

Depth of knowledge and the effect of higher order uncertainty[★]

Stephen Morris¹, Andrew Postlewaite¹, and Hyun Song Shin²

¹ Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA

² Department of Economics, University of Southampton, Southampton, ENGLAND

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Summary. A number of recent papers have highlighted the importance of uncertainty about others' information in models of asymmetric information. We introduce a notion that reflects the depth of knowledge in an information system. We show how the depth of knowledge can be used to bound the effect of higher order uncertainty in certain problems. We further provide bounds on the size of bubbles in finite horizon rational expectations models where the bounds depend on the depth of knowledge.

1. Introduction

In the last twenty five years, tremendous strides have been made in the study of informational issues in economics. The standard technique of addressing informational issues is to assume that some or all individuals in the economy receive some information, and that the information structure (the distribution of signals each individual receives) is common knowledge. While this seems like a strong assumption, an argument going back to Harsanyi (1967/68) suggests that it is without loss of generality. If one individual is uncertain of the information structure of another individual, we can consider an expanded state space in which such uncertainty is explicitly modelled. A number of authors, including Mertens and Zamir (1985) and Brandenburger and Dekel (1993), have studied the mathematical properties of that infinite hierarchy of beliefs.

In an important respect this response is not helpful. Even if there exists a universal belief space where information structures are, in a formal sense, common knowledge, the fact remains that the vast majority of work in information economics is not working with such unwieldy objects. The standard approach begins with some uncertainty space representing payoff-relevant, or "fundamental", events, and descriptions of the individuals' information about these fundamentals. It is assumed that the fundamental events and individuals' partitions of the fundamental events are common knowledge. Stated differently, the standard form of asymmetry of

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information in economic models incorporates asymmetry about fundamentals, not others' knowledge.

It is now well documented that uncertainty about others' knowledge can matter, even when everyone knows the fundamentals. Milgrom and Stokey (1982) showed that if it is common knowledge that there are no *ex ante* gains from trade among a group of individuals, then even in the presence of asymmetric information, it is impossible for them to trade with each other. However, it is possible that they may trade with each other when they all know there are no gains from trade (i.e. the relevant fundamentals), but this is not common knowledge (see also Sebenius and Geanakoplos (1983), Geanakoplos (1992) and Morris (1992)). In game theory, Milgrom and Roberts (1982) and Kreps, Milgrom, Roberts and Wilson (1982) argued that not only can a small amount of uncertainty about payoffs lead to a breakdown of backward induction in finite games, but also that even when all players know the true payoffs but that this is not common knowledge, backward induction may break down. (See also Rubinstein (1989), Shin and Williamson (1992), Morris, Rob and Shin (1995)). Aumann (1992) shows that high order mutual knowledge of rationality gives rise to predictions which diverge from those which arise given common knowledge of rationality. (A conspicuous exception to this general rule is the argument of Aumann and Brandenburger (1991) which suggests that Nash equilibrium may be justified by means of a single layer of mutual knowledge when there are two players.) In a model of financial assets, Allen, Morris and Postlewaite (1993) present an instance of a bubble in which every individual knows an asset will never yield a positive return, but nonetheless that asset trades at a positive price. Such a bubble may exist in a rational expectations equilibrium in a finite horizon economy when there are short sales constraints. While individuals have the same knowledge about fundamentals, uncertainty about others' knowledge allows this result.²

Models in which qualitative results can be reversed when uncertainty about fundamentals is augmented by higher order uncertainty about the information of others cast doubt on the lessons to be learned from models that ignore such higher order uncertainty. The assumption that uncertainty is only about fundamentals rather than about the information of others is typically for tractability, not descriptive reality; one is seldom confident that the fully detailed models, including individuals' preferences, payoffs and strategic options, are common knowledge. One response to this concern might be a hope that while there may be higher order uncertainty in most problems of interest, for some problems the higher order uncertainty is sufficiently small in some well-defined sense that equilibria of models which ignore it for reasons of tractability don't substantially differ from the equilibria in more complicated models that properly take into account the higher order uncertainty. This paper provides sufficient conditions under which this will hold.

The structure of the paper is as follows. In Section 2, we provide a framework in which we may identify the depth of knowledge inherent in a given finite information system.³ We will speak of two states differing in their fundamentals if they differ in

² See also Abel and Mailath (1994) and Kraus and Smith (1990) for related examples.

³ Haifetz and Samet (1993) independently introduce a closely related concept of depth of knowledge.

some payoff relevant way. Two states that don't differ in the fundamentals may differ with respect to individuals' knowledge in the states. For example, it may be that at one of the two states, all individuals know the fundamentals while at the second some may not. Given a set of events which represent fundamentals, we ask how many iterations of knowledge are necessary to distinguish from each other all those states which it is possible to distinguish on the basis of fundamentals and iterated knowledge of fundamentals. This exercise is of some independent interest in understanding information structures of asymmetric information problems.

