NOTES AND COMMENTS

SIMPLE FINITE HORIZON BUBBLES ROBUST TO HIGHER ORDER KNOWLEDGE

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An asymmetric information model of a finite horizon "nth order" rational asset price bubble is presented, where (all agents know that)^n the asset is worthless. Also, the model has only two agents, so the first order version of the bubble is simpler than other first order bubbles in the literature.

Keywords: Bubbles, asymmetric information, higher order knowledge.

1. INTRODUCTION

Observers often interpret wide fluctuations in asset markets, such as recent US equity price movements, as due to speculative bubbles (Shiller (2000) and Ofek and Richardson (2003)). As Warren Buffett (2001) suggests, investors in these markets sometimes resemble "Cinderella at the ball. They know that overstaying the festivities ... will eventually bring on pumpkins and mice," but they "all plan to leave just seconds before midnight. There is a problem, though: they are dancing in a room in which the clocks have no hands." Observers frequently relate this to a "greater fools" phenomenon (Kindleberger (2000, p. 111, and note 13, p. 256)), where people hold overpriced assets in hopes of selling them to greater fools before the bubble bursts.

While there are many approaches to modeling asset price bubbles in the literature,[2] the model of Allen, Morris, and Postlewaite (1993) captures this greater fools dynamic especially well. They use asymmetric information and short sale constraints to model a

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[2]Blanchard and Watson (1982) consider a model of randomly bursting bubbles, though people hold overpriced assets only because they believe they will be overpriced forever in expected value. Loewenstein and Willard (2000) model a finite horizon bubble in continuous time, on an asset with zero net supply, which cannot be short sold due to wealth constraints. They also present an example of a finite horizon bubble in positive net supply, which they seem to consider pathological because the deflated wealth process is ill behaved. Other finite horizon bubble models building on some sort of infinity include Kamihigashi (1998) and Montrucchio and Privileggi (2001). Additional bubble models include Tirole (1985), Santos and Woodford (1997), Abreu and Brunnermeier (2003), Scheinkman and Xiong (2003), Allen and Gorton (1993), Allen and Gale (2000), and Allen, Morris, and Shin (2002). For an excellent review of the literature, see Chapter 2 of Brunnermeier (2001). For opposing views on the empirical relevance of bubbles, see Shiller (2000) and Garber (2000), among many others. See also Hunter, Kaufman, and Pomerleano (2003).

rational bubble in a finite state, discrete time economy that lasts only three periods. In their bubble state, agents hold a worthless asset at a positive price in the first period, in hopes of selling it in the second period to someone else who thinks it may be worth something. In short, they develop a greater fools model of bubbles, but with everyone rational, so the standard tools of economic theory apply.

In the Allen, Morris, and Postlewaite model, people hold an overpriced asset because they think that others might believe that this assets is valuable. In many historical cases where bubbles are thought to have existed, however, the possible overpricing of assets was widely discussed at the time.[3] It is therefore plausible, not only that people believed that assets were overpriced, but that they believed that most *other* people believed that assets were overpriced. This raises the question of whether a bubble can exist if everyone knows that the asset is overpriced, everyone know that everyone *else* knows that the asset is overpriced, and so on. That is, are "higher order" bubbles possible? This question was raised, but not resolved, in Morris, Postlewaite, and Shin (1995).

This paper shows that such higher order bubbles are possible in models with rational agents and a finite number of states and periods, where, like Allen, Morris, and Postlewaite, we use asymmetric information and short sale constraints to model these bubbles. Thus, bubbles can exist even if people know that others are skeptical about asset prices. That is, bubbles are robust to agents knowing a lot about one anothers' beliefs.

The model may also be of interest because it shows that bubbles are possible even if there is common knowledge of the pattern of trade, contradicting an argument in Allen, Morris, and Postlewaite (1993). It therefore requires only two agents, while Allen, Morris, and Postlewaite believed that at least three agents were needed to model a finite horizon bubble. This makes the first order version of the present bubble model simpler than the example in Allen, Morris, and Postlewaite. In fact, the present model may be simple enough to use in applied work.

