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NOTE

Rationalizable Trade

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We formulate necessary and sufficient conditions for interim rationalizable trade between two players. *Journal of Economic Literature* Classification Numbers: D82, C72.

1. INTRODUCTION

"No-trade" theorems provide sufficient conditions for the absence of equilibrium trade between asymmetrically informed players.[1] In this paper, we examine when trade is rationalizable. Since rationalizability is a weaker solution concept than equilibrium, trade is sometimes rationalizable, even when there is no equilibrium trade. We provide necessary and sufficient conditions for rationalizable trade between two asymmetrically informed

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[1]See Appendix A of Dekel and Gul (1997) for a review. For example, a result of Sebenius and Geanakoplos (1983) showed that two asymmetrically informed, risk neutral players with a common prior will not trade in equilibrium an object of common ex post value, unless they expect zero gains from the trade. Morris (1994) provides necessary conditions on players' priors for such results.

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players. Our characterization covers situations where the no-trade theorems do not apply, for example, when there are ex post gains from trade and there is no common prior. Thus our necessary and sufficient conditions neither imply nor are implied by the standard sufficient conditions for no-equilibrium trade.

We begin by reviewing some known special arguments on rationalizable trade that our analysis will unify and relate to the general case.

1.1. *An Example of Rationalizable Trade*

Each of two players is handed a card out of a shuffled deck, and after seeing the card makes the decision to either deposit \$10 in a pot or walk away from the game. The two players make their decisions simultaneously, each without observing the other player's card or decision. If either player walks away, the game terminates without any gains or losses. If neither player walks away, the cards are revealed; if they are the same color, player one collects the \$20 in the pot, otherwise player two collects the \$20.

The decision to accept the trade in this context is to not walk away from the game. Using a plus or a minus sign to denote a positive ex post gain from trading for player one or two, respectively, the ex post gains from trade in this game can be summarized by the following sign pattern (the "cross" pattern), which will play an important role in this paper.

	b	r
b	$+$	$-$
r	$-$	$+$

Let $[b]$ be the strategy "accept the trade only if the card is black" and let $[r]$ be the strategy "accept the trade only if the card is red." Then $[b]$ is a best response for player one to strategy $[b]$ of player two; $[b]$ for player two is a best response to player one choosing $[r]$; $[r]$ is in turn a best response for player one to strategy $[r]$ of player two; and $[r]$ for player two is a best response to player one choosing $[b]$. Strategies $[b]$ and $[r]$ are therefore rationalizable for both players.[2]

1.2. *Examples with No Rationalizable Trade*

We consider a finite-signal version of the trading envelopes problem studied by Nalebuff (1989), Geanakoplos (1994), and others. Each of two players randomly selects an envelope out of a box containing indistinguishable sealed envelopes. It is common knowledge between the players that each envelope in the box contains some number of one dollar bills not exceeding some commonly known number, say 10. After the players privately open

[2] Aumann (1987) has argued that rationalizability is prima facie too weak a solution concept, precisely because it allows gains from trade in zero-sum settings such as the above example.

their envelopes, they have the option of offering to exchange their envelope with the other player's envelope. If both players offer to trade, the envelopes are swapped; otherwise the players keep their original envelopes. A simple iterative argument shows that trade is not rationalizable in this setting. A player who receives an envelope containing 10 dollars will clearly not offer to trade. Knowing that, a player who receives 9 dollars will also refuse to trade, and so on.

More generally, if trade is zero-sum and each player's gain is a monotonic function of both players' signals, then excluding equilibrium trade, for instance by assuming common prior beliefs, also excludes rationalizable trade. This is because, by monotonicity in best responses (strategic complementarities), the supremum of all rationalizable strategy profiles is an equilibrium (as in Milgrom and Roberts (1990)). Morris (1992) gave sufficient conditions for no rationalizable trade exploiting such strategic complementarity arguments.

1.3. *Necessary and Sufficient Conditions*

Unless otherwise indicated, our use of the term "rationalizability" refers to *interim* rationalizability, where different types of the same player may hold different conjectures over the opponent's actions.[3] We also assume that players reject trade when they are indifferent, and that there is a finite number of possible signal realizations. In this setting, we show that trade is rationalizable if and only if the "sign pattern" of the game (constructed in analogy to the cross pattern of the above card example) contains no "cycles." In the special case in which exactly one player always gains ex post, we show there is no rationalizable trade if and only if the game's sign pattern does not have the *cross pattern* of the card example. We also show that this condition is in fact very close to the other two conditions that we discussed above. If exactly one player always gains ex post, there is no rationalizable trade if and only if the game's sign pattern is that of some *trading envelopes problem*, which is true if and only if the game's sign pattern is that of some *monotone zero-sum trade*. This characterization is independent of the players' priors on signals.

