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Common *p*-Belief: The General Case

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We develop belief operators for information systems where individuals have an uncountable number of possible signals, and we give a general version of Monderer and Samet's (1989) theorem relating "iterative" and "fixed point" notions of common p-belief. *Journal of Economic Literature* Classification Numbers: D82.

1. INTRODUCTION

An event is "common p-belief" if everyone believes it with probability at least p, everyone believes with probability at least p that everyone believes it with probability at least p, and so on *ad infinitum*. Monderer and Samet (1989) [hereafter, MS] provide a characterization of common p-belief which relates this *iterative* definition to the following *fixed*-point definition:[1] An event is said to be p-evident if whenever it is true, everyone believes it with probability at least p. An event E is common p-belief at state ω if and only if ω is an element of some p-evident event F with the property that everyone believes E with probability at least p whenever F is true. Since common 1-belief is essentially equivalent to the usual notion of common knowledge, this result is a generalization of Aumann's classic result[2] giving a fixed point characterization of common knowledge.

Common p-belief has been shown to be a natural notion of "almost common knowledge": for p sufficiently close to one, economic outcomes

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[1] Stinchcombe (1988) independently introduced an alternative notion of almost common knowledge.

[2] Aumann (1976).

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are similar to outcomes under common knowledge.[3] Unfortunately the MS result requires each information set of each individual to have positive probability and thus each individual to have at most a countable number of possible signals. These assumptions remove the indeterminacy of conditional probability at particular states and so make it possible to define a *belief operator* which specifies at which states a given event is believed with probability p. This simplifies both the technical analysis and the interpretation, but excludes important applications.

In an analysis of common 1-belief, Nielsen (1984) identified sets which differ only by zero measure sets. In this note, we follow this approach and define belief operators on classes of equivalent events (rather than on events directly). We prove a straightforward analogue of the MS result using such belief operators. This approach has the attractive feature that all definitions and results are independent of any particular version of conditional probability: there is a unique representation of belief operators despite the multiplicity of conditional probabilities. However, we emphasize that an alternative interpretation of our results is that we are confirming that it does not matter (almost surely) if a particular conditional probability is fixed.

The note is organized as follows. After introducing some measure theoretic properties in Section 2, we then define class based belief operators (in subsection 3.1) and use them to define and characterize common p-belief (in subsection 3.2); this characterization includes an uncountable signals version of Monderer and Samet's theorem. In Section 4, we discuss the application of the approach to a simple example.

2. SETUP

Throughout the paper we will fix a set Ω, a σ-field $\mathscr{F}$ on Ω, a probability measure P on $[\Omega, \mathscr{F}]$, a finite collection of individuals $\mathscr{I}$, with sub σ-fields representing the information of each individual i, $\{\mathscr{F}_i\}_{i \in \mathscr{I}}$. We will identify events which differ only by zero probability events. We first summarize some measure theoretic facts, which are versions of standard results that can be found in, for instance, Halmos (1950). Although our terminology is slightly different, we are following Nielsen's (1984) work on knowledge (1-belief), who in turn attributes the idea to Koopman (1940) and Halmos (1944).

Say two measurable sets E, F are *equivalent* (under P) if $P(E \Delta F) = 0$, where $E \Delta F$ denotes the symmetric difference of E and F. We shall write

[3] This has been shown for games of incomplete information (MS), no disagreement results (MS), no trade results (Sonsino, 1995), and learning (Monderer and Samet, 1995).

$E \sim F$ if E and F are equivalent. It is easy to show that the relation $\sim$ is an equivalence relation on $\mathscr{F}$. Let $\mathbf{E}$ denote the equivalent class of events that contains event E. Then $E \sim F$ if and only if $\mathbf{E} = \mathbf{F}$. Denote the set of equivalent classes by $\tilde{\mathscr{F}}$; that is, $\tilde{\mathscr{F}} = \mathscr{F}/\sim$.

We write $\mathbf{E} \subset \mathbf{F}$ if there are measurable sets $E' \in \mathbf{E}$, $F' \in \mathbf{F}$ such that $E' \subset F'$. It is readily verified that $\subset$ is a reflexive and transitive relation on $\tilde{\mathscr{F}}$. Notice that the relation $\subset$ is *not* the class inclusion and that it depends on the measure P.