In Section 3, we introduce methods of approximating knowledge. Our approach is to restrict attention to a subset of the state space, and define conditional knowledge on that subset (that is, what does each agent know, conditional on that subset being true). This leads to natural notions of iterated conditional knowledge and the depth of knowledge on the subset. When this event is assigned sufficiently high probability whenever it is true, this subset will be a good approximation for the whole state space. Monderer and Samet (1989) defined this property formally: they said that an event is an "evident p -belief" if every individual assigns it probability at least p whenever it is true. In general, we will be interested in subsets of the state space that have high probability when true and have a lower depth of knowledge than the original state space. This will be the basis for identifying when problems with a high depth of knowledge can be approximated by problems with a lower depth. We will also review the relation between "common p -belief" – Monderer and Samet's notion of almost common knowledge – and evident p -belief events in this section.

In Section 4, we illustrate our approach through an example of great interest itself, the possibility of bubbles in a financial asset model with a finite horizon. Say that an event is k th order mutual knowledge if everyone knows that everyone knows that... (k times) the event holds. First, we show that in rational expectations equilibrium with short sales constraints and a finite T period horizon, a necessary condition for an asset to have a strictly positive price is that it is not T th order mutual knowledge that the final period dividend of the asset will be zero. A result by Allen, Morris and Postlewaite (1993) showed that the result is tight in the sense that there exists a rational expectations equilibrium with an asset having a positive price when there is first order mutual knowledge that the asset is valueless. In our model, a consequence of the definition of the depth of knowledge (in Section 2) is that if the depth of knowledge of an information system is n , and $m \geq n$, an event is m th order mutual knowledge if and only if it is common knowledge (k th order mutual knowledge for every k). Thus if the depth of knowledge is less than T (the number of periods remaining), T th order knowledge that the asset is valueless is sufficient to ensure that the price is zero.

We are able to approximate this result with the techniques we outlined above. Suppose there is an evident p -belief event such that the depth of knowledge restricted to that event is n . Suppose that at some state in that event it is n th order knowledge that the asset is worthless. Then there is an upper bound on the price of the asset. The upper bound is the product of the number of periods until the dividend pays, $1 - p$, and the maximum possible value of the dividend.

In the concluding section, Section 5, we show how the idea of the depth of knowledge could be extended to capture the depth of belief implicit in an information system.

2. Depth of knowledge

Let us introduce the ideas of this section with an information structure which has appeared in a number of studies of iterated knowledge, including Rubinstein's (1989) electronic mail game, the co-ordinated attack problem in the computer science literature (see Halpern (1986)) and no trade paradoxes of Geanakoplos (1992). Suppose that two individuals are concerned about whether some event E has occurred. Individual 1 knows whether E has occurred, or not. Individual 2 initially does not know. If E has occurred, individual 1 sends a message informing 2 of that fact. There is some possibility that the message gets lost, so if 2 receives such a message, he will send back a message to 1 confirming receipt of the message. Again, there is some possibility of the message getting lost, so 1 will in turn send a message confirming the confirmation, if he receives the confirmation. This sequence of confirmations continues until a maximum of six messages have been sent, which, let us say, exhausts the time available for confirmation. Note that we envisage an environment where the sending of messages is not strategic.

The uncertainty facing the individuals in this environment can be represented as follows:

Finite state space	$\Omega = \{0, 1, 2, 3, 4, 5, 6\}$
Fundamental event	$E = \Omega / \{0\} = \{1, 2, 3, 4, 5, 6\}$
1's partition	$\mathcal{P}_1 = (\{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\})$
2's partition	$\mathcal{P}_2 = (\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6\})$

At state 0, event E has not occurred and no messages have been sent. At states $\omega \in \{1, 2, 3, 4, 5, 6\}$, ω messages have been sent. For ω odd, individual 1's last message was sent but not received. For $\omega = 2$ or 4, individual 2's last message was sent but not received. For $\omega = 6$, all six messages were sent and received.

Intuitively, this state space requires a depth of knowledge of six for the following reason. Consider states 5 and 6. At both states, E is true, 1 knows that E is true, 2 knows that 1 knows that E is true, 1 knows that 2 knows that 1 knows that E is true, 2 knows that 1 knows that 2 knows that 1 knows that E is true and 1 knows that 2 knows that 1 knows that 2 knows that 1 knows E . In fact, every statement of this form that involves no more than five knowledge statements is true at both state 5 and state 6. On the other hand, at state 5, 2 does not know that 1 knows that 2 knows that 1 knows that 2 knows that 1 knows E , whereas at state 6, 2 does know that this is the case. Thus states 5 and 6 can be distinguished on the basis of sixth order statements about knowledge. Since *any* pair of states can be distinguished on the basis of sixth order statements about knowledge, it is natural to say that the depth of knowledge in this information system is six. The results of this section show how to identify the depth of knowledge of any information system.

Let Ω be the state space. We shall assume that Ω is finite. Denote by \mathcal{F} the partition of Ω generated by the "fundamentals". Thus, if two states ω and ω' belong

to the same element F of \mathcal{F} , then the description of the world in terms of the fundamentals – the payoff relevant aspects of the problem – are identical at the two states. Thus in the example, we said that the two individuals care about whether event E occurred or not, so that their payoffs depend on whether E occurred or not. Thus \mathcal{F} would consist of $\{0\}$ and $\{1, 2, 3, 4, 5, 6\}$. Write $F(\omega)$ for the element of \mathcal{F} containing ω . There is a set of *individuals*, indexed by the finite set I . The individual $i \in I$ has the information partition \mathcal{P}_i over Ω . We denote by $P_i(\omega)$ the element of \mathcal{P}_i which contains the state ω .