Section 2 introduces the basic framework. Section 3 presents an *n*th order bubble for arbitrary finite *n*, as well as an example of a first order bubble. Section 4 concludes.

2. THE BASIC FRAMEWORK

This section presents the basic asset market model with asymmetric information and short sale constraints.[4] The next section then presents the higher order bubble.

There are two risk neutral agents in this market, Players 1 and 2, and a finite set of states of the world, Ω. The typical state of the world is $\omega \in \Omega$. Player i's prior probability distribution over Ω is denoted by $\pi_i(\omega)$, $i = 1, 2$. We allow $\pi_1(\omega)$ and $\pi_2(\omega)$ to differ,

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[3]For example, in 1999, during the technology stock boom, Steve Ballmer, President of Microsoft, maintained in a *Wall Street Journal* article that "there is such an overvaluation of technology stocks that it is absurd ... and I'd put our company's stock in that category" (quoted in Bond and Cummins (2000)).

[4]This basic framework follows Allen, Morris, and Postlewaite (1993) closely, except that information partitions are replaced by public and private signals. Models of this sort go back before Kreps (1977). Note also that short sale constraints are assumed in many models of asset markets (e.g., Harrison and Kreps (1978), Tirole (1982), and Allen, Morris, and Postlewaite (1993)). On the other hand, put options can play a role very similar to that of short sales.

in order to give Players 1 and 2 an incentive to trade.[5]

The market lasts for T periods, so $1 \le t \le T$, but there is no discounting. There is a riskless asset, money, and a risky asset, X. A unit of X will ultimately pay a single dividend of $d(\omega)$ in state ω. Player i is initially endowed with state dependent amounts, $m_0^i(\omega)$ of money, and $a_0^i(\omega)$ of X.

Denote Player i's net sales of X in period t by $x_t^i(\omega)$. Thus, if $a_t^i(\omega)$ is her holdings of X at the end of period t, then $a_t^i(\omega) = a_{t-1}^i(\omega) - x_t^i(\omega)$. In the same way, if $m_t^i(\omega)$ is Player i's money holdings at the end of period t, and $p_t(\omega)$ is the price of X in period t, then $m_t^i(\omega) = m_{t-1}^i(\omega) + p_t(\omega)x_t^i(\omega)$.

There are no short sales of X, so $a_t^i(\omega) \ge 0$ for all i, t, and ω. Also, the price of the consumption good in terms of money is fixed at one. Since players are risk neutral, Player i's expected payoff can be normalized to $E^i[m_T^i(\omega) + a_T^i(\omega)d(\omega)]$, where E^i is the expectation with respect to Player i's prior π_i.

Before trade begins, Players 1 and 2 receive initial private signals, $s_1(\omega)$ and $s_2(\omega)$, respectively. Then, in each period t, both players receive a public signal $r_t(\omega)$. Players can also learn from the market price $p_t(\omega)$, but in the examples below, the information structure is chosen so that prices reveal no additional information.

A competitive equilibrium in this market consists of a state dependent price, $p_t(\omega)$, each period, and state dependent net trades, $x_t^i(\omega)$, $i = 1, 2$, each period, such that:

 (i) $p_t(\omega)$ depends only on information possessed by market participants through time t, i.e., on $s_1(\omega)$, $s_2(\omega)$, and $r_{t'}(\omega)$ for $t' \le t$;

 (ii) Player i's net sales, $x_t^i(\omega)$, in period t depend only on information he/she possesses at that time, i.e., on $s_i(\omega)$, and on $p_{t'}(\omega)$ and $r_{t'}(\omega)$ for $t' \le t$;

 (iii) the market clears, so $x_t^1(\omega) + x_t^2(\omega) = 0$; and

 (iv) each agent's net trades are optimal, given his/her information, prices, short sale constraints, and his/her (correct) beliefs about the other's strategy rule.

Next, as in Allen, Morris, and Postlewaite (1993), a *strong bubble* exists at state ω if all agents know that X is overpriced for sure, i.e., $p_t(\omega') > d(\omega')$ for all ω' any agent thinks is possible at ω. Finally, an nth order bubble exists at ω for a given finite n if, at ω, everyone knows that X is overpriced for sure, everyone knows that everyone knows that X is overpriced for sure, $\ldots$, and (everyone knows that)^n X is overpriced for sure.