The paper's characterization is also related to results on rational expectations equilibria (REE) by DeMarzo and Skiadas (1998, 1999). For example, Proposition 8 of DeMarzo and Skiadas (1999) shows that in finite two-player asymmetric information economies with one risky asset whose payoffs are contingent on the pooled signals, and one risk-free asset, the absence of the cross pattern, properly interpreted, implies the nonexistence of partially informative REE and the uniqueness of a fully informative REE.

[3]Alternative versions of rationalizability are discussed in Section 3.

2. THE RESULT

We consider a game between two players, labeled 1 and 2. The players' prior beliefs are represented by the probabilities P_1 and P_2, respectively, defined on some measurable space $(\Omega, \mathcal{F})$. Player i observes the realization of a signal τ_i: $\Omega \to T_i$, where T_i is a finite set, and $P_i[\tau_i = t_i] > 0$ for all $t_i \in T_i$ and $i \in \{1, 2\}$. We use the notation $\tau = (\tau_1, \tau_2)$ and $T = T_1 \times T_2$ throughout, while the expectation operator with respect to P_i is denoted E_i.

Based on their information, players make a decision to accept or reject a trade. If both players accept the trade, player i receives a state-contingent payoff V_i: $\Omega \to \mathbf{R}$, a P_i-integrable random variable. If either player rejects the trade, the payoff to both players is zero. In contexts in which players are not risk neutral, V_i should be thought of as the ex post utility of player i if trade occurs minus the ex post utility of player i if no trade occurs.

A *strategy* for player i is any function of the form d_i: $T_i \to [0, 1]$. We interpret $d_i(t_i)$ as the probability that player i who observes a signal value t_i will offer to trade. The space of all strategies for player i is denoted D_i, and the no-trade strategy (identically equal to zero) is denoted 0. Given any set of strategies $S \subseteq D_i$, we let $\operatorname{conv}(S)$ denote the convex hull of S.

Given player two strategy $d_2 \in D_2$, player one's *best response* is the strategy $B_1(d_2) \in D_1$, defined by

$$B_1(d_2)(t_1) = \begin{cases} 1 & \text{if } E_1[V_1 d_2(\tau_2) \mid \tau_1 = t_1] > 0; \\ 0 & \text{otherwise.} \end{cases}$$

The second player's best response function, B_2, is defined symmetrically. Implicit in these definitions is the refinement that a trade is rejected if there is zero expected gain from trade. This assumption is discussed and justified in Section 3.

A strategy profile $(d_1, d_2) \in D_1 \times D_2$ is an *equilibrium* if $d_1 = B_1(d_2)$ and $d_2 = B_2(d_1)$. No trade, $(0, 0)$, is always an equilibrium of this game. The familiar no-equilibrium trade result gives sufficient conditions for this to be the unique equilibrium:

PROPOSITION 1. *If $P_1 = P_2$ and $V_1 + V_2 \leq 0$, then the no-trade equilibrium*, $(0, 0)$, *is the unique equilibrium*.

Proof. Suppose that $(d_1, d_2) \neq (0, 0)$ is an equilibrium. Then $d_1 = B_1(d_2)$ implies $E_1[V_1 d_1(\tau_1) d_2(\tau_2)] > 0$, and $d_2 = B_2(d_1)$ implies $E_2[V_2 d_1(\tau_1) d_2(\tau_2)] > 0$. The two inequalities contradict the proposition's assumptions. ■

We consider instead the set of strategies that are rationalizable. The set of rationalizable strategies is identified by iteratively deleting strategies that are not best responses to remaining strategies of the opponent. So starting with $I_i^0 = D_i$, we recursively define I_i^{k+1} to be the set of strategies of player i such that each type of player i is choosing a best response to some belief over strategies in I_{-i}^k. In the static setting of this paper, this is equivalent to the requirement that player i is choosing a best response to some element of the convex hull of I_{-i}^k:

$$I_1^{k+1} = \{d_1 \in I_1^k \mid \forall t_1 \in T_1\colon \exists d_2 \in \mathrm{conv}(I_2^k)\colon d_1(t_1) = B_1(d_2)(t_1)\}$$
$$I_2^{k+1} = \{d_2 \in I_2^k \mid \forall t_2 \in T_2\colon \exists d_1 \in \mathrm{conv}(I_1^k)\colon d_2(t_2) = B_2(d_1)(t_2)\}.$$

The set of *interim rationalizable strategies*[4] for player i is $I_i^\infty = \bigcap_{k\geq 0} I_i^k$. This is the *interim* version of rationalizability because different types of each player are allowed to hold different conjectures about the opponent's actions. The alternative ex ante version of rationalizability is discussed in Section 3.