We are interested in the Boolean algebra of subsets of Ω generated by \mathcal{F} . Toward this end we say that a collection of sets \mathcal{C} is *closed under intersection* if $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$, and \mathcal{C} is *closed under complementation* if $A \in \mathcal{C} \Rightarrow -A \in \mathcal{C}$, where $-A$ is the complement of A in Ω . The Boolean algebra of subsets of Ω generated by \mathcal{F} is the smallest collection of sets \mathcal{C}_0 such that $\mathcal{F} \subseteq \mathcal{C}_0$ and \mathcal{C}_0 is closed under intersection and complementation.

We shall refer to \mathcal{C}_0 as the set of *events of degree 0*. In the example, \mathcal{C}_0 consists of the four events $\emptyset, \{0\}, \{1, 2, 3, 4, 5, 6\}$ and Ω . We can interpret \mathcal{C}_0 as the set of events which correspond to the set of logical propositions which describe the fundamental features of the world. The closure conditions given above reflect the closure conditions for the set of logical propositions under the usual logical connectives of negation, conjunction, disjunction and implication. The finiteness of Ω implies that any Boolean algebra is also a σ -algebra, and there is no loss in generality in restricting attention to sets of events with finite closure properties as opposed to sets of events closed under countable intersections.

For each individual $i \in I$, we define the *knowledge operator* K_i as follows. For any $A \subset \Omega$,

$$K_i(A) \equiv \{\omega \mid P_i(\omega) \subseteq A\}. \tag{2.1}$$

$K_i(A)$ is the event in which individual i knows that the true state is in A .

We defined above \mathcal{C}_0 , the fundamental events. We want to extend this set to include propositions not only about the fundamental features of the world, but also about what individuals know about those events. We say that a set \mathcal{C}_{n+1} is a *knowledge extension* of \mathcal{C}_n if \mathcal{C}_{n+1} is the smallest collection of sets such that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$, \mathcal{C}_{n+1} is closed under intersection and complementation, and $A \in \mathcal{C}_n \Rightarrow K_i(A) \in \mathcal{C}_{n+1}$, for all $i \in I$.

We shall refer to \mathcal{C}_n as the set of *events of degree n* . These events correspond to the set of propositions which describe the world not only in terms of the fundamentals, but also in terms of the knowledge of the individuals in I up to degree n . Thus, \mathcal{C}_n corresponds to propositions which are rich enough to describe the n th order iterated knowledge of the individuals. In the example, \mathcal{C}_1 consists of all unions and intersections of the events $\{0\}, \{1\}$ and $\{2, 3, 4, 5, 6\}$; \mathcal{C}_2 consists of all unions and intersections of the events $\{0\}, \{1\}, \{2\}$, and $\{3, 4, 5, 6\}$; et cetera. Although no explicit logic has been invoked in this construction, the sequence of Boolean algebras defines a logic in the manner of Shin (1993).

The interpretation of the above sequence of events gives rise to the following definition.

Definition. Ω has *depth of knowledge n* if $\mathcal{C}_{n-1} \neq \mathcal{C}_n$, but $\mathcal{C}_n = \mathcal{C}_l$, for all $l \geq n$.

We require n th order statements in order to distinguish states on the basis of iterated knowledge of fundamentals. Since Ω is finite, the depth of knowledge is well-defined. It will be useful to define the operator K_* on the set of events.

$$K_*(A) \equiv \bigcap_{i \in I} K_i(A).$$

$K_*(A)$ is the event which corresponds to the proposition “everyone knows A ”. We will say that the event A is *mutual knowledge* in the event $K_*(A)$. We shall denote by $K_*^m(A)$ the m -iterated application of K_* to event A . That is,

$$K_*^m(A) \equiv K_*(K_*(\dots K_*(A)\dots)).$$

$K_*^m(A)$ is the event which corresponds to the proposition “everyone knows that everyone knows that \dots that everyone knows A ”, where the degree of iteration is m . In our earlier example, $K_*({1, 2, 3, 4, 5, 6}) = {2, 3, 4, 5, 6}$ since $K_1({1, 2, 3, 4, 5, 6}) = {1, 2, 3, 4, 5, 6}$ and $K_2({1, 2, 3, 4, 5, 6}) = {2, 3, 4, 5, 6}$; $K_*^2({1, 2, 3, 4, 5, 6}) = {3, 4, 5, 6}$ since $K_1({2, 3, 4, 5, 6}) = {3, 4, 5, 6}$ and $K_2({2, 3, 4, 5, 6}) = {2, 3, 4, 5, 6}$.

We say that an event A is *common knowledge at state ω* if, for any composition of knowledge operators $\{K_i(\cdot) | i \in I\}$ given by $K(\cdot)$, we have $\omega \in K(A)$. This definition of common knowledge is now standard, and can be traced to Aumann (1976). Milgrom (1981) and Bacharach (1985) have developed this definition. We shall be particularly interested in those events that are common knowledge whenever they are true. We shall refer to them as *evident events*. That is, an event A is *evident* if, whenever $\omega \in A$, A is common knowledge at ω . Note that the empty event is evident trivially.