Binary relations on $\tilde{\mathscr{F}}$ analogous to union and intersection are defined as follows:

$$\mathbf{E} \cup \mathbf{F} = \{E' \cup F' : E' \in \mathbf{E}, F' \in \mathbf{F}\};$$

$$\mathbf{E} \cap \mathbf{F} = \{E' \cap F' : E' \in \mathbf{E}, F' \in \mathbf{F}\}.$$

It can be readily verified that $\mathbf{E} \cup \mathbf{F}$ and $\mathbf{E} \cap \mathbf{F}$ are equivalence classes, containing $E \cup F$ and $E \cap F$, respectively. The relative complement is also defined analogously: $\mathbf{E} \setminus \mathbf{F} = \{E' \setminus F' : E' \in \mathbf{E}, F' \in \mathbf{F}\}$, which is an equivalence class, containing $E \setminus F$. The union and intersection of *countably* many classes are also well defined:

$$\text{for } \mathbf{E}_n,\ n = 1, 2, \ldots,\quad \bigcup_{n=1}^{\infty} \mathbf{E}_n = \left\{ \bigcup_{n=1}^{\infty} E_n' : E_n' \in \mathbf{E}_n \right\},$$

$$\bigcap_{n=1}^{\infty} \mathbf{E}_n = \left\{ \bigcap_{n=1}^{\infty} E_n' : E_n' \in \mathbf{E}_n \right\}.$$

It is easy to show that $\bigcup_{n=1}^{\infty} \mathbf{E}_n$ (resp. $\bigcap_{n=1}^{\infty} \mathbf{E}_n$) is an equivalence class, containing $\bigcup_{n=1}^{\infty} E_n$ (resp. $\bigcap_{n=1}^{\infty} E_n$). To sum up, the relations $\cap$, $\cup$, and $\setminus$ work exactly as their set theoretic counterparts as long as we are dealing with countable operations on $\tilde{\mathscr{F}}$ and the choice of representative element in each class does not matter.

We say $F \in \mathscr{F}$ is essentially $\mathscr{F}_i$-measurable if there is $F' \in \mathscr{F}_i$ such that $F \sim F'$. By construction, if F is essentially $\mathscr{F}_i$-measurable, then the class $\mathbf{F}$ consists of essentially $\mathscr{F}_i$-measurable sets, and we say $\mathbf{F}$ is $\mathscr{F}_i$-measurable. It is easy to see that $\mathscr{F}_i$-measurable classes are closed under countable operations of $\cup$, $\cap$, and $\setminus$. A real-valued $\mathscr{F}$-measurable function f on Ω is said to be essentially $\mathscr{F}_i$-measurable if f coincides with an $\mathscr{F}_i$-measurable function almost surely.

Let $E \in \mathscr{F}$. A conditional probability function of event $E \in \mathscr{F}$ with respect to $\mathscr{F}_i$ is an essentially $\mathscr{F}_i$-measurable function f such that $\int_A f(\omega)\, dP = \int_A 1_E(\omega)\, dP$ $(= P(E \cap A))$ for any $A \in \mathscr{F}_i$, where 1_E is the indicator function of E. Since functions that are identical almost everywhere have identical integrals, a conditional probability function is essentially unique in the sense that f is a conditional probability of E if and only if any f' that coincides with f almost everywhere is a conditional probability of E.

Conditional probability is countably additive in the following sense:

Fact 1. Let $E_1, E_2, \ldots, E_n, \ldots \in \mathscr{F}$ be a sequence of disjoint sets. Suppose for each n, f_n is a conditional probability function of E_n. Then the function $\sum_{n=1}^{\infty} f_n$ is well defined and it is a conditional probability function of $\bigcup_{n=1}^{\infty} E_n$.

DEFINITION 2. The posterior probability of a class $\mathbf{E}$ with respect to $\mathscr{F}_i$ (denoted by $\Pi_i(\mathbf{E})$) is the collection of all conditional probability functions of all sets $E' \in \mathbf{E}$, with respect to $\mathscr{F}_i$.

To summarize the above discussion, we have:

Fact 3. $f \in \Pi_i(\mathbf{E})$ if and only if f is $\mathscr{F}_i$-measurable and $\int_A f(\omega)\, dP = \int_A 1_{E'}(\omega)\, dP$ for any $E' \in \mathbf{E}$ and $A \in \mathscr{F}_i$. Therefore, in particular, $f \in \Pi_i(\mathbf{E})$ if and only if any f' that coincides with f P-almost surely belongs to $\Pi_i(\mathbf{E})$.

