1}. Hence $\overline{J^{k+1}} = \overline{J^k} \cup \{i^{k+1}\}$ in this case. Consider the Nash equilibrium strategies of cadet i^{k+1} and cadet j. If $s^*_{i^{k+1}} = \emptyset$, then by relation (44) cadet j can secure herself an assignment of (b, t^+) by reporting a strategy of $s_j = b$, which would mean cadet i^{k+1} has to remain unassigned, since by Lemma 3 no cadet in $(I^0 \cup I^1) \setminus \{i^1, \ldots, i^k\}$ can envy the assignment of cadet i^{k+1} at Nash equilibria and by relation (48) all cadets in $\overline{J^k}$ receive an assignment of (b, t^+).[36] In contrast, reporting a strategy of $s_{i^{k+1}} = b$ assures that cadet i^{k+1} receives a position, which is preferred at any price to remaining unmatched by assumption $(b, t^+) \succ_{i^{k+1}} \emptyset$. Therefore, $s^*_{i^{k+1}} = b$, and hence

$$(b, t^+) \succ_{i^{k+1}} \emptyset \implies \begin{cases} \varphi_{i^{k+1}}^{2020}(s^*) = (b, t^+), \text{ and} \\ \varphi_i^{2020}(s^*) = (b, t^+) \text{ for any } i \in \overline{J^{k+1}} = \overline{J^k} \cup \{i^{k+1}\}. \end{cases} \tag{49}$$

Next assume that $\emptyset \succ_{i^{k+1}} (b, t^+)$. In this case $J^{k+1} = J^k$ and $\overline{J^{k+1}} = \overline{J^k} \cup \{j\}$. By Lemma 3, no cadet in $I^0 \cup I^1 \setminus \{i^1, \ldots, i^k\}$ can envy the assignment of cadet i^{k+1} at Nash equilibria. Therefore, since all cadets in $\overline{J^k}$ receive an assignment of (b, t^+) by relation (48), a strategy of $s_{i^{k+1}} = b$ means that cadet i^{k+1} receives an assignment of (b, t^+), which is inferior to remaining unmatched by assumption. Therefore $s^*_{i^{k+1}} = \emptyset$. Moreover reporting a strategy of $s_j = \emptyset$ means that cadet j remains unmatched, whereas reporting a strategy of $s_j = b$ assures that she receives an assignment of (b, t^+), which is preferred to remaining unmatched since $j \in J^k$. Therefore, $s^*_{i^{k+1}} = \emptyset$, and hence

$$\emptyset \succ_{i^{k+1}} (b, t^+) \implies \begin{cases} \varphi_{i^{k+1}}^{2020}(s^*) = \emptyset, \text{ and} \\ \varphi_i^{2020}(s^*) = (b, t^+) \quad \text{for any } i \in \overline{J^{k+1}} = \overline{J^k} \cup \{j\}. \end{cases} \tag{50}$$

Relations (49) and (50) complete the proof for $\ell = k+1$, and conclude the proof of Lemma 4. $\diamondsuit$

We are ready to complete prove the theorem for our last two cases:

[36] Since $\left|(I^0 \cup I^1) \setminus \{i^1, \ldots, i^k\}\right| = (q-k)$ and $\left|\overline{J^k}\right| = k$, this basically means cadets i^{k+1} and j are competing for a single position.

Case 2. $n \in \{1, \ldots, q_b^f - 1\}$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{DP},

$$\left|\{j \in J^n : (j, t^+) \ \omega_b^+ \ (i^{n+1}, t^0)\}\right| = n. \tag{51}$$