Theorem 2.1. *Suppose Ω has depth of knowledge n and suppose $A \in \mathcal{F}$. Then, for all $m > n$, the event $K_*^m(A)$ is evident.*

We can re-state the content of the theorem as follows. Since $K_*(A) \subseteq A$ and $K_*^{m+1}(A) \equiv K_*(K_*^m(A))$, it is clear that $K_*^{m+1}(A) \subseteq K_*^m(A)$, that is, the sets $\{K_*^m(A)\}$ are weakly decreasing. Since they are all finite, they must converge as $m \rightarrow \infty$, hence for some m , $K_*^{m+j}(A) = K_*^m(A)$, for all $j \geq 0$. Hence $K_*^m(A)$ is evident. Thus there must exist a smallest m such that $K_*^m(A)$ is evident, for every event A . The theorem shows that this m is precisely the depth of knowledge, as we defined it.

In proving this theorem, we need some preliminary definitions and results. To begin with, note that the knowledge operator satisfies the following properties.

$$K_i(A \cap B) = K_i(A) \cap K_i(B). \tag{K0}$$

$$K_i(A) \subseteq A. \tag{K1}$$

To verify (K0), $\omega \in K_i(A \cap B) \Leftrightarrow P_i(\omega) \subseteq (A \cap B) \Leftrightarrow P_i(\omega) \subseteq A$ and $P_i(\omega) \subseteq B \Leftrightarrow \omega \in K_i(A)$ and $\omega \in K_i(B)$. To verify (K1), $\omega \in K_i(A) \Rightarrow \omega \in P_i(\omega) \subseteq A \Rightarrow \omega \in A$.

We shall refer to any function:

$$f: \{1, 2, \dots, k\} \rightarrow I$$

as an *indexing function of degree k* . Thus, the sequence $f(1), f(2), \dots, f(k)$ is some sequence of names of individuals in I . We denote by $F(k)$ the set of all indexing functions of degree k . Using this notation, we can demonstrate the following lemma.

Lemma 2.1. $K_*^m(A) = \bigcap_{f \in F(m)} K_{f(m)}(K_{f(m-1)}(\dots K_{f(1)}(A)\dots))$.

Proof. The proof is by induction on m . For $m = 1$, $K_*^m(A) = K_*(A)$, and $K_*(A) = \bigcap_{i \in I} K_i(A)$. Since $F(1)$ consists of all functions f such that $f(1) \in I$, we have:

$$K_*(A) = \bigcap_{f \in F(1)} K_{f(1)}(A).$$

For the inductive step, assume that the statement of the lemma holds for $m - 1$.

Then,

$$\begin{aligned} K_*^m(A) &= K_*(K_*^{m-1}(A)) \\ &= K_* \left[\bigcap_{f \in F(m-1)} K_{f(m-1)}(\dots K_{f(1)}(A)\dots) \right] \\ &= \bigcap_{i \in I} K_i \left[\bigcap_{f \in F(m-1)} K_{f(m-1)}(\dots K_{f(1)}(A)\dots) \right] \\ &= \bigcap_{i \in I} \bigcap_{f \in F(m-1)} K_i(K_{f(m-1)}(\dots K_{f(1)}(A)\dots)), \text{ by (K0)} \\ &= \bigcap_{f \in F(m)} K_{f(m)}(\dots K_{f(m-1)}(\dots K_{f(1)}(A)\dots)). \end{aligned}$$

Now, consider the following subset of Ω for some state $\omega \in \Omega$.

$$Q_k(\omega) \equiv \bigcap \{A \mid A \in \mathcal{C}_k \text{ and } \omega \in A\}.$$

In other words, $Q_k(\omega)$ is the event for which, if ω' and ω'' are elements of $Q_k(\omega)$, then ω' and ω'' agree on every event in \mathcal{C}_k . We may interpret $Q_k(\omega)$ as the event which corresponds to the proposition which is the maximally specific description of the state ω in terms of propositions which involve iterated knowledge of degree k or lower. We have the following lemma.

Lemma 2.2. *If $\mathcal{C}_k = \mathcal{C}_{k+1}$, then for all $\omega \in \Omega$ and all $i \in I$, either $K_i(Q_k(\omega))$ is empty or $K_i(Q_k(\omega)) = Q_k(\omega)$.*

Proof. We prove the contrapositive. Suppose that there is some state ω and some individual i such that:

$$\begin{aligned} K_i(Q_k(\omega)) &\neq \emptyset, \\ K_i(Q_k(\omega)) &\neq Q_k(\omega). \end{aligned}$$

By the knowledge axiom (K1), this pair of conditions is equivalent to:

$$\emptyset \neq K_i(Q_k(\omega)) \subset Q_k(\omega),$$

where the inclusion is strict. Then, $K_i(Q_k(\omega)) \notin \mathcal{C}_k$, since by definition there is no event in \mathcal{C}_k which includes ω but is strictly finer than $Q_k(\omega)$. However, we know that $K_i(Q_k(\omega))$ belongs to the next highest depth of events \mathcal{C}_{k+1} . Thus, $\mathcal{C}_k \neq \mathcal{C}_{k+1}$. \square

We also have the following lemma.