3. p-BELIEF OPERATORS: A GENERAL APPROACH

3.1. *Individual Belief Operators*

For any function $f\colon \Omega \to [0,1]$, let

$$F^p(f) = \{\omega : f(\omega) \geq p\}.$$

Then from Fact 3, the following can be readily verified:

Fact 4. For any $f, f' \in \Pi_i(\mathbf{E})$, $F^p(f) \sim F^p(f')$. If $A \in \mathscr{F}$ satisfies $A \sim F^p(f)$ for some $f \in \Pi_i(\mathbf{E})$, then there exists $g \in \Pi_i(\mathbf{E})$ such that $F^p(g) = A$.

The following describes the *class* of events where the individual p-believes E.

$$\tilde{B}_i^p(\mathbf{E}) = \{F^p(f) : f \in \Pi_i(\mathbf{E})\}.$$

The following proposition summarizes basic properties of the p-belief operator $\tilde{B}_i^p$, which are completely analogous to the finite or the countable case (as studied by MS), except that we are now dealing with classes of events.

PROPOSITION 5. *The following properties hold*:

(B0) $\tilde{B}_i^p(\mathbf{E})$ *is a class*;

(B1) $\tilde{B}_i^p(\mathbf{E})$ *is* $\mathscr{F}_i$*-measurable*;

(B2) *if* $\mathbf{E}$ *is* $\mathscr{F}_i$*-measurable, then* $\mathbf{E} = \tilde{B}_i^p(\mathbf{E})$;

(B3) *if* $\mathbf{E} \subset \mathbf{F}$, *then* $\tilde{B}_i^p(\mathbf{E}) \subset \tilde{B}_i^p(\mathbf{F})$;

(B4) *if* $\mathbf{E}$ *is* $\mathscr{F}_i$*-measurable, then* $\tilde{B}_i^p(\mathbf{E} \cap \mathbf{F}) = \mathbf{E} \cap \tilde{B}_i^p(\mathbf{F})$;

(B5) *if* $\mathbf{F}^n$ *is a decreasing sequence of classes* (*i.e.*, $\mathbf{F}^{n+1} \subset \mathbf{F}^n$ *for all* $n = 1, 2, \dots$), *then* $\tilde{B}_i^p(\bigcap_{n=1}^{\infty} \mathbf{F}^n) = \bigcap_{n=1}^{\infty} \tilde{B}_i^p(\mathbf{F}^n)$.

Proof. (B0) By fact 4.

(B1) By construction.

(B2) If $\mathbf{E}$ is $\mathscr{F}_i$-measurable, then there exists $E' \in \mathbf{E}$ with $E' \in \mathscr{F}_i$; thus $1_E \in \Pi_i(\mathbf{E})$ and the result follows.

(B3) Let $\mathbf{E} \subset \mathbf{F}$ and $f \in \Pi_i(\mathbf{E})$. Choose $E' \in \mathbf{E}$ and $F' \in \mathbf{F}$ such that $E' \subseteq F'$. Choose a conditional probabilities (with respect to $\mathscr{F}_i$) f of E' and g of F'. Then by definition, $g \geq f$ on E' almost surely, thus $\tilde{B}_i^p(\mathbf{E}) \subset \tilde{B}_i^p(\mathbf{F})$.

(B4) Choose any $E' \in \mathbf{E}$, $F' \in \mathbf{F}$. Recall that if E' is $\mathscr{F}_i$-measurable, then $1_{E'}$ is a conditional probability of E'. Let f be a conditional probability function of F'. Then $g(\omega) = \min(1_{E'}(\omega), f(\omega))$ is a conditional probability of $E' \cap F'$. Indeed, for any A, $\int_A g = \int_{A \cap E'} f = \int_{A \cap E'} 1_{F'} = \int_A 1_{E' \cap F'}$. Since $g(\omega) \geq p$ if and only if $\omega \in E'$ and $f(\omega) \geq p$, (B4) holds.