Consider cadet i^{n+1}. There are $q - (n+1)$ cadets with higher π_b-priority, and by relation (51) there are n cadets in J^n whose increased price assignments have higher ω_b^+ priority under the price responsiveness policy than the base-price assignment for cadet i^{n+1}. For any other cadet $i \in I \setminus \Big(J^n \cup I^0 \cup \big(I^1 \setminus \{i^1, \ldots, i^{n+1}\}\big)\Big)$ with $(i, t^+) \ \omega_b^+ \ (i^{n+1}, t^0)$, we must have $\emptyset \succ_i (b, t^+)$ since $J^n \supseteq J^0$. Therefore none of these individuals can receive an assignment of (b, t^+) under a Nash equilibrium strategy, and hence the number of cadets who can have higher $\pi_b^+(s^*)$-priority than cadet is i^{n+1} is at most $q - (n+1) + n = q - 1$ under any Nash equilibrium strategy. That is, cadet $i^{n+1} \in I^+(s^*)$ regardless of her submitted strategy, and therefore,

$$\varphi_{i^{n+1}}^{2020}(s^*) = (b, t^0), \tag{52}$$

since her best response $s^*_{i^{n+1}}$ to $s^*_{-i^{n+1}}$ results in an assignment of (b, t^0). Moreover, Lemma 3 and relation (52) imply that, for any cadet $i \in I^0 \cup \big(I^1 \setminus \{i^1, \ldots, i^{n+1}\}\big)$,

$$\varphi_i^{2020}(s^*) = (b, t^0). \tag{53}$$

Hence Lemma 4 and relations (52), (53) imply $\varphi^{2020}(s^*) = \phi^{DP}(\succ)$.

Finally, the strategy profile s' where $s'_i = b$ for any cadet $i \in J^n$ and $s'_j = \emptyset$ for any cadet $j \in I \setminus J^n$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{DP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 2. ■

Case 3. $n = q_b^f$

Since at most q_b^f positions can be assigned at the increased price t^+, Lemma 3 and Lemma 4 immediately imply $\varphi^{2020}(s^*) = \phi^{DP}(\succ)$.

Finally the strategy profile s' where $s'_i = b$ for any cadet $i \in J^{q_b^f} \cup I^0$ and $s'_j = \emptyset$ for any cadet $j \in I \setminus \big(J^n \cup I^0\big)$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{DP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 3, and the proof of the proposition. ■ □

Proof of Corollary 1: Since *BRADSO-IC* is implied by *strategy-proofness*, Corollary 1 is a direct implication of Theorem 1 and Proposition 2. □

C Formal Description of USMA-2006 Mechanism and Individual-Proposing Deferred Acceptance Algorithm

C.1 USMA-2006 Mechanism

The USMA-2006 mechanism is a quasi-direct mechanism with the following strategy space:

$$\mathcal{S}^{2006} = \left(\mathcal{P} \times 2^B\right)^{|I|}.$$

The following construction is useful to formulate the outcome function for the USMA-2006 mechanism:

Given an OML π and a strategy profile $s = (P_i, B_i)_{i \in I} \in \mathcal{S}^{2006}$, for any branch $b \in B$ construct the following adjusted priority order $\pi_b^+ \in \Pi$ on the set of cadets I. For any pair of cadets $i, j \in I$,

1. $b \in B_i$ and $b \in B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j \iff i \; \pi \; j,$
2. $b \notin B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j \iff i \; \pi \; j,$ and
3. $b \in B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j.$

Under the adjusted priority order π_b^+, any pair of cadets are rank ordered through the OML π if they have indicated the same willingness to pay the increased price for branch b, and through the ultimate price responsiveness policy $\overline{\omega}_b^+$ (which gives higher priority to the cadet who has indicated to pay the increases price) otherwise.

Given an OML $\pi \in \Pi$ and a strategy profile $s = (P_i, B_i)_{i \in I} \in \mathcal{S}^{2006}$, the outcome $\varphi^{2006}(s)$ of the **USMA-2006 mechanism** is obtained with the following sequential procedure:

> ***Branch assignment***: At any step $\ell \geq 1$ of the procedure, the highest π-priority cadet i who is not tentatively on hold for a position at any branch applies to her highest-ranked acceptable branch b under her submitted branch preferences P_i that has not rejected her from earlier steps.[37]
>
> Branch b considers cadet i together with all cadets it has been tentatively holding both for its q_b^0 base-price positions and also for its q_b^f flexible-price positions, and
>
> 1. it tentatively holds (up to) q_b^0 highest π-priority applicants for one of its q_b^0 base-price positions,
> 2. among the remaining applicants it tentatively holds (up to) q_b^f highest π_b^+-priority applicants for one of its q_b^f flexible-price positions, and
> 3. it rejects any remaining applicant.