Lemma 2.3. *For any $A \in \mathcal{F}$, if $\omega \in K_*^m(A)$, then $K_*^m(A) = Q_m(\omega)$.*

Proof. From the distributive property (K0) of the knowledge operator, every event A in \mathcal{C}_m can be expressed as:

$$A = \bigcap \{B \mid B \in \mathcal{C}_{m-1} \text{ or } B = K_i(C) \text{ or } B = -K_i(C), \text{ where } C \in \mathcal{C}_{m-1}, i \in I\}$$

By repeated applications of (K0), any event $A \in \mathcal{C}_m$ can be expressed as the intersection of events in the set consisting of \mathcal{C}_0 and the events of the form:

$$Z_{f(l)}(Z_{f(l-1)}(\cdots(Z_{f(1)}(E))\cdots)), \quad (2.2)$$

where $l \leq m$, $E \in \mathcal{C}_0$, and $Z_{f(i)}$ is either $K_{f(i)}$ or $-K_{f(i)}$. Define the operator $L_i(\cdot)$ as:

$$L_i(B) \equiv -K_i(-B).$$

$L_i(B)$ is the event in which i does not know that B is false. In other words, $L_i(B)$ is the event in which i allows that B is possible. Then, the event (2.2) can be expressed as:

$$\tilde{Z}_{f(l)}(\tilde{Z}_{f(l-1)}(\cdots(\tilde{Z}_{f(1)}(\tilde{E}))\cdots)), \quad (2.3)$$

where $\tilde{Z}_{f(i)}$ is either $K_{f(i)}$ or $L_{f(i)}$, and \tilde{E} is either E or $-E$. By property (K1), we have $B \subseteq L_i(B)$ for all i and event B . Thus, we have the set inclusion:

$$K_{f(l)}(K_{f(l-1)}(\cdots(K_{f(1)}(\tilde{E}))\cdots)) \subseteq \tilde{Z}_{f(l)}(\tilde{Z}_{f(l-1)}(\cdots(\tilde{Z}_{f(1)}(\tilde{E}))\cdots)).$$

Now, consider $Q_m(\omega)$. Since $Q_m(\omega)$ is the intersection of events in \mathcal{C}_m , it can be represented as the intersection of events in the set consisting of \mathcal{C}_0 and the events of the form (2.3). For any $E \in \mathcal{C}_0$ for which $\omega \in E$, we have $\omega \in K_*^m(A) \subseteq A \subseteq E$, since $A \in \mathcal{F}$. By hypothesis, $\omega \in K_*^m(A)$. By lemma 1 and the inclusion above, we have $K_*^m(A) \subseteq Q_m(\omega)$. However, the reverse inclusion also holds, since by definition, $Q_m(\omega)$ is the finest event in \mathcal{C}_m which includes ω . Thus, $K_*^m(A) = Q_m(\omega)$. \square

We now have all the necessary steps in the proof of Theorem 2.1. Thus, suppose Ω has depth of knowledge n . In other words, $\mathcal{C}_{n-1} \neq \mathcal{C}_n$ but $\mathcal{C}_n = \mathcal{C}_m$ for all $m \geq n$. Consider the event $K_*^m(A)$. Either this event is empty or it is not. If it is empty, then it is evident trivially. Thus, suppose $K_*^m(A)$ is non-empty. Then, by Lemma 2.3, $K_*^m(A) = Q_m(\omega)$, for some state ω . By Lemma 2.2 and the fact that $K_*^m(A)$ is non-empty,

$$K_i(K_*^m(A)) = K_*^m(A).$$

Thus, $K_*^m(A)$ is evident. This concludes the proof of Theorem 2.1.

3. Approximating knowledge

In Section 2, we identified the depth of knowledge which is needed to interpret a given information system. As argued in the introduction, most economic models will implicitly assume a depth of knowledge of zero (so there is no uncertainty about the knowledge of other economic actors). While this is surely untrue in virtually every economic problem of interest (there is inevitably some uncertainty about others' knowledge), we would like to know when we can safely ignore such additional uncertainty as exists.

Suppose we have an uncertainty space with depth of knowledge n , and we have results for information systems with depth of knowledge $m < n$. Suppose that we can

find an event E , a subset of the state space, which is a good approximation to the whole state space (in a sense to be defined), such that the depth of knowledge conditional on event E is m . Then it may be reasonable to restrict analysis to event E .

In this section, we pursue such an approach. It turns out that an event will be a sufficiently good approximation if it is an “evident p -belief” event – that is, an event which is always believed with probability at least p whenever it is true. Monderer and Samet (1989) introduced this notion in characterizing common p -belief. We will review some of their results below.

For some event $E \subset \Omega$, define *conditional partitions*, *conditional knowledge* and *conditional mutual knowledge operators* as follows:

$$\begin{aligned}
 P_i(\omega|E) &= P_i(\omega) \cap E \\
 K_i(F|E) &= \{\omega \in E | P_i(\omega|E) \subset F\} \\
 K_*(F|E) &= \bigcap_{i \in I} K_i(F|E)
 \end{aligned}$$

Conditional common knowledge and the conditional depth of knowledge are then defined in the natural way.