(B5) We can choose a decreasing sequence of measurable sets $F^n \in \mathbf{F}^n$, and set $F = \bigcap_{n=1}^{\infty} F^n$. $F \in \bigcap_{n=1}^{\infty} \mathbf{F}^n$ by construction. Let f^n, f be conditional probability functions of F^n, F, respectively. Since F^n is decreasing, applying fact 1 to $E^n = F^n \setminus F^{n+1}$, we have that f^n converges monotonically, almost surely to f. In particular, $\{\omega : f(\omega) \geq p\} \sim \bigcap_{n=1}^{\infty} \{\omega : f^n(\omega) \geq p\}$; that is, $\tilde{B}^p(\mathbf{F}) = \bigcap_{n=1}^{\infty} \tilde{B}^p(\mathbf{F}^n)$, as desired. ■

3.2. *Common p-Belief Operator*

We now want to consider belief operators for many individuals and formulate the concept of common p-belief in this general framework.

Define the "everyone believes" operator by the rule

$$\tilde{B}_*^p(\mathbf{E}) = \bigcap_{i \in \mathscr{I}} \tilde{B}_i^p(\mathbf{E}).$$

LEMMA 6. $\tilde{B}_*^p(\tilde{B}_*^p(\mathbf{E})) \subset \tilde{B}_*^p(\mathbf{E})$.

Proof.

$$\begin{aligned}
\tilde{B}_*^p\left(\tilde{B}_*^p(\mathbf{E})\right) &= \bigcap_{i \in \mathscr{I}} \left\{\tilde{B}_i^p\left(\tilde{B}_*^p(\mathbf{E})\right)\right\} \\
&= \bigcap_{i \in \mathscr{I}} \left\{\tilde{B}_i^p\left\{\bigcap_{j \in \mathscr{I}} \tilde{B}_j^p(\mathbf{E})\right\}\right\}, \qquad \text{so by (B1) and (B4)}, \\
&= \bigcap_{i \in \mathscr{I}} \left\{\tilde{B}_i^p(\mathbf{E}) \cap \tilde{B}_i^p\left(\bigcap_{j \neq i} \tilde{B}_j^p(\mathbf{E})\right)\right\}, \qquad \text{so by (B3)}, \\
&\subset \bigcap_{i \in \mathscr{I}} \left\{\tilde{B}_i^p(\mathbf{E})\right\} \\
&= \tilde{B}_*^p(\mathbf{E}). \quad \blacksquare
\end{aligned}$$

Define "common p-belief operator" $\tilde{C}^p$ by the rule

$$\tilde{C}^p(\mathbf{E}) = \tilde{B}_*^p(\mathbf{E}) \cap \tilde{B}_*^p\left(\tilde{B}_*^p(\mathbf{E})\right) \cap \tilde{B}_*^p\left(\tilde{B}_*^p\left(\tilde{B}_*^p(\mathbf{E})\right)\right) \cap \cdots .$$

Because $\tilde{C}^p(\mathbf{E})$ is a countable intersection of classes, it must itself be a class. In this framework, it no longer makes sense to ask if an event or class of events is common p-belief at a particular state ω, unless the singleton set $\{\omega\}$ occurs with positive probability. We rather define when a class $\mathbf{E}$ is common p-belief at some other class $\mathbf{F}$.

DEFINITION 7. Class $\mathbf{E}$ is common p-belief at $\mathbf{F}$ if $\mathbf{F} \subset \tilde{C}^p(\mathbf{E})$.

This *iterative* definition will be related to the following *fixed-point* characterization.

DEFINITION 8. Class $\mathbf{E}$ is p-evident if $\mathbf{E} \subset \tilde{B}_*^p(\mathbf{E})$.

The following result is thus an uncountable version of MS's result, which is in turn a generalization (from 1-belief to p-belief) of Aumann's (1976) characterization of common knowledge.

THEOREM 9. *The class* $\mathbf{E}$ *is common p-belief at* $\mathbf{A}$ *if and only if there is a p-evident class* $\mathbf{F}$ *such that* $\mathbf{A} \subset \mathbf{F} \subset \tilde{B}_*^p(\mathbf{E})$.

We can clarify the theorem by restating it in two parts. Pick any class $\mathbf{E}$; then (a) for any p-evident class $\mathbf{F}$ such that $\mathbf{F} \subset \tilde{B}_*^p(\mathbf{E})$, we have

$$\mathbf{F} \subset \tilde{C}^p(\mathbf{E}).$$

Conversely, (b) for any class $\mathbf{A}$ such that $\mathbf{A} \subset \tilde{C}^p(\mathbf{E})$, there exists a p-evident class $\mathbf{F}$ such that $\mathbf{A} \subset \mathbf{F} \subset \tilde{B}^p_*(\mathbf{E})$.