We write L_i for the set of signal profiles, where player i expects a strict gain from trade:

$$L_i = \{t \in T\colon E_i[V_i 1_{\{\tau=t\}}] > 0\}, \qquad i \in \{1, 2\}, \tag{2.1}$$

where $1_{\{\tau=t\}}$ denotes the random variable that is equal to one on the event $\{\tau = t\}$ and zero otherwise.

Given any pair $(L_+, L_-) \in 2^T \times 2^T$ (for example, (L_1, L_2)), an *n-cycle* of (L_+, L_-) is any set of the form $\{t^1, t^2, \ldots, t^n\} \subseteq L_+$ such that $(t_1^{k+1}, t_2^k) \in L_-$ for all $k \in \{1, \ldots, n-1\}$, and $(t_1^1, t_2^n) \in L_-$. Thus writing a $+$ for elements of L_+ and a $-$ for elements of L_-, the following are examples of a 1-cycle, a 2-cycle, and a 4-cycle, respectively:

$$\begin{array}{c|c|}
 & t_2 \\ \cline{2-2}
t_1 & \pm \\ \cline{2-2}
\end{array}
\qquad
\begin{array}{c|c|c|}
 & t_2 & t_2' \\ \cline{2-3}
t_1 & + & - \\ \cline{2-3}
t_1' & - & + \\ \cline{2-3}
\end{array}
\qquad
\begin{array}{c|c|c|c|c|}
 & t_2^1 & t_2^2 & t_2^3 & t_2^4 \\ \cline{2-5}
t_1^1 & + & & & - \\ \cline{2-5}
t_1^2 & - & + & & \\ \cline{2-5}
t_1^3 & & - & + & \\ \cline{2-5}
t_1^4 & & & - & + \\ \cline{2-5}
\end{array}$$

(L_+, L_-) is *acyclic* if there is no n-cycle for any n.

The following is the paper's main result.

THEOREM 2. *The following conditions are equivalent*:

1. *There is no interim rationalizable trade* ($I_1^\infty = I_2^\infty = \{0\}$).
2. (L_1, L_2) *is acyclic*.

[4] Bernheim (1984) and Pearce (1984) defined rationalizability for complete information games. Static incomplete information games can always be represented in extensive form by a move of nature, followed by simultaneous action choices. Applying Pearce's notion of *extensive form rationalizability* to that game gives interim rationalizability.

3. *There exists a partition* (L_+, L_-) *of* T *without* 2*-cycles, such that* $L_1 \subseteq L_+$ *and* $L_2 \subseteq L_-$.

4. *There exist functions* f_i: $T_i \to \{1, 2, \ldots\}$ *such that* $L_1 \subseteq \{t \in T: f_2(t_2) > f_1(t_1)\}$ *and* $L_2 \subseteq \{t \in T: f_1(t_1) > f_2(t_2)\}$.

5. *There is exists a function* x: $T \to \mathbf{R}$ *that is weakly increasing in each argument under some ordering of* T_1 *and* T_2, *and such that* $L_1 \subseteq \{t \in T: x(t) > 0\}$ *and* $L_2 \subseteq \{t \in T: x(t) < 0\}$.

A complete proof is given in the Appendix. The necessity of condition 2 can be seen along these lines: for trade to be rationalizable for some type t_1^1 of player 1, there must be at least one type of player 2, t_2^1, that makes trade profitable for player 1 and that type t_1^1 of player 1 considers possible. But for type t_2^1 of player 2 to trade, there must be some type t_1^2 of player 1, that makes trade profitable for player 2 and that type t_2^1 of player 2 considers possible. Continuing in this way, one constructs a sequence $\{(t_1^k, t_2^k)\}_{k=1}^{\infty}$. Since T is finite, we can always extract a cycle from this sequence.