Lemma 3.1. $[K_*]^k F \cap E \subset [K_*(\cdot|E)]^k F$, for all events E, F and integers k .

Thus if there is k th order (unconditional) mutual knowledge that F has occurred, and E is true, then it is k th order E conditional mutual knowledge that F has occurred.

Proof. First note from definitions that $K_*(F|E) = K_*(F \cap E|E)$ for all events F and that the lemma is true for $k = 1$. Suppose true for $k \leq c$. Then

$$[K_*]^c F \cap E \subset [K_*(\cdot|E)]^c F$$

Thus

$$[K_*]^{c+1} F \cap E \subset K_*(\cdot|E)[K_*]^c F = K_*(\cdot|E)([K_*]^c F \cap E) \subset [K_*(\cdot|E)]^{c+1} F$$

Now the lemma holds by induction on c . \square

In order to consider when E is a good approximation to Ω , we introduce belief operators. Suppose each individual i has some strictly positive prior π_i on the state space. Define the “ p belief operator” for player i by

$$B_i^p F = \{\omega \in \Omega | \pi_i[F | P_i(\omega)] \geq p\}$$

$B_i^p F$ is the set of states where event F is believed with at least probability p by i . Also define a “mutual p -belief operator” by

$$B_*^p F = \bigcap_{i \in I} B_i^p F$$

Event F is mutual p -belief if every individual believes F with probability at least p . Event F is said to be k th order mutual p -belief at state ω if $\omega \in [B_*^p]^k F$, so that every individual believes with probability p that (k times) ... that every individual believes F with probability at least p . Event F is said to be common p -belief at state ω if $\omega \in [B_*^p]^k F$, for all $k \geq 1$.

Monderer and Samet’s key result was a characterization of common p -belief closely related to Aumann’s (1976) classic characterization of common knowledge. Event F is said to be “evident p -belief” if it is believed with probability at least p by every player whenever it occurs, so that $F \subset B_i^p F$ for each i .

Theorem⁴. [Monderer and Samet (1989)] *Event E is common p -belief at state ω if and only if there exists an evident p -belief event F such that $\omega \in F$ and $F \subset B_i^p E$ for all i .*

The following lemma makes clear a key connection between conditional knowledge and evident p -belief.

Lemma 3.2. *If E is an evident p -belief event, then $K_*(F|E) \subset B_*^p F$.*

Proof. $\omega \in K_*(F|E)$ implies $\omega \in E$ implies (by E evident p -belief) $\pi_i[E|P_i(\omega)] \geq p$ for all i . But $\omega \in K_*(F|E)$ implies $\pi_i[F|E] = 1$. So $\pi_i[F|P_i(\omega)] \geq p$ and $\omega \in B_*^p F$. \square

In the next section, we consider an economic example concerning the existence of bubbles in rational expectations equilibria. In this example, we can use the ideas of the depth of knowledge, conditional knowledge and evident p -belief to give necessary conditions for bubbles as well as bounds on their size.

4. Bubbles

In this section, we will illustrate the ideas above through their application to the question of the existence of bubbles in a rational expectations finite horizon model. We will be concerned with the evolution of knowledge through time. Let there be time periods $1, 2, \dots, T$ and let individual i ’s period t information partition be \mathcal{P}_{it} . Define knowledge operators by $K_{it}F = \{\omega \in \Omega | P_{it}(\omega) \subset F\}$. $K_{it}F$ is the set of states where event F is known to be true by i at time t . Also define a mutual knowledge operator by

$$K_{*t}F = \bigcap_{i \in I} K_{it}F$$

Event F is said to be k th order mutual knowledge at state ω at time t if $\omega \in [K_{*t}]^k F$. Event F is said to be common knowledge at state ω at time t if $\omega \in [K_{*t}]^k F$, for all k . We assume that information improves through time, so that $P_{it}(\omega) \subset P_{is}(\omega)$ if $t \geq s$. This implies $K_{is}F \subset K_{it}F$ if $t \geq s$, and so $K_{*s}F \subset K_{*t}F$ if $t \geq s$. Also assume $P_{iT}(\omega) = \{\omega\}$ for all $\omega \in \Omega$, so that $K_{*T}F = K_{iT}F = F$, for all events F . Thus all uncertainty is resolved at time T .

We also introduce the following notation for the expectation of any random variable $x: \Omega \rightarrow \mathbf{R}_+$ conditional on i ’s information in state ω at time t .

$$E_{it}[x|\omega] = \frac{\sum_{\omega' \in P_{it}(\omega)} \pi_i(\omega')x(\omega')}{\sum_{\omega' \in P_{it}(\omega)} \pi_i(\omega')}$$

⁴ Following Aumann (1976), Monderer and Samet actually defined common p -belief in terms of evident p -belief events and showed the equivalence to the iterated notion.

We are interested in the price paths for an asset we might see in a rational expectations equilibrium with short sales constraints, as individuals' information evolves through time. For simplicity, consider the case where all individuals are risk neutral, but they have differing prior beliefs over states of the world. Allen, Morris and Postlewaite [AMP] (1993) showed that qualitatively similar results hold if individuals share the same prior beliefs, and the ex ante motivation for trade comes from differences in risk aversion or endowments.

