Online Appendix for "Central Bank Forward Guidance and the Signal Value of Prices"

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In the text, we carried our analysis under the informal assumption that the state θ was uniformly distributed on the real line. Here we solve the model under the more general assumption that θ has a proper normal prior distribution with mean $\overline{\theta}$ and precision δ. The objective of the central bank is then to minimize the variance of $(r-\theta)$, given the prior mean $\overline{\theta}$. This problem is algebraically involved. Our purpose in the section is threefold. First, we lay out the logic of the general solution even though it becomes algebraically intractable. We confirm that the improper prior analysis in the paper corresponds to the limit as the precision δ tends to 0, i.e., as the prior approaches the improper limit. And we show that as the semi-public signal becomes uninformative, the central bank is able to perfectly target the state by conditioning policy on the fully public prior mean $\overline{\theta}$. As discussed in the text, we don't think this is the ecnomically relevant case for analysis, but we study the problem in order to fully appreciate the logic of this model.

1 General Analysis

We assume that the central bank commits to a linear rule

$$r = c + \lambda \overline{a} + \mu z \tag{1}$$

Market participant i's optimal strategy must satisfy

$$\begin{aligned} a_i &= wE_i(r) + (1-w)E_i(\theta) \\ &= wE_i(c + \lambda\overline{a} + \mu z) + (1-w)E_i(\theta) \quad (2)\\ &= wc + w\lambda E_i(\overline{a}) + w\mu z + (1-w)E_i(\theta) \quad (3) \end{aligned}$$

Assuming a linear equilibrium strategy

$$a_i = \kappa + \xi x_i + \nu y$$

we will have

$$\overline{a} = \kappa + \xi\theta + \nu y$$

Substituting into the equilibrium condition, we have

$$\begin{aligned} a_i &= wc + w\lambda E_i(\overline{a}) + w\mu E_i(z) + (1-w)E_i(\theta) \\ &= wc + w\lambda E_i(\kappa + \xi\theta + \nu y) + w\mu E_i(\theta) + (1-w)E_i(\theta) \\ &= wc + w\lambda\kappa + w\lambda\nu y + (w\lambda\xi + w\mu + 1 - w)E_i(\theta) \\ &= wc + w\lambda\kappa + w\lambda\nu y + (w\lambda\xi + w\mu + 1 - w)\frac{\alpha y + \beta x_i + \delta\overline{\theta}}{\alpha+\beta+\delta} \\ &= w(c + \lambda\kappa) + (w\lambda\xi + w\mu + 1 - w)\frac{\delta}{\alpha+\beta+\delta}\overline{\theta} + (w\lambda\xi + w\mu + 1 - w)\frac{\beta}{\alpha+\beta+\delta}x_i \\ &+ \left(w\lambda\nu + (w\lambda\xi + w\mu + 1 - w)\frac{\alpha}{\alpha+\beta+\delta}\right)y \end{aligned}$$

Matching coefficients, we have

$$\begin{aligned}
\kappa &= w\left(c+\lambda\kappa\right)+\left(w\lambda\xi+w\mu+1-w\right)\frac{\delta}{\alpha+\beta+\delta}\overline{\theta}\\
\xi &= \left(w\lambda\xi+w\mu+1-w\right)\frac{\beta}{\alpha+\beta+\delta}\\
&= \left(w\lambda\xi+1-w\left(1-\mu\right)\right)\frac{\beta}{\alpha+\beta+\delta}\\
\nu &= w\lambda\nu+\left(w\lambda\xi+w\mu+1-w\right)\frac{\alpha}{\alpha+\beta+\delta}
\end{aligned}$$

Thus

$$\begin{aligned}
\xi &= \frac{\left(1-\left(1-\mu\right)w\right)\frac{\beta}{\alpha+\beta+\delta}}{1-w\lambda\frac{\beta}{\alpha+\beta+\delta}}\\
&= \frac{\beta\left(1-w\left(1-\mu\right)\right)}{\alpha+\delta+\beta(1-w\lambda)} \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\nu &= \frac{\alpha(1-w(1-\mu-\lambda\xi))}{(1-w\lambda)(\alpha+\beta+\delta)}\\
&= \frac{\alpha\left(1-w\left(1-\mu-\lambda\frac{\beta(1-w(1-\mu))}{\alpha+\delta+\beta(1-w\lambda)}\right)\right)}{(1-w\lambda)(\alpha+\beta+\delta)}\\
&= \frac{\alpha(1-w(1-\mu))}{(1-w\lambda)(\alpha+\delta+\beta(1-w\lambda))} \qquad (5)\\
\kappa &= \frac{cw+\frac{\delta}{\alpha+\beta+\delta}\overline{\theta}(1-w(1-\mu-\lambda\xi))}{1-w\lambda}\\
&= \frac{cw+\frac{\delta}{\alpha+\beta+\delta-\beta\lambda w}\overline{\theta}(1-w(1-\mu))}{1-w\lambda} \qquad (6)
\end{aligned}$$