[37]The USMA-2006 mechanism can also be implemented with a variant of the algorithm where each cadet who is not tentatively holding a position simultaneously apply to her next choice branch among branches that has not rejected her application.

The procedure terminates when no applicant is rejected. Any cadet who is not tentatively on hold at any brach remains unmatched, and all tentative branch assignments are finalized.

Price assignment: For any branch $b \in B$,

1. any cadet $i \in I$ who is assigned one of the q_b^0 base-price positions at branch b is charged the base price t^0, and
2. any cadet $i \in I$ who is assigned one of the q_b^f flexible-price positions is charged
   (a) the increased price t^+ if $b \in B_i$, and
   (b) the base price t^0 if $b \notin B_i$.

C.2 Individual-Proposing Deferred Acceptance Algorithm

The USMA-2020 mechanism was based on the individual-proposing deferred acceptance algorithm (Gale and Shapley, 1962). Given a ranking over branches, the individual-proposing deferred acceptance algorithm (DA) produces a matching as follows.

Individual-Proposing Deferred Acceptance Algorithm (DA)

Step 1: Each cadet applies to her most preferred branch. Each branch b tentatively assigns applicants with the highest priority until all cadets are chosen or all q_b slots as assigned and permanently rejects the rest. If there are no rejections, then stop.

Step k: Each cadet who was rejected in Step k-1 applies to her next preferred branch, if such a branch exists. Branch b tentatively assigns cadets with the highest priority until all all cadets are chosen or all q_b slots are assigned and permanently rejects the rest. If there are no rejections, then stop.

The algorithm terminates when there are no rejections, at which point all tentative assignments are finalized.

D Cadet Data and Survey Appendix

D.1 Data Appendix

Our data cover the West Point Classes of 2014 through 2021. We present two tables about data processing. The first table reports summary statistics on branches for the Class of 2020 and Class of 2021. The second table presents summary information about mechanism replication for the Classes of 2014-2021.

Table 1: Branches and Applications for Classes of 2020 and 2021

	Class of 2020			Class of 2021		
	Number Assigned	Ranked First	BRADSO Willing	Number Assigned	Ranked First	BRADSO Willing
Branch	(1)	(2)	(3)	(4)	(5)	(6)
Air Defense	61	5	16	54	11	8
Adjutant General	2	57	30	19	43	25
Armor	112	80	19	97	75	34
Aviation	109	129	93	88	116	1
Chemical	2	9	9	8	5	2
Cyber	40	42	32	40	51	40
Engineer	139	147	49	124	145	62
Explosive Ordnance Disposal				10	27	7
Field Artillery	173	147	17	154	77	17
Finance	1	6	16	5	8	6
Infantry	237	206	26	207	197	74
Military Intelligence	58	72	92	65	87	58
Military Police	18	27	32	12	18	13
Medical Services	7	8	0	19	27	28
Ordinance	36	37	30	14	19	23
Quartermaster	26	26	49	17	57	37
Signal Corp	43	67	60	44	26	22
Transportation Corp	25	24	27	17	5	10
Total	1089	1089	597	994	994	467

Notes. This table reports information on branches for the Class of 2020 and 2021. Number Assigned equals the capacity of the branch. Ranked First is the number of cadets ranking the branch as their highest rank choice. BRADSO Willing is the number of cadets who rank a BRADSO contract at the branch anywhere on their rank order list. Explosive Ordnance Disposal was not a branch option for the Class of 2020.