Rather than formally describing rational expectations equilibria (REE) of such an economy, we will focus on necessary conditions for such a REE. Suppose that risk neutral individuals are trading a riskless asset (which will pay out one unit with certainty in period T), and also a risky asset. Let $d: \Omega \rightarrow \mathbf{R}_+$ be the dividend of the risky asset, paying out in period T . Suppose each individual is short sales constrained in his holding of the risky asset, but is not short sales constrained in the riskless asset. Individuals have some initial private information, which may be refined by information revealed by equilibrium prices. Let \mathcal{P}_t be the equilibrium information of individual i at time t (i.e. incorporating the information revealed by prices). Write $q_t: \Omega \rightarrow \mathbf{R}_+$ for the price of the risky asset (in terms of a riskless asset) in period t . Then we have:

Lemma (AMP). *At any rational expectations equilibrium, $q_T = d$ and prices in earlier periods can be calculated by backward induction:*

$$\text{i.e. } q_t(\omega) = \max_{i \in I} E_{i_t}[q_{t+1} | \omega] \text{ for all } \omega \in \Omega, \quad t = 1, \dots, T.$$

Notice that if d^* is the maximum possible value of the asset [i.e. $d^* = \max_{\omega \in \Omega} d(\omega)$], we have (again by backward induction) $0 \leq q_t(\omega) \leq d^*$, for all ω, t . The idea of the proof of the lemma is as follows. If the price of the asset was strictly less than any individual's expectation of the price of the asset in the next period, he could not be maximizing, since he could always buy another unit of the asset. On the other hand, if the price of the asset was strictly greater than every individual's expectation of the price of the asset in the next period, no one would hold the asset.

AMP argued that while there was some ambiguity in defining what was meant by a bubble in such a context, any reasonable definition of a bubble would include the case that it was known to all individuals that the dividend would be zero, but the asset traded for a strictly positive price nonetheless. It was said that a rational expectations equilibrium exhibited a strong bubble at a state for which this was true. We will generalize that approach here. Write Ω_T for the set of states where the dividend is zero, i.e. $\Omega_T = \{\omega \in \Omega | d(\omega) = 0\}$. Now we give results which show how the time period and depth of knowledge can be linked together to give a precise characterization of when there is a strong bubble in the rational expectations equilibrium.

Theorem 4.1 $\{\omega \in \Omega | q_t(\omega) = 0\} = K_{*t} K_{*,t+1} \cdots K_{*,T-1} \Omega_T$.

Thus the price of the asset is zero in period t if and only if every individual knows that every individual will know in period $t + 1$ that every individual will know in

period $t + 2$ that... every individual will know in period $T - 1$ that the asset is worthless.

This result is significant in that it characterizes the conditions under which a finite bubble may exist. However, it should be seen as a “reduced form” result in the following sense. The distinguishing feature of a rational expectations equilibrium is that the prices reflect information. Hence the information partitions which underlie the knowledge operators in Theorem 4.1 are *endogenous*. Given only agents’ initial information (before observing the information revealed prices), it is not possible to use Theorem 4.1 to directly check for bubbles. Nevertheless, this result gives us a handle on what to look for in searching for bubbles. Bubbles exist when and only when the sequence of partitions generated by the price system is coarse enough to block iterated knowledge of the appropriate order (as in the example in AMP).

Proof. Let Ω_t be the set of states where the price is zero at time t , i.e. $\Omega_t = \{\omega \in \Omega \mid q_t(\omega) = 0\}$. Now $E_{it}[q_{t+1} \mid \omega] = 0$ if and only if $P_{it}(\omega) \subset \Omega_{t+1}$ for all i , i.e. $\omega \in K_{*t}\Omega_{t+1}$. \square

Corollary 4.1. $[K_{*t}]^{T-t}\Omega_T \subset \{\omega \in \Omega \mid q_t(\omega) = 0\}$.

If it is $(T - t)$ th mutual knowledge at time t that the asset is worthless, then the price is zero. The statement of Theorem 4.1 involved knowledge statements at different time periods. The power of the corollary is that it depends on a statement about knowledge at time $T - t$ only. An example in AMP shows that this corollary is tight in the following sense. They constructed a three-period example in which in the first period everyone knows that the dividend will be zero, but the price is positive nonetheless. Corollary 4.1 shows that it could not be the case that everyone knew that everyone knew that the dividend would be zero for such an example, if everyone knows that everyone knows that the dividend is zero, and the price is strictly positive, there must be at least three periods to go until the dividend is paid (i.e. $T \geq 4$).

The proof of Corollary 4.1 exploits the fact that knowledge can only improve through time.

Proof. $K_{*s}F \subset K_{*t}F$ and $K_{*t}F \subset K_{*t}G$, for all $s \leq t$ and events $F \subset G$. Thus $[K_{*t}]^{T-t}\Omega_T \subset K_{*t}K_{*t+1}\dots K_{*T-1}\Omega_T$. \square

We can use Corollary 4.1 to make a connection between the depth of knowledge and the existence of bubbles. By Theorem 2.1, if the depth of knowledge of an information system is n , an event is k th order mutual knowledge and $k \geq n$, then the event is evident (and therefore m th order mutual knowledge for any finite m). Thus we have:

Corollary 4.2. If it is k th order mutual knowledge at time t that the asset is valueless [i.e. $\omega \in [K_{*t}]^k\Omega_T$], the “depth of knowledge” at time t is n , and $k \geq \min\{n, T - t\}$, then the asset has zero price.