Proof. Now the "if" statement follows from (a), and the "only if" part follows from (b).

Proof of (a): Suppose a p-evident class $\mathbf{F}$ satisfies $\mathbf{F} \subset \tilde{B}^p_*(\mathbf{E})$. By (B3),

$$\left(\tilde{B}^p_*\right)^n(\mathbf{F}) \subset \left(\tilde{B}^p_*\right)^{n+1}(\mathbf{E}), \qquad \text{for } n = 0, 1, 2, \ldots$$

and so

$$\begin{aligned} \tilde{C}^p(\mathbf{F}) &= \bigcap_{n=1}^{\infty} \left(\tilde{B}^p_*\right)^n(\mathbf{F}) \\ &\subset \bigcap_{n=0}^{\infty} \left(\tilde{B}^p_*\right)^n(\mathbf{F}) \qquad \text{since } \mathbf{F} \text{ is } p\text{-evident}, \\ &\subset \bigcap_{n=1}^{\infty} \left(\tilde{B}^p_*\right)^n(\mathbf{E}) \\ &= \tilde{C}^p(\mathbf{E}). \end{aligned}$$

By the definition of p-evident, we also have $\mathbf{F} \subset \tilde{B}^p_*(\mathbf{F})$ and thus, by (B3),

$$\left(\tilde{B}^p_*\right)^n(\mathbf{F}) \subset \left(\tilde{B}^p_*\right)^{n+1}(\mathbf{F}), \qquad \text{for } n = 0, 1, 2, \ldots,$$

$$\text{and } \mathbf{F} \subset \left(\tilde{B}^p_*\right)^n(\mathbf{F}), \qquad \text{for } n = 1, 2, \ldots$$

$$\text{So } \mathbf{F} \subset \bigcap_{n=1}^{\infty} \left(\tilde{B}^p_*\right)^n(\mathbf{F}) = \tilde{C}^p(\mathbf{F}) \subset \tilde{C}^p(\mathbf{E}).$$

Proof of (b): Let $\mathbf{F} = \tilde{C}^p(\mathbf{E})$. By construction of $\tilde{C}^p$, $\mathbf{A} \subset \mathbf{F} \subset \tilde{B}^p_*(\mathbf{E})$. So it is enough to show that $\mathbf{F}$ is p-evident. Let $\mathbf{E}^0 = \mathbf{E}$, and define $\mathbf{E}^k = \tilde{B}^p_*(\mathbf{E}^{k-1})$ iteratively. By construction, $\mathbf{F} = \bigcap_{k=1}^{\infty} \mathbf{E}^k$.

By Lemma 6, $\mathbf{E}^k$, $k = 1, 2, \ldots,$ is a decreasing sequence of classes. Therefore, by (B5), $\tilde{B}^p_i(\mathbf{F}) = \bigcap_{k=1}^{\infty} \tilde{B}^p_i(\mathbf{E}^k)$, so

$$\begin{aligned} \tilde{B}^p_*(\mathbf{F}) &= \bigcap_{i \in \mathscr{I}} \tilde{B}^p_i(\mathbf{F}), \qquad \text{using the relation above}, \\ &= \bigcap_{i \in \mathscr{I}} \left\{ \bigcap_{k=1}^{\infty} \tilde{B}^p_i(\mathbf{E}^k) \right\}, \qquad \text{and exchanging the order}, \\ &= \bigcap_{k=1}^{\infty} \left\{ \bigcap_{i \in \mathscr{I}} \tilde{B}^p_i(\mathbf{E}^k) \right\} \\ &= \bigcap_{k=1}^{\infty} \tilde{B}^p_*(\mathbf{E}^k). \end{aligned}$$

On the other hand, since $\mathbf{E}^k$ is decreasing, $\bigcap_{k=1}^{\infty} \tilde{B}^p_*(\mathbf{E}^k) = \bigcap_{k=2}^{\infty} \tilde{B}^p_*(\mathbf{E}^k) = \bigcap_{k=1}^{\infty} \mathbf{E}^k = \mathbf{F}$. Therefore, we have $\mathbf{F} = \tilde{B}^p_*(\mathbf{F})$, so in particular $\mathbf{F}$ is p-evident as desired. ∎