Now the central bank chooses c, λ and μ, anticipating the the private sector will best response with equilibrium κ, ξ and ν defined above. Now

$$\begin{aligned}
r-\theta &= c+\lambda\overline{a}+\mu z-\theta\\
&= c+\lambda\kappa+\lambda\xi\theta+\lambda\nu y+\mu z-\theta\\
&= c+\lambda\kappa+\lambda\xi\theta+\lambda\nu\left(\theta+\eta\right)+\mu\left(\theta+\zeta\right)-\theta\\
&= c+\lambda\kappa+\left(\lambda\xi+\lambda\nu+\mu-1\right)\theta+\lambda\nu\eta+\mu\zeta\\
&= c+\lambda\kappa+(\lambda\xi+\lambda\nu+\mu-1)\overline{\theta}+\left(\lambda\xi+\lambda\nu+\mu-1\right)\left(\theta-\overline{\theta}\right)+\lambda\nu\eta+\mu\zeta
\end{aligned}$$

The variance of this is

$$\left(c+\lambda\kappa+(\lambda\xi+\lambda\nu+\mu-1)\overline{\theta}\right)^2+\left(\lambda\xi+\lambda\nu+\mu-1\right)^2\frac{1}{\delta}+\lambda^2\nu^2\frac{1}{\alpha}+\mu^2\frac{1}{\gamma}$$

Now observe that c impacts κ, but not ξ and ν. Thus given λ, μ, ξ and ν, we must choose c to set

$$c+\lambda\kappa+(\lambda\xi+\lambda\nu+\mu-1)\overline{\theta}=0$$

so

$$c+\lambda\left(\frac{wc+\left(w\lambda\xi+w\mu+1-w\right)\frac{\delta}{\alpha+\beta+\delta}\overline{\theta}}{1-w\lambda}\right)+(\lambda\xi+\lambda\nu+\mu-1)\overline{\theta}=0$$

and

$$\begin{aligned}
c &= \overline{\theta}\frac{1-\lambda\xi-\lambda\nu-\mu-\frac{\lambda\left(w\lambda\xi+w\mu+1-w\right)\frac{\delta}{\alpha+\beta+\delta}}{1-w\lambda}}{1+\frac{\lambda w}{1-w\lambda}}\\
&= \overline{\theta}\left(\left(1-\lambda\xi-\lambda\nu-\mu\right)\left(1-w\lambda\right)-\lambda\left(w\lambda\xi+w\mu+1-w\right)\frac{\delta}{\alpha+\beta+\delta}\right)
\end{aligned}$$

With this, the variance will be

$$
\begin{aligned}
&(\lambda\xi+\lambda\nu+\mu-1)^2\frac{1}{\delta}+\lambda^2\nu^2\frac{1}{\alpha}+\mu^2\frac{1}{\gamma}\\
&=\left(\lambda\left(\frac{\beta\left(1-w\left(1-\mu\right)\right)}{\alpha+\delta+\beta(1-w\lambda)}+\frac{\alpha(1-w(1-\mu))}{(1-w\lambda)(\alpha+\delta+\beta(1-w\lambda))}\right)+\mu-1\right)^2\frac{1}{\delta}+\lambda^2\left(\frac{\alpha(1-w(1-\mu))}{(1-w\lambda)(\alpha+\delta+\beta(1-w\lambda))}\right)^2\frac{1}{\alpha}+\mu^2\frac{1}{\gamma}\\
&=\left(\frac{\lambda\left(1-w\left(1-\mu\right)\right)}{\alpha+\delta+\beta(1-w\lambda)}\left(\beta+\frac{\alpha}{(1-w\lambda)}\right)+\mu-1\right)^2\frac{1}{\delta}+\lambda^2\left(\frac{\alpha(1-w(1-\mu))}{(1-w\lambda)(\alpha+\delta+\beta(1-w\lambda))}\right)^2\frac{1}{\alpha}+\mu^2\frac{1}{\gamma}
\end{aligned}
$$

Note that this function is quadratic in μ and therefore the optimal μ can be found as the solution to FOC

$$
\begin{aligned}
0=&\frac{\mu}{\gamma}+\frac{\alpha\lambda^2w(1-(1-\mu)w)}{(1-w\lambda)^2(\alpha+\delta+\beta(1-w\lambda))^2}\\
&+\frac{(\alpha+\beta+\delta-\lambda w(\beta+\delta))(\alpha(\lambda+\mu-1)+(1-w\lambda)(\beta(\lambda+\mu-1)-\delta(1-\mu)))}{\delta(1-w\lambda)^2(\alpha+\delta+\beta(1-w\lambda))^2}
\end{aligned}
$$