Proof. If $k \geq T - t$, then true by Corollary 4.1. If $k \geq n$, then k th order knowledge implies common knowledge (i.e. m th order mutual knowledge for all m), by Theorem 2.1, so again true by Corollary 4.1. \square

Now we show how to combine the ideas of belief operators and the depth of knowledge in characterizing bubbles. First note the following approximation results which does *not* have to do with the depth of knowledge. Let $\Omega_t(p)$ be the set of states where the dividend is less than $(1 - p)(T - t)d^*$, i.e. $\Omega_t(p) = \{\omega \in \Omega \mid q_t(\omega) \leq (1 - p)(T - t)d^*\}$. Note that $\Omega_T(p) = \Omega_T = \{\omega \in \Omega \mid d(\omega) = 0\}$.

Theorem 4.2. $B_{*t}^p B_{*t+1}^p \dots B_{*T-1}^p \Omega_T \subset \Omega_t(p)$.

Thus if every individual p -believes that every individual will p -believe in period $t + 1$ that every individual will p -believe in period $t + 2$ that... every individual will p -believe in period $T - 1$ that the asset is worthless, then the price of the asset is no more than $(1 - p)(T - t)d^*$.

Proof. It is sufficient to prove that $B_{*t}^p \Omega_{t+1}(p) \subset \Omega_t(p)$, for all t . Suppose $\omega \in B_{*t}^p \Omega_{t+1}(p)$. Then each individual i assigns at most probability $1 - p$ to states not in $\Omega_{t+1}(p)$, where the price is at most d^* . At states in $\Omega_{t+1}(p)$, the price is at most $(1 - p)(T - t - 1)d^*$. Thus, for each i , the expectation of the price in the next period is no more than $(1 - p)d^* + p(1 - p)(T - t - 1)d^* \leq (1 - p)d^*[1 + T - t - 1] = (1 - p)d^*(T - t)$. So $\omega \in \Omega_t(p)$. \square

But suppose instead of it being evident p -belief that the price is zero, it is evident p -belief that we are in a subset of the state space where the depth of knowledge is at most k . Then we obtain an alternative approximation result (which uses the above result).

Theorem 4.3. Suppose that, at state ω^* , (i) it is k th order mutual knowledge at time t that the asset is worthless ($\omega^* \in [K_{*t}]^k \Omega_t$), (ii) there exists a subset of the state space, E , such that E is evident p -belief at every date following t [$E \subset B_{*t'}^p E$, for all $t' \geq t$], (iii) E is true [$\omega^* \in E$] and (iv) the depth of knowledge conditional on E is less than or equal to k . Then the price of the asset at state ω^* at time t is at most $(1 - p)(T - t)d^*$.

Proof. By (i), (iii) and Lemma 3.1, it is k th order E conditional mutual knowledge at state ω^* at time t that the asset is valueless. But then by (iv), it is E conditional common knowledge at state ω^* at time t that the asset is valueless, since all k th order statements must be true for all $c \geq k$. In particular, it must be $(T - t)$ th order E conditional mutual knowledge at state ω^* at time t that the asset is valueless (for all t, T). Thus

$$\omega^* \in [K_t^*(\cdot | E)]^{T-t} \Omega_T$$

by the argument of Corollary 4.1, $[K_t^*(\cdot | E)]^{T-t} \Omega_T \subset K_t^*(\cdot | E) K_{t+1}^*(\cdot | E) \dots K_{T-1}^*(\cdot | E) \Omega_T$

by Lemma 3.2, $K_t^*(\cdot | E) K_{t-1}^*(\cdot | E) \dots K_{T-1}^*(\cdot | E) \Omega_T \subset B_{*t}^p B_{*t+1}^p \dots B_{*T-1}^p \Omega_T$

by Theorem 4.2, $B_{*t}^p B_{*t+1}^p \dots B_{*T-1}^p \Omega_T \subset \Omega_t(p)$

so $\omega^* \in \Omega_t(p)$ \square

5. Conclusion

This paper focusses on a relatively narrow question. Suppose we are interested in an economic problem where the qualitative outcomes depend critically on the depth of

knowledge. How do we formally define the depth of knowledge and once we have defined it, when do we expect a model with a high depth of knowledge to behave like a model with a lower depth of knowledge? In the bubbles example of Section 4, these questions made sense. The depth of knowledge was the critical property for the existence of bubbles. We found conditions under which we could bound the size of the bubbles by looking at an evident p -belief subset of the state space where the depth of knowledge was reduced.

More generally, the methods of this paper potentially can be applied to analyze the interaction between asymmetric information about fundamentals and asymmetric information about economic actors' knowledge and beliefs. For some problems such as the bubble problem above, there is good intuition as to why higher orders of knowledge affect the analysis both qualitatively and quantitatively. For other problems, we conjecture that increasing the depth of knowledge should have no significant effect on our understanding of those problems. An analysis of the interaction between uncertainty about fundamentals and knowledge will help identify those problems for which it is important to take seriously the depth of knowledge.

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