Table 2: Mechanism Replication Rate

			Percent Correct	
	Total Applicants	Number Incorrect	Branch	BRADSO
Applicant Class	(1)	(2)	(3)	(4)
2014	1006	28	97.2%	98.1%
2015	976	4	99.6%	100.0%
2016	951	11	98.8%	99.6%
2017	944	2	99.8%	100.0%
2018	963	11	98.9%	99.6%
2019	931	4	99.6%	100.0%
2020	1089	0	100.0%	100.0%
2021	994	0	100.0%	100.0%
All	7854	60	99.2%	99.7%

Notes. This table reports the replication rate of the USMA assignment mechanism across years. The USMA-2006 mechanism is used for the Classes of 2014-2019, USMA-2020 mechanism is used for the Class of 2020, and the Dual-Price Cumulative Offer mechanism is used for the Class of of 2021. Number incorrect are the number of cadets who obtain a different assignment under our replication. Branch percent correct is the number of branch assignments that we replicate. BRADSO percent correct is the number of BRADSO assignments we replicate.

D.2 Cadet Survey Questions and Answers

In fall 2020, the Army administered a survey of cadets. This survey asked two questions related to assignment mechanisms, one on cadet understanding of USMA-2020 and the other on cadet preferences over assignment mechanisms. This section reports the questions and the distribution of survey responses.

Question 1. *What response below best describes your understanding of the impact of volunteering to BRADSO for a branch in this year's branching process?*

A. I am more likely to receive the branch, but I am only charged a BRADSO if I would have failed to receive the branch had I not volunteered to BRADSO. (43.3% of respondents)

B. I am charged a BRADSO if I receive the branch, regardless of whether volunteering to BRADSO helped me receive the branch or not. (9.5% of respondents)

C. I am more likely to receive the branch, but I may not be charged a BRADSO if many cadets who receive the same branch not only rank below me but also volunteer to BRADSO. (38.8% of respondents)

D. I am more likely to receive the branch, but I do not know how the Army determines who is charged a BRADSO. (6.7% of respondents)

E. I am NOT more likely to receive the branch even though I volunteered to BRADSO. (1.8 percent of respondents)

38.8% of cadets answered the correct answer (answer C). 43.3% of cadets believed that the 2020 mechanism would only charge a BRADSO if required to receive the branch (answer A)

Question 2. *A cadet who is charged a BRADSO is required to serve an additional 3 years on Active Duty. Under the current mechanism, cadets must rank order all 17 branches and indicate if they are willing to BRADSO for each branch choice. For example:*

- **Current Mechanism Example**:
  - **1: AV/BRADSO, 2: EN, 3: CY**

Under an alternative mechanism, cadets could indicate if they prefer to receive their second branch choice without a BRADSO charge more than they prefer to receive their first branch choice with a BRADSO charge. For example:

- **Alternative Mechanism Example**:
  - **1: AV, 2: EN, 3: AV/BRADSO, 4: CY**

When submitting branch preferences, which mechanism would you prefer?

- A. Current Mechanism (21.4% of respondents)
- B. Alternative Mechanism (49.7% of respondents)
- C. Indifferent (24.2% of respondents)
- D. Do Not Understand (4.8% of respondents)

E Potential Applications of Price Responsive Policies in Priority-Based Assignment

Our model and main application are inspired by the reform of the U.S. Army's cadet assignment system, but we believe that price responsive policies could have several other potential applications.

E.1 Talent Alignment and Retention in Priority-Based Assignment Markets

1. *Diplomat / Foreign Service Officer Placement*

   Each year, thousands of applicants compete for diplomatic positions at more than 285 U.S. embassies and consulates around the world. Prioritization is based on scores on the foreign service officer test, with additional points given for applicants based on veterans or disability status and foreign language ability (US State Department, 2019). In this market, a price-responsiveness policy where willingness to work for an extended tour in exchange for a priority boost could help manage retention and talent alignment.

2. *Civil Service Placement*

   Governments around the world use centralized systems to place personnel into positions. For example, Khan, Khwaja, and Olken (2019) describe the use of a centralized assignment mechanism to assign property tax inspectors in Pakistan. They designed a scheme where priority was determined by past performance as an inspector. In such a scheme, a price-responsiveness policy where an willingness to sign an extended service commitment generates increased priority in assignment could help manage retention and talent alignment.