3. HIGHER ORDER BUBBLES

This section presents a model of an nth order bubble. The model involves only two agents, so bubbles can exist even though the pattern of trade is common knowledge, contrary to a suggestion by Allen, Morris, and Postlewaite (1993).[6] We also illustrate

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[5]As Allen, Morris, and Postlewaite (1993) show, motives for trade could also be modeled by assuming that different agents have different marginal utilities of consumption in different states of the world, due, say, to random future labor income. Thus, while we assume that potential gains from trade arise from differing prior probabilities, one could just as easily have assumed that they arise from risk sharing or insurance motives.

[6]Morris, Postlewaite, and Shin (1995) show that at least $T = n + 2$ periods are required to construct an nth order bubble. Our model involves *exactly* $n + 2$ periods, so this bound is tight. On the other hand, the model involves more states than needed, to make it more symmetrical. Thus,

the general model with the case of a first order bubble. This example may be of independent interest since it is simpler than previous models of first order bubbles in the literature.

We first present the state space and information structure in (1)–(4), and then give an intuitive interpretation of this structure.[7] Next we present the prior probabilities, describe the equilibrium, and check the equilibrium conditions.

The model has two players, Player 1 and Player 2, $T = n + 2$ periods, and the following $T^2 + T - 1$ states of the world:

$$(1) \qquad \Omega = \left\{ \omega_{22}^1, \omega_{j(j+1)}^\tau, \omega_{(j+1)j}^\tau : 2 \le j \le T, 1 \le \tau \le j \right\}.$$

Thus, the state space consists of state ω_{22}^1, together with the states ω_{jk}^τ with $|j - k| = 1$ and $1 \le \tau \le \min(j, k)$. Here ω_{22}^1 will be the nth order bubble state. Players 1 and 2 initially own one unit of X each in all states, so $a_0^i(\omega) = 1$ for $i = 1, 2$ and all $\omega \in \Omega$. X only pays a dividend in states $\omega_{T(T+1)}^T$ and $\omega_{(T+1)T}^T$, and this dividend is 2^{T-1}, so

$$(2) \qquad d(\omega) = \begin{cases} 2^{T-1} & \text{if } \omega = \omega_{T(T+1)}^T \text{ or } \omega_{(T+1)T}^T, \\ 0 & \text{otherwise.} \end{cases}$$

The players' information structure is given as follows. In state ω_{jk}^τ, Players 1 and 2 are initially told j and k, respectively, so Players 1 and 2 receive the signals

$$(3) \qquad s_1(\omega_{jk}^\tau) = j \quad \text{and} \quad s_2(\omega_{jk}^\tau) = k,$$

respectively. Then, in each period t both players are told $r_t(\omega)$, defined as

$$(4) \qquad r_t(\omega_{jk}^\tau) = \begin{cases} 1 & \text{if } \tau \ge t, \\ 0 & \text{if } \tau < t. \end{cases}$$

Thus, in state ω_{jk}^τ, $r_t(\omega_{jk}^\tau) = 1$ through period $t = \tau$. Then, in period $t = \tau + 1$, $r_t(\omega_{jk}^\tau)$ falls to zero and stays there. Here, $r_t(\omega) = 0$ is bad news, since, when $r_t(\omega) = 0$, it becomes common knowledge that ω is not $\omega_{T(T+1)}^T$ or $\omega_{(T+1)T}^T$, so X pays no dividend, and the price should collapse. Conversely, $r_t(\omega) = 1$ is good news.

The signals $s_1(\omega_{jk}^\tau) = j$ and $s_2(\omega_{jk}^\tau) = k$ determine how optimistic Players 1 and 2 are, based on their initial information. Thus, Player 1 initially believes that the news might stay positive through period $t = \min(j, T)$, and Player 2 believes it might stay positive through period $t = \min(k, T)$. Thus, if $j = k + 1$, then Player 1 is more optimistic than Player 2, and conversely if $j = k - 1$. Suppose, e.g., that the state is $\omega_{j(j+1)}^j$, with $j < T$. Then in period $t = j$, Player 1 knows that the news will turn bad next period, but the more optimistic Player 2 does not know this. Thus, Player 1 will want to sell the asset to Player 2 in period j, and Player 2 might be willing to buy it.