To see why the existence of a cycle implies rationalizable trade, note that [never trade] is always a rationalizable strategy for each type. Thus the strategy [trade if and only if type t_2] is a best response to the strategy [trade if and only if type t_1] whenever type t_2 gains from trade at (t_1, t_2) and considers type t_1 possible. If (L_1, L_2) is cyclic, it is possible to construct a sequence of such strategies all of which will survive an arbitrary number of rounds of iterated deletion of dominated strategies.

The last three conditions of Theorem 2 are characterizations of acyclicity that allow us to relate the theorem to the examples of the introduction. Suppose that (L_1, L_2) is a partition of T, which is the case if, for any signal realization, exactly one of the players strictly gains ex post, and all signal profile realizations are possible (that is, for every $t \in T$, $E[V_1 \mid \tau = t]$ and $E[V_2 \mid \tau = t]$ are nonzero and of opposite sign, and $P_i[\tau = t] \neq 0$ for each i). In this case, Theorem 2 has the following implications:

2.1. *The Cross Pattern*

Generalizing the card game example, condition 3 implies that the absence of the cross pattern (a 2-cycle) is necessary and sufficient for no interim rationalizable trade.

2.2. *Trading Envelopes Game*

We noted that an easy sufficient condition for no interim rationalizable trade is that the trade has the structure of a trading envelopes problem, that is, there exist functions f_i: $T_i \to \{1, 2, \ldots\}$ with the interpretation that $f_i(t_i)$ is the number of dollars in the envelope of type t_i of player i; and so $V_1 = -V_2 = f_2(\tau_2) - f_1(\tau_1)$. By condition 4, there is no interim rationalizable trade if and only if (L_1, L_2) is the sign pattern of some trading envelopes game.

2.3. *Monotonicity*

We also noted that a sufficient condition for no interim rationalizable trade is that the trade is zero-sum and monotonic in both players' signals, that is, $V_1 = -V_2 = x(\tau_1, \tau_2)$, where x is weakly increasing in both signals. By condition 5, there is no interim rationalizable trade if and only if (L_1, L_2) is the sign pattern of some monotone zero-sum trade.

3. DISCUSSION

3.1. *Priors and the Size of Ex Post Gains from Trade Do Not Matter*

Players' prior beliefs over the signal space T are irrelevant to the existence of interim rationalizable trade; all that matters is which elements of T each player thinks possible. Similarly, the size of the ex post gain from trade $E_i[V_i \mid \tau = t]$ is irrelevant; all that matters is whether it is strictly positive. This implies that attitudes toward risk and endowments are irrelevant to the existence of interim rationalizable trade as long as trades depend only on the realized signal profile. Specifically, suppose that player i has a strictly increasing von Neumann–Morgenstern utility function u_i, and an endowment given by the random variable e_i. The proposed dollar trade is given by the function $x\colon T \to \mathbf{R}$, where $x(t)$ represents the amount that player two will pay player one in the event $\{\tau = t\}$. The implied net utility gains are

$$V_1 = u_1(e_1 + x(\tau)) - u_1(e_1) \quad \text{and} \quad V_2 = u_2(e_2 - x(\tau)) - u_2(e_2).$$

But calculating the L_1 and L_2 sets for V_1 and V_2 gives

$$L_1 = \{t \in T\colon P_1[\tau = t] > 0 \text{ and } x(t) > 0\}$$

$$L_2 = \{t \in T\colon P_2[\tau = t] > 0 \text{ and } x(t) < 0\}.$$

Thus utility functions and endowments are irrelevant for the existence of interim rationalizable trade.

3.2. *Interim versus Ex Ante Rationalizability*

In applying rationalizability in an incomplete information setting, we face a modeling choice. We can think of players choosing their actions at the interim stage (after observing their private signals), in which case it is natural to allow different types of the same player to have different conjectures over their opponent's strategy. Alternatively, we can think of players choosing their strategies at the ex ante stage (before observing their private signals), in which case it is natural to require different types of the same player to have the same conjecture over their opponent's strategy. The first alternative leads to the interim sense of rationalizability that we described earlier. The second alternative gives the following definition of ex ante rationalizability. Starting with $X_i^0 = D_i$, we recursively define the strategy sets:

$$X_1^{k+1} = \{d_1 \in X_1^k \mid \exists d_2 \in \operatorname{conv}(X_2^k)\colon \forall t_1 \in T_1\colon d_1(t_1) = B_1(d_2)(t_1)\}$$
$$X_2^{k+1} = \{d_2 \in X_2^k \mid \exists d_1 \in \operatorname{conv}(X_1^k)\colon \forall t_2 \in T_2\colon d_2(t_2) = B_2(d_1)(t_2)\}.$$