4. DISCUSSION

4.1. *A Fixed Conditional Probability Approach*

Nielsen (1984) gave a characterization of common 1-belief for uncountable state spaces using the class based approach which we pursue here. Brandenburger and Dekel (1987) pursued the alternative approach (for common 1-belief) of fixing a particular conditional probability, and continuing to analyze common 1-belief about *events* defined at *states*. We believe that while the former approach (which we have pursued here) is mathematically more elegant, it is often useful in applications to use a natural conditional probability. One interpretation of the results presented here is that as long as we interested in probability 1 statements, there is no loss of generality in fixing a conditional probability. We therefore conclude by briefly summarizing how to do so.

For each i, fix a regular conditional probability P_i: $\mathscr{F} \times \Omega \to \Re$: that is, $P_i(E|\cdot) \in \Pi_i(\mathbf{E})$ for all i and $E \in \mathscr{F}$, and $P_i(\cdot|\omega)$ is a probability measure with probability one [such regular conditional probabilities exist if Ω is a separable metric space and $\mathscr{F}$ is the Borel field]. For these fixed P_i, define event belief operators $B_i^p(E) = \{\omega : P_i(E|\omega) \geq p\}$, $B_*^p(E) = \bigcap_{i \in \mathscr{I}} B_i^p(E)$, and $C^p(E) = B_*^p(E) \cap B_*^p(B_*^p(E)) \cap B_*^p(B_*^p(B_*^p(E))) \cap \cdots$. Then by (B1) of Proposition 5 and construction, it can be readily verified that for any $E \in \mathscr{F}$, $E' \in \tilde{B}_*^p(\mathbf{E})$ holds if and only if $B_*^p(E) \sim E'$, and so $\mathbf{E}$ is p-evident if and only if there is $E' \in \mathbf{E}$ such that for all i, $B_i^p(E') \setminus E'$ is a null set. So we say that an event E is p-evident if $P_i(E|\omega) \geq p$ holds for almost every $\omega \in E$. Also $\tilde{C}^p(\mathbf{E})$ is the class containing $C^p(E)$. Thus if we say that E is common p-belief at ω if $\omega \in C^p(E)$, we can rephrase Theorem 9 as follows:

COROLLARY 10. *For any event E, the following statement is true for almost every ω: Event E is common p-belief at ω if and only if there exists a p-evident event F such that $\omega \in F \subset B_*^p(E)$.*

In the remainder of this note, we will informally present an example which illustrates the usefulness of using fixed conditional probabilities to characterize common p-belief.

4.2. *The Noise Example*

The following example was analyzed by Morris *et al.* (1993). There are two individuals 1 and 2. Individual 1 observes a number $x_1 \in [0,1)$ and individual 2 observes a number $x_2 \in [0,1)$. So let $\Omega = [0,1)^2$ with its Borel sets and $\mathscr{F}_i$ be the collection of the sets of the form $\{(x_1, x_2)\colon x_i = \bar{x}\}$, for some $\bar{x} \in [0,1)$. By convention, we will identify any number with its decimal part, so that if $\alpha > \beta$, $[\alpha, \beta] = [\alpha, 1) \cup [0, \beta]$ and $|\alpha - \beta| =$

$\min\{\alpha - \beta, 1 - (\alpha - \beta)\}$. We consider probability measures indexed by $\varepsilon \in [0, \frac{1}{2}) : P^{\varepsilon}$ is the uniform measure on the set $\{(x_1, x_2) \colon |x_1 - x_2| \leq \varepsilon\}$; in particular, P^0 is uniform on the diagonal set $D = \{(x_1, x_2) \colon x_1 = x_2\}$. Thus the marginal distribution of x_i is uniform on the interval $[0, 1)$. Given x_i, individual j observes a signal x_j which is distributed uniformly on the interval $[x_j - \varepsilon, x_j + \varepsilon]$.