Note also that, if the price is initially positive at ω_{22}^1, then this represents an nth order bubble. For, at ω_{22}^1, Player 1 knows that $s_1(\omega) = 2$, so the state is one of $\omega_{22}^1, \omega_{23}^1$, or ω_{23}^2. Similarly, Player 2 knows that $s_2(\omega) = 2$, so the state is one of $\omega_{22}^1, \omega_{32}^1$, or ω_{32}^2. Thus, both players know that both players' signals, $s_1(\omega) = j$ and $s_2(\omega) = k$, are at most 3.

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while the first order bubble example below involves eleven states of the world, it is possible to construct an example involving only eight states of the world. Details are available upon request.

[7]A referee's suggestion has improved this presentation dramatically.

Therefore, the state cannot be $\omega^T_{T(T+1)}$ or $\omega^T_{(T+1)T}$, since $T+1 = n+3 \geq 4$. The asset is therefore worthless, so a positive price represents a bubble.

Also, since both know that $j, k \leq 3$, both know that both know that j and k are at most 4, and so on. Proceeding in this way, (both know that)^n j and k are at most $n + 2 = T$. Thus, both know to this order that the state is not $\omega^T_{T(T+1)}$ or $\omega^T_{(T+1)T}$. They therefore know to this order that the asset pays no dividend, and so, is worthless. Thus, if the price is initially positive at ω^1_{22}, then there is an nth order bubble at ω^1_{22}.

Finally, we choose prior probabilities for the two players to make a bubble equilibrium work. Following Allen, Morris, and Postlewaite, assume that Players 1 and 2 have different prior probabilities over the states of the world. These different priors are used to give the agents an incentive to trade, though other approaches also work (see footnote 5). For Player 1 these priors are:

(5)
$$\pi_1(\omega^1_{22}) = \pi_1(\omega^1_{23}) = A, \quad \pi_1(\omega^2_{23}) = 2A,$$

$$\pi_1\big(\omega^\tau_{j(j-1)}\big) = \pi_1\big(\omega^\tau_{j(j+1)}\big) = 2^{j-\tau-1}A \quad \text{for} \quad 3 \leq j \leq T, 1 \leq \tau \leq j-1,$$

$$\pi_1\big(\omega^j_{j(j+1)}\big) = 2A \quad \text{for} \quad 3 \leq j \leq T,$$

$$\pi_1\big(\omega^\tau_{(T+1)T}\big) = 2^{T-\tau-1}A \quad \text{for} \quad 1 \leq \tau \leq T-1, \pi_1\big(\omega^T_{(T+1)T}\big) = A,$$

where $A = 1/(2^{T+1} + 2^{T-1} - 4)$. Player 2's prior probabilities are given symmetrically.

We now consider the following prices and net trades, and show that they form an equilibrium. First, the price in period t is

(6)
$$p_t(\omega) = 2^{t-1}r_t(\omega).$$

Thus, the price starts in period 1 at $p_1(\omega) = 1$ for all ω. It then doubles each period as long as $r_t(\omega) = 1$. If $r_t(\omega)$ ever falls to zero, $p_t(\omega)$ also falls to zero, and stays there forever. Next, the net trades in state ω^τ_{jk} are as follows:

(a) In period 1, there is no trade.

(b) In period t, $2 \leq t \leq T-1$, if $r_t(\omega) > 0$, and t and j have the same parity (both even or both odd), then Player 1 sells all she owns of X to Player 2.

(c) In period t, $2 \leq t \leq T-1$, if $r_t(\omega) > 0$, and t and j have *opposite* parity (one even and one odd), then Player 2 sells all he owns of X to Player 1.

(d) If $r_t(\omega) = 0$, so $p_t(\omega) = 0$, there is no trade.

(e) In period T there is no trade.