The set of *ex ante rationalizable strategies* for player i is $X_i^\infty = \bigcap_{k\geq 0} X_i^k$.[5] The difference between the above recursion and that in the definition of interim rationalizability is the transposition of the existential and universal quantifiers. Clearly, $X_i^k \subseteq I_i^k$ for all k, and $X_i^\infty \subseteq I_i^\infty$. So, if (L_1, L_2) is acyclic, there is also no ex ante rationalizable trade. On the other hand, the following example demonstrates that acyclicity of (L_1, L_2) is not necessary for the impossibility of ex ante rationalizable trade.

EXAMPLE 3. Suppose that, for $i \in \{1, 2\}$, $T_i = \{t_i^1, t_i^2, t_i^3\}$, $P_i[\tau = t] = 1/9$ for all $t \in T$, and $V_1 = -V_2 = V$, where V is given as a function of types by the following table:

	t_2^1	t_2^2	t_2^3
t_1^1	0	1	1
t_1^2	-1	$-\varepsilon$	ε
t_1^3	-1	ε	$-\varepsilon$

If $\varepsilon \in (0, 1/3)$, then ex ante rationalizable trade is impossible, even though (L_1, L_2) has a 2-cycle.

However, a partial converse is possible. Suppose that (L_1, L_2) has a cycle, $\{t^1, \ldots, t^n\}$. Modifying P_i so that $P_i[\tau = t] = 1/2n$ for all t associated with that cycle (that is, any t of the form (t_1^k, t_2^k) or (t_1^{k+1}, t_2^k) or (t_1^1, t_2^n)) while

[5] Both ex ante and interim rationalizability assume that the players' priors over the state space are common knowledge. Battigalli (1998) introduced a notion of incomplete information rationalizability that does not depend on players' prior beliefs about types, and is in general weaker than interim rationalizability (in static games). But since our characterization of interim rationalizable trade was independent of the prior beliefs, our interim rationalizability characterization would hold unchanged for Battigalli's notion. An early version of Battigalli (1998) noted the impossibility of rationalizable trade (in his sense) in the finite signal trading envelopes example.

the conditional probabilities $P_i[\cdot \mid \tau = t]$ remain the same,[6] it follows easily that all strategies of the form $1_{\{\tau_i = t^k\}}$ are ex ante rationalizable under the new priors.

3.3. *Refinements*

We assumed that trade was rejected if a player is indifferent between accepting and rejecting. We have in mind that there is a small cost associated with accepting trade (whether or not the trade is implemented). All of the analysis of this paper would remain the same in the presence of such a sufficiently small cost, which was therefore omitted from our formal setup. This assumption ensures that a player's best response to no trade is no trade. If we did not impose this refinement, anything would be a best response to no trade, and therefore any behavior would be interim rationalizable. Thus some form of refinement is required to obtain any interesting results.

Our particular choice of refinement implies that no trade is always interim rationalizable for every type. It is *this* property that ensures that the existence of interim rationalizable trade is independent of the players' prior beliefs about signals. Suppose instead we deleted one round of weak dominated strategies before iterated deletion of strictly dominated strategies (Dekel and Fudenberg, 1990). Then weak dominance considerations would sometimes require that accepting trade was the only possible best response by the opponent. This could make a significant difference, as in the following example.[7]

EXAMPLE 4. $T_1 = \{t_1\}$, $T_2 = \{t_2, t_2'\}$, $P_i[\tau = t] = \frac{1}{2}$ for each $i \in \{1, 2\}$ and $t \in T$, and (V_1, V_2) is given as a function of types by the following table:

	t_2	t_2'
t_1	$1, 1$	$-2, 1$

Trade is interim rationalizable at (t_1, t_2), since (t_1, t_2') is a 1-cycle. However, deleting weak dominated strategies would imply that both types of player 2 must trade. But now accepting trade cannot be a best response for the unique type of player 1.

Thus, while the refinement we chose seems natural for this trading game, it is important to note the subtle role that it plays.

[6] We could also assume that $P_i[\tau = t] > 0$ for all $t \in T$ by letting $P_i[\tau = t] = 1/2n - \varepsilon > 0$ for all $t \in \{t^1, \ldots, t^n\}$, and sufficiently small $\varepsilon > 0$.