The solution is generically unique (unless the denominator is zero) and given by

$$
\mu^*_\lambda(\alpha,\beta,\gamma,\delta,w)=\frac{\gamma\left(\alpha^2(1-\lambda)+\alpha(2\beta(1-\lambda)(1-w\lambda)+\delta(\lambda(w(\lambda w-2)-1)+2))+(\beta+\delta)(1-w\lambda)^2(\beta(1-\lambda)+\delta)\right)}{\left[\begin{array}{c}(1-w\lambda)^2\left((\beta+\delta)^2(\gamma+\delta)+\beta^2\delta\lambda^2w^2-2\beta\delta\lambda w(\beta+\delta)\right)+\alpha^2\left(\gamma+\delta(1-w\lambda)^2\right)\\+\alpha\left(\delta\left(\gamma(\lambda w(\lambda w-2)+2)+2\delta(1-w\lambda)^2\right)+2\beta(1-w\lambda)\left(\gamma+\delta(1-w\lambda)^2\right)\right)\end{array}\right]} \tag{7}
$$

Substituting (7) into the expression for the variance, we have a closed form expression for the variance as a function of λ. The CB problem corresponds to finding the λ minimizing this expression, which then pins down the value of μ and c in the optimal policy.

2 The Improper Prior Limit

We consider what happens as $\delta\to 0$, and thus the prior approaches the improper prior considered in the text. Observe that

$$\lim_{\delta\to 0}\mu^*_\lambda(\alpha,\beta,\gamma,\delta,w)=1-\lambda$$

The intuition is that under an improper prior, the CB must put wieghts adding up to 1 on $\bar{a}$ and z. Now the variance reduces to

$$\lambda^2\left(\frac{\alpha}{\alpha+\beta(1-w\lambda)}\right)^2\frac{1}{\alpha}+(1-\lambda)^2\frac{1}{\gamma} \tag{8}$$

which is the variance minimized in the text.

3 Removing the Semi-Public Signal

We now consider what happens when we remove the semi-public signal, or equivalently let $\alpha\to 0$. Rather than solving for the optimum policy, we will identify a policy that will make variance arbitrarily small. This policy involves putting a small weight on ε on z, and then putting a weight on $\bar{a}$ that will induce the private sector to fully reveal their private information.

Suppose that the CB follows the rule

$$r=(1-\varepsilon)\left(\bar{a}+\frac{\delta}{\beta}\left(\bar{a}-\bar{\theta}\right)\right)+\varepsilon z$$

This corresponds to setting $c=-(1-\varepsilon)\frac{\delta}{\beta}\bar{\theta}$, $\lambda=(1-\varepsilon)\frac{\beta+\delta}{\beta}$ and $\mu=\varepsilon$ in equation (1). Substituting these parameters and $\alpha=0$, we can derive the optimal policy rule from equations (4), (5) and (6), we obtain

$$\xi=\frac{\beta\left(1-w\left(1-\mu\right)\right)}{\alpha+\delta+\beta\left(1-w\lambda\right)}=\frac{\beta\left(1-w\left(1-\varepsilon\right)\right)}{\delta+\beta\left(1-w\left(1-\varepsilon\right)\frac{\beta+\delta}{\beta}\right)}=\frac{\beta}{\beta+\delta}$$

$$\kappa = \frac{cw + \frac{\delta}{\alpha+\beta+\delta}\overline{\theta}(1 - w(1-\mu-\lambda\xi))}{1 - w\lambda} = \frac{-w\left(1-\varepsilon\right)\frac{\delta}{\beta}\overline{\theta} + \frac{\delta}{\beta+\delta}\overline{\theta}(1 - w(1-\varepsilon-(1-\varepsilon)))}{1 - w\left(1-\varepsilon\right)\frac{\beta+\delta}{\beta}} = \frac{\delta}{\beta+\delta}\bar{\theta}$$

and $\nu = 0$,

giving linear equilibrium strategy

$$a_i = \frac{\delta}{\beta+\delta}\bar{\theta} + \frac{\beta}{\beta+\delta}x_i;$$

thus in equilibrium,

$$\bar{a} = \frac{\delta}{\beta+\delta}\bar{\theta} + \frac{\beta}{\beta+\delta}\theta$$

and

$$r = \left(1-\varepsilon\right)\theta + \varepsilon z$$

Now the expectation of $\left(r-\theta\right)^2$ is $\frac{\varepsilon^2}{\gamma}$. Thus the central bank can make the variance arbitrarily small by setting ε sufficieintly small.

As noted in the text, we can see that this is an equilibrium direct, without appealing to the more general solution. Notice that this is an equilibrium for any value of $\varepsilon \geq 0$.