Thus, as long as the price is positive between periods $t = 2$ and $t = T - 1$, Players 1 and 2 trade X back and forth. Note that, in period 2, *one* unit of X is traded, while, from period 3 on, *two* units of X are traded back and forth.

To see that this is an equilibrium, first note that prices and net trades depend only on each player's information. Prices depend only on the public signal $r_t(\omega)$, and net trades depend only on the public signal $r_t(\omega)$, and the parities of j and k, which both players know, since j and k have opposite parities whenever trade occurs. Thus, conditions (i) and (ii) of an equilibrium are met. It is also obvious that markets clear (condition (iii)), and there are no short sales.

Next, consider the problem of Player 1, and suppose Player 1 sees the signal $s_1(\omega) = j$. If $j = 2$, then Player 1 knows the state is one of ω^1_{22}, ω^1_{23}, or ω^2_{23}. Thus,

$p_1(\omega) = 1$, and Player 1 believes that $p_t(\omega)$ will either double to 2 or fall to 0 in period 2. Also, the probability $p_t(\omega)$ doubles is

$$\pi_1(\omega_{23}^2)/\pi_1(\omega_{22}^1, \omega_{23}^1, \omega_{23}^2) = 2A/(A + A + 2A) = 1/2,$$

so the expected value of $p_2(\omega)$ is 1. Since Player 1 is risk neutral, she is then willing to not trade in period 1, as in the equilibrium (part (a)). In period 2, if the price is still positive, Player 1 knows that the state is ω_{23}^2, so the price will fall to zero by period 3. Thus, Player 1 strictly prefers to sell all of her endowment, as called for by the equilibrium (part (b); note that $t = 2$ has the same parity as $j = 2$). She can sell no more than this since she is short sale constrained.

Next suppose $3 \leq j \leq T$. Consider first period t with $t \leq j - 1$. If the price is positive in this period, then it is 2^{t-1}. Also, Player 1's expected period $t + 1$ price is

$$(7) \qquad E_t^1 p_{t+1}(\omega) = 2^t \pi_1(r_{t+1}(\omega) = 1 | r_t(\omega) = 1) = (2^t)(1/2) = 2^{t-1},$$

where the second step follows since

$$(8) \qquad \pi_1(r_{t+1}(\omega) = 1 | r_t(\omega) = 1) = \frac{\pi_1(\{\omega_{j(j-1)}^\tau, \omega_{j(j+1)}^\tau : \tau \geq t+1\})}{\pi_1(\{\omega_{j(j-1)}^\tau, \omega_{j(j+1)}^\tau : \tau \geq t\})} = \frac{1}{2},$$

from (5). Thus, since Player 1 is risk neutral, she is indifferent between buying, selling, and doing nothing in period $t \leq j - 1$. Next, in period $t = j$, if the price is still positive, then Player 1 knows that the state is $\omega_{j(j+1)}^j$, so the price next period will be zero. Thus, Player 1 strictly prefers to sell all she owns of X, as called for by the equilibrium (part (b), where $t = j$ so t and j have the same parity). She can sell no more than this by the short sale constraint.

Next, when $j = T + 1$, calculations like (7) and (8) show that, if $p_t(\omega) > 0$ in period $t \leq T - 1$, then Player 1 is indifferent between buying, selling, and doing nothing, as called for by the equilibrium. In period T, if $p_T(\omega) > 0$, Player 1 knows the state is $\omega_{(T+1)T}^T$, so the dividend is 2^{T-1}, which equals the price in this case. Thus, Player 1 is willing to hold X in this period. Finally, if $r_t(\omega) = 0$, so $p_t(\omega) = 0$, then it is common knowledge that X is worthless, so Player 1 is willing to not trade.

In short, Player 1 is willing to follow the supposed equilibrium. The same argument applies to Player 2, so the above is an equilibrium. Since this equilibrium starts with a positive price at ω_{22}^1, there is therefore an nth order bubble at ω_{22}^1.