[7] We are grateful to a referee for pointing out the implications of the Dekel–Fudenberg (1990) procedure in this example.

3.4. *Infinite Signal Spaces*

The proof of Theorem 2 remains valid if T_1 is finite and T_2 is countably infinite, but Theorem 2 is not valid if both T_1 and T_2 are countably infinite. For example, consider a countably infinite version of the trading envelopes game, where $T_1 = T_2 = \{1, 2, 3, \ldots\}$, $V = \tau_1 - \tau_2$, and $P_i[\tau_{-i} > \tau_i \mid \tau_i = t_i] > 0$, for all $t_i \in T_i$, $i \in \{1, 2\}$. In this case, (L_1, L_2) is acyclic, but all strategies are interim rationalizable.[8]

APPENDIX: PROOF OF THE THEOREM

The paper's theorem is a consequence of the following two lemmas.

LEMMA 5. *For any pair, (L_1, L_2), of subsets of T, the following conditions are equivalent*:

A. (L_1, L_2) *is acyclic*.

B. *There exists a partition*, (L_+, L_-), *of T without cycles such that* $L_1 \subseteq L_+$ *and* $L_2 \subseteq L_-$.

C. *There exists a partition*, (L_+, L_-), *of T without 2-cycles such that* $L_1 \subseteq L_+$ *and* $L_2 \subseteq L_-$.

D. *There exist functions* $f_i\colon T_i \to \{1, 2, \ldots\}$ *such that* $L_1 \subseteq \{t \in T\colon f_2(t_2) > f_1(t_1)\}$ *and* $L_2 \subseteq \{t \in T\colon f_1(t_1) > f_2(t_2)\}$.

E. *There exist bijections* $\phi_1\colon T_1 \to \{1, 2, \ldots, n_1\}$ *and* $\phi_2\colon T_2 \to \{1, 2, \ldots, n_2\}$ *and a (weakly) increasing function*, $x\colon \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \to \mathbf{R}$, *such that* $L_1 \subseteq \{t \in T\colon x(\phi_1(t_1), \phi_2(t_2)) > 0\}$ *and* $L_2 \subseteq \{t \in T\colon x(\phi_1(t_1), \phi_2(t_2)) < 0\}$.

Proof. $(A \Rightarrow B)$ Suppose that (L_1, L_2) has no cycles and $t \notin L_1 \cup L_2$. We claim that either $(L_1 \cup \{t\}, L_2)$ or $(L_1, L_2 \cup \{t\})$ must then have no cycles as well. Intuitively, suppose that attaching t to either L_1 or L_2 creates

[8] The importance of allowing an infinite number of signals is well known from discussions of the trading envelopes problem (Nalebuff, 1989; Geanakoplos, 1994). Bhattacharya and Lipman (1995) showed that, with an infinite number of signals, trade is possible even in interim *equilibrium* with risk neutral agents, a common prior, and zero-sum trades. These assumptions are sufficient to rule out trade if there are only a finite number of signals, since the assumptions imply no ex ante gains from trade. But if there are an infinite number of possible signals and unbounded interim utilities from the trade, ex ante utilities may not be well defined, and the usual no-equilibrium-trade argument cannot be applied. They use a version of the trading envelopes problem to make this point.

a cycle. An illustration of this situation appears below.

$$
\begin{array}{c|c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{5}{c}{t_2} \\
\cline{2-6}
 & + & & - & & \\
\cline{2-6}
 & - & + & & & \\
\cline{2-6}
t_1 & & - & * & & + \\
\cline{2-6}
 & & & + & - & \\
\cline{2-6}
 & & & & + & - \\
\cline{2-6}
\end{array}
$$

It is then clear that the union of these two incomplete cycles forms a cycle without t, contradicting the assumption that (L_1, L_2) contains no cycles.

More formally, suppose that $(L_1 \cup \{t\}, L_2)$ has the cycle $(t^1, \ldots, t^m)$, and $(L_1, L_2 \cup \{t\})$ has the cycle $(s^1, \ldots, s^n)$, each chosen to be of minimal cardinality. Since (L_1, L_2) has no cycles, both of these cycles contain t, and since they were chosen to be minimal, they only contain t once. Without loss in generality, we assume that $t = (t_1^1, t_2^1) = (s_1^1, s_2^n)$. This implies that $(s_1^1, t_2^m) = (t_1^1, t_2^m) \in L_2$ and $(t_1^2, s_2^n) = (t_1^2, t_2^1) \in L_1$. But then the definition of a cycle implies that $(t^2, \ldots, t^m, s^1, \ldots, s^n)$ is a cycle of (L_1, L_2), a contradiction. Repeating this procedure, all elements of T can be signed without creating cycles, confirming condition B.