   Bar, Bayer, Rim, Rivera, and Sidibe (2021) describe the process used to assign police officers to positions in other districts in Chicago. The priority is based on officer seniority. A challenge in this setting is lack of demand for working in unsafe neighborhoods and oversubscription in safe neighborhoods. The officer assignment board may be able to use this oversubscription to increase retention by awarding desirable positions to officers who are willing to extend their time in a positing in exchange for higher priority.

3. *Centralized Teacher Assignment*

   Centralized schemes are used in teacher placement in several countries including in Czech Republic, France, Germany, Mexico, Peru, Portugal, Turkey, and Uruguay Combe, Terceiux, and Terrier (2020). In these markets, teachers priority is often based on seniority. The central administration aspires to assign teachers respecting their preferences, while at the same time avoiding a surplus of inexperienced teachers in disadvantaged areas. Ajzenman, Bertoni, Elacqua, Marotta, and Vargas (2020) and Bertoni, Elacqua, Hincapié, Méndez, and Paredes (2021) use data to describe Peru's national teacher selection process. In that system, teachers can rank up to 5 schools and performance on a standardized test is used for prioritization.

Since there is oversubscription in advantaged regions of the country, a price responsiveness policy where lower performing teachers can buy priority by extending their service commitment could cause some more experienced teachers to be assigned to less advantaged regions.

4. *Other Military Sectors: Marines Corps and Air Force*

   Centralized placement is also widespread in the military, aside from the United States Army. Graduates of the U.S. Air Force Academy obtain their career field using a centralized mechanism where cadets rank fields (Armacost and Lowe, 2005). The Air Force judges success of their placement process based on retention-related outcomes and an Airman's fit (NASEM, 2021). Likewise, the U.S. Marine Corps struggles with turnover of marines, and a 2021 manpower report describes creating a digital talent marketplace to address this retention concern and balance the needs of units (United States Marine Corps, 2021). Both of these markets are situations where the flexibility of a price responsiveness policy may facilitate a balance between talent alignment and retention.

E.2 Priority-Based Assignment with or without Amenities

Our first examples use a price-responsiveness policy as a tool to manage retention-related outcomes. Here we describe two examples where the mechanism could unbundle the assignment into an assignment under two terms to manage resource constraints. First, nearly 15,000 officers and 500 units in the Army participate in the Army Talent Alignment Process (U.S. Department of the United States Army, 2019b).[38] In 2020, this system used officer preferences and a version of the deferred acceptance algorithm for placement into units (Greenberg, Crow, and Wojtaszek, 2020). In this market, when an officer is assigned outside the U.S., they must reside in government-controlled military family housing. However, not all officers may wish to bring their families abroad and may not require this housing. Hence, the system could offer job assignment with and without family housing, with the base price corresponding to housing and the increased price corresponding to no family housing. In places where there is scarcity of family housing options, a price responsiveness policy could allow an officer who is willing to forego family housing to buy priority for a position over an officer who needs family housing. The same concept could apply for college admissions, where a student can be assigned with the right to on-campus housing or without the right to on-campus housing.

Second, consider student assignment at K-12 as in Abdulkadiroğlu and Sönmez (2003). In that framework, students are assigned schools, and each position at a school is identical. However, a school position can be offered to a student under different terms. For example, for kindergarten and pre-kindergarten, a school can sometimes offer a full-day or half-day option. These two terms correspond to the base price and the increased price. A price responsiveness policy where an

[38]The cadet-branch assignment process is used for the first assignment and this process is used for unit assignment and is managed by Human Resources Command, and not the OEMA office.

applicant can buy priority if she is willing be assigned a half-day option is an instrument that would allow certain lower priority applicants to access a sought-after school for a half-day that they could not otherwise access. It is possible to envision similar ideas, like offering options for a school with an early start time or late start time (a common way to manage overcrowding), or offering a school with meal or without meal service. A price responsiveness policy in these cases would allow applicants willing to take the increased cost option (e.g., starting school early for some or attending school without free breakfast) in exchange for increased priority. If these ideas are used within the context of a centralized mechanism, then our axioms are natural and imply that the DPCO is the only possible mechanism.