Intuitively, if Player 1 sees $s_1(\omega_{jk}^i) = j \leq T$, say, then, as long as $t < j$, Player 1 does not know whether she is more or less optimistic than Player 2. The probabilities are then such that she is willing to buy, sell, or hold, as called for by the equilibrium. In period $t = j$, if $r_t(\omega)$, and so $p_t(\omega)$, is still positive, Player 1 knows that the state is $\omega_{j(j+1)}^j$. She thus knows the price will crash, and so wants to sell. In this state, however, Player 2 is more optimistic than Player 1, and thinks the state might be $\omega_{(j+2)(j+1)}^{(j+1)}$. He is therefore willing to buy, and the equilibrium holds together.

We now illustrate all this with an example of a first order bubble. This bubble is simpler than that in Allen, Morris, and Postlewaite (1993), and so, should be useful in applied work.

EXAMPLE: Let $n = 1$, so $T = 3$. The above construction then yields eleven states:

$$\Omega = \{\omega_{22}^1, \omega_{23}^1, \omega_{23}^2, \omega_{32}^1, \omega_{32}^2, \omega_{34}^1, \omega_{34}^2, \omega_{34}^3, \omega_{43}^1, \omega_{43}^2, \omega_{43}^3\}.$$

TABLE I

PRIOR PROBABILITIES IN THE FIRST ORDER BUBBLE EXAMPLE

State	ω_{22}^1	ω_{23}^1	ω_{23}^2	ω_{32}^1	ω_{32}^2	ω_{34}^1	ω_{34}^2	ω_{34}^3	ω_{43}^1	ω_{43}^2	ω_{43}^3
Player 1	A	A	$2A$	$2A$	A	$2A$	A	$2A$	$2A$	A	A
Player 2	A	$2A$	A	A	$2A$	$2A$	A	A	$2A$	A	$2A$

The dividend is given by $d(\omega) = 4$ if $\omega = \omega_{34}^3$ or ω_{43}^3, and is zero otherwise. The prior probabilities for Players 1 and 2 are given in Table I, where $A = 1/16$.

In state ω_{jk}^τ, Player 1 is initially told $s_1(\omega_{jk}^\tau) = j$ and Player 2 is initially told $s_2(\omega_{jk}^\tau) = k$. Thus, Player 1's initial information partition is given by

$$P_{11} = \{\omega_{22}^1, \omega_{23}^1, \omega_{23}^2\}, \quad P_{21} = \{\omega_{32}^1, \omega_{32}^2, \omega_{34}^1, \omega_{34}^2, \omega_{34}^3\},$$

$$P_{31} = \{\omega_{43}^1, \omega_{43}^2, \omega_{43}^3\},$$

with a similar partition for Player 2 (see Figure 1). For example, cell P_{11} contains states where $s_1(\omega) = 2$. In period 2 both players are told whether or not $r_2(\omega) = 1$, and in period 3 they are told whether or not $r_3(\omega) = 1$.

The equilibrium is then given as follows. The first period price, $p_1(\omega) = 1$ always. The second period price, $p_2(\omega_{jk}^\tau) = 2$ if $\tau \geq 2$, and equals zero otherwise. The third period price, $p_3(\omega) = 4$ if ω is ω_{34}^3 or ω_{43}^3, and equals zero otherwise. There is no trade in periods 1 or 3. In period 2, Player 1 sells her endowment to Player 2 in states ω_{23}^2, ω_{43}^2, and ω_{43}^3, where $s_1(\omega) = j$ and t have the same (even) parity, while Player 2 sells his endowment to Player 1 in states ω_{32}^2, ω_{34}^2, and ω_{34}^3, where j and t have opposite parities. Note that this follows parts (b) and (c) of the equilibrium, respectively.