$(B \Rightarrow C)$ Immediate.

For the next part of this proof, we use an argument similar to that of Proposition 7 in DeMarzo and Skiadas (1999).

$(C \Rightarrow D)$ Suppose (L_+, L_-) is a partition of T without 2-cycles such that $L_1 \subseteq L_+$ and $L_2 \subseteq L_-$. Define the sets

$$F_1(t_1) = \{t_2' \in T_2\colon (t_1, t_2') \in L_+\} \text{ and}$$

$$F_2(t_2) = \{t_2' \in T_2\colon (t_1', t_2') \in L_+ \text{ for some } (t_1', t_2) \in L_-\},$$

and let $f_1(t_1) = K - 2|F_1(t_1)|$ and $f_2(t_2) = K - 1 - 2|F_2(t_2)|$, for some large positive integer K (where the notation $|F|$ represents the cardinality of the set F). By definition, $(t_1, t_2) \in L_-$ implies $F_1(t_1) \subseteq F_2(t_2)$. Thus $K - f_1(t_1) = 2|F_1(t_1)| \leq 2|F_2(t_2)| = K - 1 - f_2(t_2)$; so $f_1(t_1) - f_2(t_2) \geq 1$. Also, $(t_1, t_2) \in L_+$ implies $F_2(t_2) \subseteq F_1(t_1) \backslash \{t_2\}$ and $t_2 \in F_1(t_1)$. Thus $K - 1 - f_2(t_2) = 2|F_2(t_2)| \leq 2(|F_1(t_1)| - 1) = K - f_1(t_1) - 2$; so $f_2(t_2) - f_1(t_1) \geq 1$.

$(D \Rightarrow E)$ Suppose $f_i\colon T_i \to \{1, 2, \ldots\}$ satisfy $L_1 \subseteq \{t \in T\colon f_2(t_2) > f_1(t_1)\}$ and $L_2 \subseteq \{t \in T\colon f_1(t_1) > f_2(t_2)\}$. Choose bijections $\phi_1\colon T_1 \to \{1, 2, \ldots, n_1\}$ and $\phi_2\colon T_2 \to \{1, 2, \ldots, n_2\}$ such that $f_2(t_2) > f_2(t_2') \Rightarrow \phi_2(t_2) > \phi_2(t_2')$ and $f_1(t_1) > f_1(t_1') \Rightarrow \phi_1(t_1) < \phi_1(t_1')$. Let $x(i, j) =$

$f_2([\phi_2]^{-1}(j)) - f_1([\phi_1]^{-1}(i))$. By construction, (ϕ_1, ϕ_2, x) satisfy condition E.

$(E \Rightarrow A)$ Immediate. ■

LEMMA 6. *Suppose* L_1 *and* L_2 *defined by Eq.* (2.1). *Then there is no interim rationalizable trade if and only if* (L_1, L_2) *is acyclic.*

Proof. Starting with $T_i^0 = T_i$, we recursively define the sets

$$T_1^{k+1} = \{t_1 \in T_1^k\colon (t_1, t_2) \in L_1 \text{ for some } t_2 \in T_2^k\}$$
$$T_2^{k+1} = \{t_2 \in T_2^k\colon (t_1, t_2) \in L_2 \text{ for some } t_1 \in T_1^k\},$$