Let $\mathscr{F}^*$ be the collection of events which exclude some open interval on the diagonal, i.e.,

$$\mathscr{F}^* = \left\{ E \in \mathscr{F} \colon \quad \begin{array}{l} \text{there exists } \bar{x} \in [0,1) \text{ and } \delta > 0 \\ \text{such that } E \subset \{(x_1, x_2) : \|(x_1, x_2) - (\bar{x}, \bar{x})\| > \delta\} \end{array} \right\}.$$

We will show that for *any* $p > \frac{1}{2}$, any event E in $\mathscr{F}^*$ is never common p-belief for all sufficiently small $\varepsilon > 0$. However, if $\varepsilon = 0$, such events are p-evident and thus common p-belief whenever they are true, for any $p > 0$. We will show how the analysis of this paper can be used to make these claims precise and independent of the choice of conditional probability.

Consider first the case with $\varepsilon \in (0, \frac{1}{2})$. The conditional density is

$$f_i(x_j | x_i) = \begin{cases} \dfrac{1}{2\varepsilon}, & \text{if } x_j \in [x_i - \varepsilon, x_i + \varepsilon] \\ 0, & \text{otherwise} \end{cases},$$

so, writing λ for the Lebesgue measure on $[0, 1]$, the "natural" regular conditional probability is

$$P_i(E | x_i) = \frac{1}{2\varepsilon} \lambda\left(\{x_j' \colon (x_i, x_j') \in E \text{ and } |x_j' - x_i| \leq \varepsilon\}\right).$$

The associated event belief operators are denoted by $B_i^p(E)$. Now suppose $E \in \mathscr{F}^*$, with $E \subset \{(x_1, x_2) : \|(x_1, x_2) - (\bar{x}, \bar{x})\| > \delta\}$. Consider any $p > \frac{1}{2}$ and $\varepsilon \in (0, \delta/3)$ and let $j \neq i$; now if $x_i \in (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$, individual i assigns probability 1 to x_j being in the interval $(\bar{x} - 2\varepsilon, \bar{x} + 2\varepsilon)$. But $\|(x_i, x_j) - (\bar{x}, \bar{x})\| \leq \sqrt{\varepsilon^2 + (2\varepsilon)^2} = (\sqrt{5})\varepsilon < \delta$ if $x_i \in (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ and $x_j \in (\bar{x} - 2\varepsilon, \bar{x} + 2\varepsilon)$; so individual i assigns probability 0 to (x_i, x_j) being an element of E if $x_i \in (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ and so

$$B_i^p(E) \subset \{(x_1, x_2) \colon x_i \notin (x - \varepsilon, x + \varepsilon)\}$$

$$B_*^p(E) \subset \{(x_1, x_2) \colon x_i \notin (x - \varepsilon, x + \varepsilon) \text{ for } i = 1, 2\}$$

$$[B_*^p]^n(E) \subset \left\{ (x_1, x_2) \colon \begin{array}{c} x_i \notin (x - g(\varepsilon, p, n), x + g(\varepsilon, p, n)) \\ \text{for } i = 1, 2 \end{array} \right\},$$

where $g(\varepsilon, p, n) = \varepsilon(1 + (n - 1)(2p - 1))$. Thus $C^p(E) = \emptyset$. Thus for any $E \in \mathscr{F}^*$, $C^p(E) = \emptyset$ for all $p > \frac{1}{2}$, for ε sufficiently small.

Now consider the case where $\varepsilon = 0$, and thus probability is distributed uniformly on the diagonal. The "natural" conditional probability has individual i observing x_i assigning probability 1 to individual 2 observing x_i; the corresponding event belief operator is therefore

$$B_i^p(E) = \{(x_1, x_2) : (x_i, x_i) \in E\},$$

for all $p \in (0, 1]$. Thus any E is p-evident and $C^p(E) = D \cap E$ for all $E \in \mathscr{F}$ and all $p \in (0, 1]$.

The above analysis (like that of Morris *et al.*, 1993) fixed the conditional probabilities. But the analysis of this note shows that the choice of conditional probability does not matter. So if $\varepsilon = 0$, we have $\tilde{C}^p(\mathbf{E}) = \mathbf{E}$ for all $p \in (0, 1]$ and all classes $\mathbf{E}$; but if ε is positive but sufficiently small, $\tilde{C}^p(\mathbf{E})$ is the null class for all $E \in \mathscr{F}^*$ and all $p \in (\frac{1}{2}, 1]$.

This extreme sensitivity of common p-belief to noise is important in a number of applications; see, for example, Carlsson and van Damme (1993).

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