Intuitively, P_{11} represents states where Player 1 is a "bad seller," who knows that the asset is worthless, and P_{31} represents states where Player 1 is a "good seller," who thinks that the asset may be valuable, i.e., may pay a positive dividend. Also, Player 2 cannot distinguish between the good states from P_{31}, and two of the bad states, ω_{23}^1 and ω_{23}^2, from P_{11} (since $s_2(\omega) = 3$ in all these states). This is why Player 2 is willing to buy from Player 1 in some of the states where Player 1 is bad.
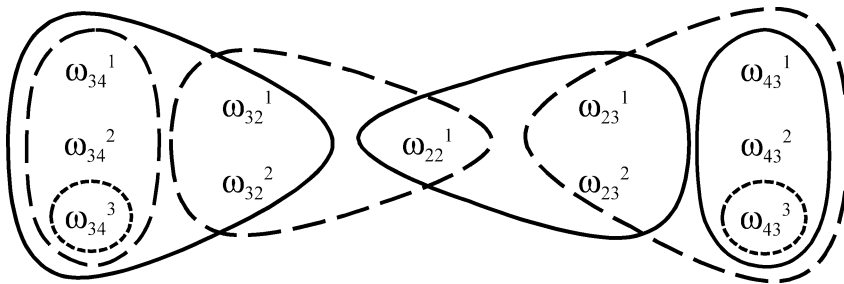

FIGURE 1.—The two players' information sets in the first order bubble example. Solid curves: Player 1's information sets. Dashed curves: Player 2's information sets. Dotted curves: Dividend paying states.

4. DISCUSSION AND CONCLUSION

First, since there are only two agents in these examples, the pattern of trade is common knowledge (Player 1's sales are Player 2's purchases and visa versa). A bubble nevertheless exists. That is, bubbles are consistent with common knowledge of the pattern of trade, contrary to a suggestion in Allen, Morris, and Postlewaite (1993). The necessary conditions for bubbles in models with a finite number of states and time periods, discussed in Allen, Morris, and Postlewaite (1993), can therefore be shortened from four to three: (a) asymmetric information, (b) potential gains from trade between agents, and (c) short sale constraints. The fourth condition suggested by Allen, Morris, and Postlewaite, that agents' trades not be common knowledge, is *not* necessary for a bubble.[8] This means that bubbles are possible with only two agents, so it is easier to construct models of speculative bubbles.

To see the role played by the three remaining conditions, note that bubbles only occur if people have different information, so it is possible for someone to be optimistic enough to buy an asset, even if someone else knows that it is overpriced (condition (a)). However, a buyer only risks buying from a bad seller who knows an asset is overpriced, if that buyer believes he *might* be buying from a good seller who does *not* believe the asset is overpriced, and with whom there are *nontrivial gains from trade* (condition (b)). Finally, short sale constraints (condition (c)) prevent sellers who know an asset is overpriced from flooding the market, and so, revealing their private information.

Next, whenever $\max(j, k) \leq T - 1$, the states ω_{jk}^{τ} are also bubble states, involving bubbles of order $T - \max(j, k)$. In addition, prices rise in these states through period $t = \tau$, inclusive. It is therefore possible to model bubbles of arbitrarily high order in this framework, which last for arbitrarily many periods.

Finally, in these states, for $2 \leq t \leq \tau$, people are *trading* the asset even though everyone knows it is overpriced. In fact, agents actually *buy* an asset they know is overpriced, in hopes of selling it in the future. Thus, even if everyone knows an asset is overpriced, and everyone knows that everyone else knows this, etc., people may still be willing to buy the asset in hopes of selling it in the future, before prices crash.

In future work it would be interesting to model a bubble that is robust to small changes in parameters. Like the Allen, Morris, and Postlewaite (1993) model, the bubbles in this paper are not robust in this way. This is because, if one changes the probabilities slightly, for example, prices will vary with states in a way that reveals too much private information to other players. To make a bubble robust, one must introduce noise into the model, to hide this private information. See Conlon (2003) for a start in this direction.

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[8]The Allen, Morris, and Postlewaite argument builds on Geanakoplos (1994). Geanakoplos shows that, if the pattern of trade is common knowledge, then there is a cruder symmetric information structure that yields the same pattern of trade. Since information is symmetric under this cruder structure, there can be no bubble under this cruder structure. The reason the bubble no longer exists, however, is because agents no longer know, given their cruder information, that the asset is worthless in the former bubble state. This does not prevent that state from being a bubble state under the original *finer* information structure. Details are available upon request.

This result also has a (negative) policy implication. If common knowledge of trades had prevented bubbles, one could have eliminated bubbles by informing agents of each others' trades. Since bubbles are consistent with common knowledge of trades, however, making trades public may not be enough to eliminate bubbles.

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