and we let $T_i^\infty = \bigcap_{k\geq 0} T_i^k$. Given any set $T_i' \subseteq T_i$, we let $D_i(T_i')$ denote the set of all $d_i \in D_i$ such that $d_i(t_i) = 0$ if $t_i \notin T_i'$. Now we may verify by induction that $I_i^k = D_i(T_i^k)$ for all k; this is true by definition for $k = 0$. Suppose that it is true for k. Now if $t_1 \notin T_1^{k+1}$, then $(t_1, t_2) \notin L_1$ for all $t_2 \in T_2^k$. Thus $E_1[V_1 d_2 | \tau_1 = t_1] \leq 0$ for all $d_2 \in I_2^k$, and not offering to trade is the best response for player 1 given signal t_1. On the other hand, if $t_1 \in T_1^{k+1}$, then $(t_1, t_2) \in L_1$ for some $t_2 \in T_2^k$. The strategy of accepting only given signal t_2 ($d_2(t_2') = 1_{\{t_2'=t_2\}}$) is in T_2^k. So offering to trade is a best response for player 1 given signal t_1. But the strategy of never offering to trade is also in T_2^k. So not offering to trade is also a best response for player 1 given signal t_1. Thus $I_1^{k+1} = D_1(T_1^{k+1})$. A symmetric argument implies $I_2^{k+1} = D_2(T_2^{k+1})$. So $I_i^\infty = D_i(T_i^\infty)$, $i \in \{1, 2\}$. Thus there is interim rationalizable trade if and only if $T_i^\infty \neq \emptyset$.

Now if (L_1, L_2) has a cycle, that cycle clearly survives all steps in the above elimination, and therefore trade is interim rationalizable.

To prove the converse, observe that

$$t_1 \in T_1^\infty \quad \Rightarrow \quad (t_1, t_2) \in L_1 \qquad \text{for some } t_2 \in T_2^\infty$$
$$t_2 \in T_2^\infty \quad \Rightarrow \quad (t_1, t_2) \in L_2 \qquad \text{for some } t_1 \in T_1^\infty.$$

Suppose now that $t_1^0 \in T_1^\infty$. Then there exists $t_2^0 \in T_2^\infty$ such that $(t_1^0, t_2^0) \in L_1$. Similarly, there exists $t_1^1 \in T_1^\infty$ such that $(t_1^1, t_2^0) \in L_2$. Repeating this argument, we generate a sequence $\{t^k\colon k = 0, 1, \ldots\}$ such that $(t_1^k, t_2^k) \in L_1$ and $(t_1^{k+1}, t_2^k) \in L_2$ for all k. If (L_1, L_2) is acyclic, then this sequence must take an infinite number of values, contradicting our assumption that T is finite. ■

REFERENCES

Aumann, R. (1987). "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica* **55**, 1–18.

Battigalli, P. (1998). “Rationalizability in Incomplete Information Games,” working paper, Princeton University.

Bernheim, D. (1984). “Rationalizable Strategic Behavior,” *Econometrica* **52**, 1007–1028.

Bhattacharya, S., and Lipman, B. (1995). “Ex Ante versus Interim Rationality and the Existence of Bubbles,” *Econ. Theory* **6**, 469–494.

Dekel, E., and Fudenberg, D. (1990). “Rational Behavior with Payoff Uncertainty,” *J. Econ. Theory* **52**, 243–267.

Dekel, E., and Gul, F. (1997). “Rationality and Common Knowledge in Game Theory,” in *Advances in Economics and Econometrics: Seventh World Congress of the Econometric Society*, Vol. 1. Cambridge, UK: Cambridge Univ. Press.

DeMarzo, P., and Skiadas, C. (1998). “Aggregation, Determinacy, and Informational Efficiency for a Class of Economies with Asymmetric Information,” *J. Econ. Theory* **80**, 123–152.

DeMarzo, P., and Skiadas, C. (1999). “On the Uniqueness of Rational Expectations Equilibria,” *Econ. Theory* **13**, 1–24.

Geanakoplos, J. (1994). “Common Knowledge,” in *Handbook of Game Theory* Vol. 2, Chap. 40. (R. Aumann and S. Hart, Eds.), Amsterdam/New York: Elsevier Science.

Milgrom, P., and Roberts, J. (1990). “Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities,” *Econometrica* **58**, 1255–1278.

Morris, S. (1992). “Acceptance Games and Protocols,” CARESS working paper 92-22, University of Pennsylvania.

Morris, S. (1994). “Trade with Heterogeneous Prior Beliefs and Asymmetric Information,” *Econometrica* **62**, 1327–1347.

Nalebuff, B. (1989). “Puzzles: The Other Person’s Envelope Is Always Greener,” *J. Econ. Perspect.* **3**, 171–181.

Pearce, D. (1984). “Rationalizable Strategic Behavior and the Problem of Perfection,” *Econometrica* **52**, 1029–1050.

Sebenius, J., and Geanakoplos, J. (1983). “Don’t Bet On It: Contingent Agreements with Asymmetric Information,” *J. Am. Stat. Assoc.* **78**, 424–426.